Remarks on form factor bounds

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Improved model independent upper bounds on the weak transition form factors are derived using inclusive sum rules. A comparison of the new bounds with the old ones is made for the form factors h_{A_1} and h_V in $B \to D^*$ decays.

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A set of model independent bounds has been derived to provide a restriction on the shape of weak transition form factors [1–3]. They have been extensively used to bound weak decay form factors and the decay spectrum of heavy hadrons [2–5].[1] Here we provide a more stringent upper bound without any further assumptions. This upper bound differs from the one derived previously at order $1/m_Q^2$ or α_s/m_Q. Though this is only a small improvement, it is worth doing because it can give a tighter bound from above if one includes higher order corrections.

The bounds are derived from sum rules that relate the inclusive decay rate, calculated using the operator product expansion (OPE) [8,9] and perturbative QCD, to the sum of exclusive decay rates. To be complete, we will derive both the upper and lower bounds, though the lower bound is the same as the previous one.

Without loss of generality, we take for example the decay of a B meson into an H meson, with the underlying quark process $b \to f$, where f could be either a heavy or light quark. First, consider the time ordered product of two weak transition currents taken between two B mesons in momentum space:

$$\begin{aligned} T^{\mu\nu} &= \frac{i}{2M_B}\int d^4x\, e^{-iq\cdot x}\langle B(v)|T(J^{\mu\dagger}(x)J^{\nu}(0))|B(v)\rangle \\ &= -g^{\mu\nu}T_1 + v^\mu v^\nu T_2 + i\epsilon^{\mu\nu\alpha\beta}q_\alpha v_\beta T_3 + q^\mu q^\nu T_4 \\ &\quad + (q^\mu v^\nu + v^\mu q^\nu)T_5, \end{aligned} \tag{1}$$

where J^μ is a $b \to f$ weak transition current. The time ordered product can be expressed as a sum over hadronic or partonic intermediate states. The sum over hadronic states includes the matrix element $\langle H|J|B\rangle$. After inserting a complete set of states and contracting with a four-vector pair $a_\mu^* a_\nu$, we obtain

$$\begin{aligned} T(\epsilon) &= \frac{1}{2M_B}\sum_X (2\pi)^3\delta^3(\vec{p}_X+\vec{q})\frac{|\langle X|a\cdot J|B\rangle|^2}{E_X - E_H - \epsilon} \\ &\quad + \frac{1}{2M_B}\sum_X (2\pi)^3\delta^3(\vec{p}_X-\vec{q})\frac{|\langle B|a\cdot J|X\rangle|^2}{\epsilon + E_X + E_H - 2M_B}, \end{aligned} \tag{2}$$

where $T(\epsilon) \equiv a_\mu^* T^{\mu\nu} a_\nu$, $\epsilon = M_B - E_H - v\cdot q$, and the sum over X includes the usual $\int d^3p/2E_X$ for each particle in the state X. We choose to work in the rest frame of the B meson, $p = M_B v$, with the z axis pointing in the direction of $\vec{q}$. We hold q_3 fixed while analytically continuing $v\cdot q$ to the complex plane. $E_H = \sqrt{M_H^2 + q_3^2}$ is the H meson energy. There are two cuts in the complex ϵ plane, $0 < \epsilon < M_B - E_H$, corresponding to the decay process $b \to f$, and $-\infty < \epsilon < -2E_H$, corresponding to two b quarks and a $\bar{f}$ quark in the final state. The second cut will not be important for our discussion.

The integral over ϵ of the time ordered product $T(\epsilon)$, times a weight function $\epsilon^n W_\Delta(\epsilon)$ can be computed perturbatively in QCD [2,3]. For simplicity, we pick the weight function $W_\Delta(\epsilon) = \theta(\Delta - \epsilon)$, which corresponds to summing over all hadronic resonances up to the excitation energy Δ with equal weight. Relating the integral with the hard cutoff to the exclusive states requires local duality at the scale Δ. Therefore, Δ must be chosen large enough so that the structure functions can be calculated perturbatively.

Taking the zeroth moment of $T(\epsilon)$, we get

$$\begin{aligned} M_0 &\equiv \frac{1}{2\pi i}\int_C d\epsilon\, \theta(\Delta - \epsilon)\, T(\epsilon) \\ &= \frac{|\langle X|a\cdot J|B\rangle|^2}{4M_B E_H} + \sum_{X\neq H}{}' \; \theta(E_X - E_H - \Delta)(2\pi)^3\delta^3(\vec{q}+\vec{p}_X)\frac{|\langle X|a\cdot J|B\rangle|^2}{2M_B}, \end{aligned}$$

where the primed summation means a sum over all the kinematically allowed states except the H meson. So,

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[1] See, however, [6,7] for model independent parametrizations of the form factors.

$$\frac{|\langle X|a\cdot J|B\rangle|^2}{4M_BE_H\epsilon}=M_0-\sum_{X\neq H}{}' \ \theta(E_X-E_H-\Delta)(2\pi)^3\delta^3(\vec{q}+\vec{p}_X)\frac{|\langle X|a\cdot J|B\rangle|^2}{2M_B}. \tag{3}$$

On the other hand, the first moment of $T(\epsilon)$ gives

$$\begin{aligned}M_1&\equiv\frac{1}{2\pi i}\int_C d\epsilon\,\epsilon\,\theta(\Delta-\epsilon)T(\epsilon)\\&=\sum_{X\neq H}{}' \ \theta(\Delta-E_X+E_H)(E_X-E_H)\,(2\pi)^3\delta^3(\vec{p}_X+\vec{q})\frac{|\langle X|a\cdot J|B\rangle|^2}{4M_BE_X}\\&\begin{cases}\leqslant(E_{max}-E_H)\sum_{X\neq H}{}' \ \theta(\Delta-E_X+E_H)(2\pi)^3\delta^3(\vec{p}_X+\vec{q})\frac{|\langle X|a\cdot J|B\rangle|^2}{4M_BE_X},\\ \geqslant(E_1-E_H)\sum_{X\neq H}{}' \ \theta(\Delta-E_X+E_H)(2\pi)^3\delta^3(\vec{p}_X+\vec{q})\frac{|\langle X|a\cdot J|B\rangle|^2}{4M_BE_X},\end{cases}\end{aligned} \tag{4}$$

where E_{max} and E_1 denote the highest energy state kinematically allowed and the first excited state that is more massive than H meson, respectively. Here the validity of the second inequality relies on the assumption that multiparticle final states with energy less than E_1 contribute negligibly. This assumption is true in large N_c, and is also confirmed by current experimental data. However, the first inequality is valid without any further assumption.

From Eq. (3) and the first inequality in Eq. (4), one can get an upper bound on the matrix element $|\langle H|a\cdot J|B\rangle|^2/4M_BE_H$:

$$\frac{|\langle H|a\cdot J|B\rangle|^2}{4M_BE_H}\leqslant\frac{1}{2\pi i}\int_C d\epsilon\,\theta(\Delta-\epsilon)\,T(\epsilon)\left(1-\frac{\epsilon}{E_{max}-E_H}\right). \tag{5}$$

Dropping $\epsilon/(E_{max}-E_H)$ on the right hand side gives the previously derived upper bound [1–3]. Since $E_{max}-E_H$ is of order m_Q and the first moment, M_1, is of order $1/m_Q$ and positive definite, this extra term makes the new upper bound smaller than the old one at order $1/m_Q^2$. Perturbative corrections will also modify the new bound at order α_s/m_Q.

Similarly, a lower bound can be formed by combining Eq. (3) and the second inequality in Eq. (4) to be

$$\frac{|\langle H|a\cdot J|B\rangle|^2}{4M_BE_H}\geqslant\frac{1}{2\pi i}\int_C d\epsilon\,\theta(\Delta-\epsilon)\,T(\epsilon)\left(1-\frac{\epsilon}{E_1-E_H}\right). \tag{6}$$

Therefore, we find the bounds

$$\begin{aligned}\frac{1}{2\pi i}&\int_C d\epsilon\,\theta(\Delta-\epsilon)\,T(\epsilon)\left(1-\frac{\epsilon}{E_1-E_H}\right)\\&\leqslant\frac{|\langle H(v')|a\cdot J|B(v)\rangle|^2}{4M_BE_H}\\&\leqslant\frac{1}{2\pi i}\int_C d\epsilon\,\theta(\Delta-\epsilon)\,T(\epsilon)\left(1-\frac{\epsilon}{E_{max}-E_H}\right).\end{aligned} \tag{7}$$

Since $1/(E_1-E_H)\sim1/\Lambda_{\rm QCD}$, the lower bounds will be good to one less order in $1/m_Q$ than the upper bound.

As emphasized in [3], the old upper bound is essentially model independent while the lower bound relies on the assumption about the final state spectrum. The new upper bound provided here is also model independent. These bounds are valid for both heavy mesons and baryons. (For baryons, a spin sum $[M_H/(2j+1)]\Sigma_{S,S'}$ needs to be included in front of the bounded factor.)

Great interest has been paid to the semileptonic exclusive decay rate of $B\to D^*l\bar{\nu}$ from which $|V_{cb}|$ can be extracted [10]. As an example, we now focus on the case that H is the D^* meson and give, in particular, the upper bounds on the form factors h_{A_1} and h_V. The hadronic matrix element for the semileptonic decay of a B meson into a vector meson D^* may be parametrized as

$$\begin{aligned}&\frac{\langle D^*(v',\varepsilon)|V^\mu-A^\mu|B(p)\rangle}{\sqrt{M_{D^*}M_B}}\\&\quad=-h_{A_1}(\omega)(\omega+1)\varepsilon^{*\mu}+[h_{A_2}(\omega)v^\mu+h_{A_3}(\omega)v'^\mu]v\cdot\varepsilon^*\\&\qquad+ih_V(\omega)\epsilon^{\mu\nu\alpha\beta}\varepsilon^*_\nu v'_\alpha v_\beta,\end{aligned} \tag{8}$$

where v' is the velocity of the final state meson, and the variable $\omega=v\cdot v'$ is a measure of the recoil. One may relate ω to the momentum transfer q^2 by $\omega=(M_B^2+M_{D^*}{}^2-q^2)/(2M_BM_{D^*})$. Therefore, with a proper choice of the current J^μ and the four vector a^μ, one may readily single out the form factors, h_{A_1} and h_V, and establish corresponding bounds, as was done in Refs. [2–4]. Nonperturbative corrections to the structure functions can be found in Refs. [11–13], whereas complete $\mathcal{O}(\alpha_s)$ corrections are given in Refs. [3,4].

To obtain the bounding curves within the kinematic range, $1<\omega\lesssim1.25$, we will expand in α_s, $\Lambda_{\rm QCD}/m_Q$ and $\omega-1$. For both the upper and lower bounds, we will keep perturbative corrections up to order $\alpha_s(\omega-1)$, but drop

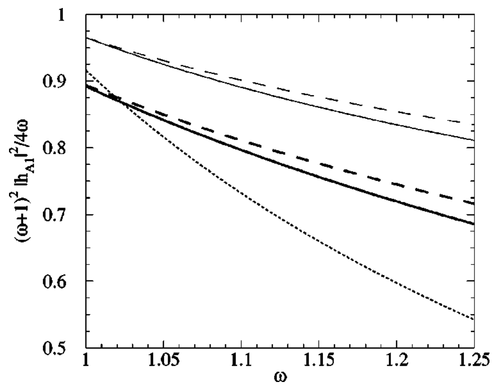

FIG. 1. The upper bound on $(\omega+1)^2|h_{A_1}(\omega)|^2/(4\omega)$. The thick solid (dashed) curve is the new (old) upper bound to $\mathcal{O}(1/m_Q^2)$ including perturbative corrections. The thin solid (dashed) curve is the upper bound to $\mathcal{O}(1/m_Q^2)$ without perturbative corrections. The dotted line is the lower bound to $\mathcal{O}(1/m_Q)$ including perturbative corrections.

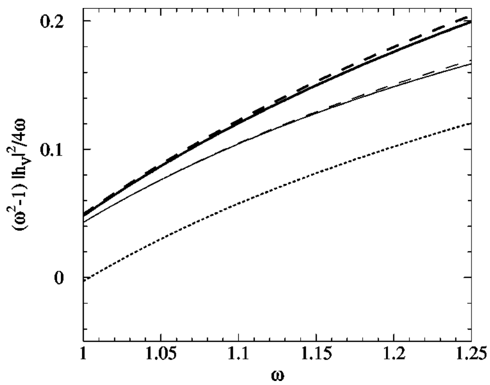

FIG. 2. The upper bound on $(\omega^2-1)|h_V(\omega)|^2/(4\omega)$. The thick solid (dotted) curve is the new (old) upper bound to $\mathcal{O}(1/m_Q^2)$ including perturbative corrections. The thin solid (dashed) curve is the upper bound to $\mathcal{O}(1/m_Q^2)$ without perturbative corrections. The dotted line is the lower bound to $\mathcal{O}(1/m_Q)$ including perturbative corrections.

terms of order $\alpha_s(\omega-1)^2$, α_s^2, and $\alpha_s\Lambda_{\mathrm{QCD}}/m_Q$. We will calculate to order $1/m_Q^2$ for the upper bounds, but only to order $1/m_Q$ for the lower bounds.

Both the old and new upper bounds along with the lower bound on h_{A_1} are shown[2] in Fig. 1. In this and the next example, the corresponding first excited state more massive than D^* that contributes to the sum rule is the $J^P=1^+$ state, i.e., the D_1 meson, and E_{max} is taken to be M_B in the limit of no energy transfer to the leptonic sector. The upper and lower bounds for $(\omega^2-1)|h_V(\omega)|^2/(4\omega)$ are shown in Fig. 2.

[2]For the figures we take $m_b=4.8$ GeV, $m_c=1.4$ GeV, $\alpha_s=0.3$ (corresponding to a scale of about 2 GeV), $\overline{\Lambda}=0.4$ GeV, $\lambda_1=-0.2$ GeV^2, $\lambda_2=0.12$ GeV^2 and $\Delta=1$ GeV.

In both diagrams, the thick solid (dashed) curve is the new (old) upper bound including perturbative corrections. The thin solid (dashed) curve is the upper bound without perturbative corrections. At large recoil, the new bound improves the upper limit by more than 4% in Fig. 1 and by about 3% in Fig. 2.

This work provides tighter upper bounds on weak decay form factors. The new upper bounds are compared with the old ones on, in particular, the $B\to D^*$ form factors, h_{A_1} and h_V. Their difference is due to the $1/m_Q^2$ nonperturbative corrections and α_s corrections that are suppressed by $1/M_Q$. The difference of higher order $1/m_Q$ corrections between the old and new bounds will be more significant.

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