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Potential for the phase of the Wilson line at nonzero quark density

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The contribution of quarks to the effective potential for the phase of the Wilson line is computed at nonzero temperature and quark chemical potential to one and two loop order. At zero temperature, regardless of the value of the quark chemical potential, the effective potential for the phase of the Wilson line vanishes. At nonzero temperature, for special values of the phase the free energy of quarks equals that of bosons; at nonzero chemical potential, such quarks can ''Bose condense,'' albeit with negative density.

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At nonzero temperature, $SU(N)$ gauge theories in the absence of dynamical fermions have a deconfining phase transition related to the spontaneous breaking of a global $Z(N)$ symmetry [1]. The order parameter for this transition is the Wilson line: its vacuum expectation value vanishes in the confined, low temperature phase, and is nonzero in the deconfined, high temperature phase. Because of the global $Z(N)$ symmetry, in the deconfined phase there are N equivalent vacua, with the phase of the Wilson line equal to one of the N^{th} roots of unity [2–5]. The interface tension between these N vacua is computable semiclassically [6–9], by computing the effective potential for the phase of the Wilson line. This interface tension is measurable from numerical simulations on the lattice [10]. In the presence of dynamical quarks in the fundamental representation, the Wilson line acquires a vacuum expectation value at all temperatures, and the vacuum is unique, with zero phase for the Wilson line. Nevertheless, depending upon the number of colors and quark flavors, metastable states in which the phase of the Wilson line is nonzero can arise [8], and may be of physical interest (although this is controversial [9]).

The effective potential for the phase of the Wilson line for an $SU(N)$ gauge theory has been computed at one [2] and two [4–7] loop order at a nonzero temperature, T. Here we extend this result to a nonzero quark chemical potential, μ [11].

The Wilson line is

$$\Omega(\vec{x}) = \mathcal{P}\exp\left(ig\int_0^\beta A_0(\vec{x},\tau)\,d\tau \right), \tag{1}$$

where $\mathcal{P}$ denotes path ordering, T is the temperature, g the gauge coupling constant, and the imaginary time $\tau{:}\,0\to\beta = 1/T$. To represent a nonzero phase of the Wilson line in the deconfined phase, we can take a constant background value for the time-like component of the gauge potential, A_0 [2–9]. After a global color rotation, this can be chosen to be a diagonal matrix

$$A_0 = \frac{2\pi T}{g}\begin{bmatrix} q_1 & 0 & \dots & 0 \\ 0 & q_2 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & q_N \end{bmatrix}, \quad \sum_{j=1}^{N} q_j = 0. \tag{2}$$

The sum of the q_j must vanish so that A_0 is an $SU(N)$, and not a $U(N)$, matrix. For example, consider $Z(N)$ degenerate vacua [6] in the pure glue theory. In the trivial vacuum the phase of the Wilson line vanishes, $\Omega = \mathbf{1}$, so all $q_j = 0$. The equivalent vacuum in which the Wilson line is the first N^{th} root of unity, $\Omega = \exp(2\pi i/N)\mathbf{1}$, is given by $q_1 = \cdots q_{N-1} = q/N$, and $q_N = -(N-1)q/N$.

At zero temperature, a constant A_0 field can be eliminated by a global gauge transformation. At nonzero temperature, the classical action is independent of A_0, but this degeneracy is lifted by quantum effects, as the required gauge transformation alters the boundary conditions for the fields in imaginary time. Consequently, at one loop order and beyond, a potential for A_0, or more properly for the phase of the Wilson line, is generated at nonzero temperature. In terms of partition functions [3], this potential is generated because one is summing over states which differ by a global gauge transformation.

We compute the quark contribution to the effective action for a single, massless flavor. At one loop order, this is just the free energy in the presence of a background A_0 field:

$$\mathcal{S}^1 = -\mathrm{Tr}\,\log i[\not{D}^{cl} - \mu\gamma_0]; \tag{3}$$

$\not{D}^{cl} = \not{\partial} - ig\not{A}$ is the covariant derivative, and the trace includes color, Dirac indices, and a loop integral. We include

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the effects of a Fermi sea for quarks by introducing a chemical potential for baryon number, μ.

In the following it helps to introduce two types of variables related to μ and the q_j. One set of variables are N color-dependent ''chemical potentials'':

$$\mu_j = \mu + 2\pi i T\, q_j\,, \quad j = 1, \ldots, N. \tag{4}$$

Notice that the q_j act like an imaginary chemical potential for color [3]. Since the μ_j are complex, the effective action is as well.

Alternately, we also use the variables C_j:

$$C_j = \frac{\mu_j}{2\pi i T} + \frac{1}{2} = q_j + \frac{1}{2} + \frac{\mu}{2\pi i T}, \quad j = 1, \ldots, N. \tag{5}$$

The μ_j are useful in comparing to the limit in which $\mu \neq 0$ and $A_0 \to 0$; the C_j are handy in the limit that $A_0 \neq 0$ and $\mu \to 0$. The 1/2 in the definition of C_j arises as follows. In the imaginary time formalism at nonzero temperature, bosonic energies k^0 are even multiples of πT, while fermionic energies are odd multiples. In diagrams, $2\pi T C_j$ enters like an energy, k^0; the 1/2 then turns C_j from a fermionic into a bosonic type energy. We find this convenient because our expressions for the effective potential of the Wilson line are written most naturally in terms of contributions from gluons, which are bosons.

As the background gauge field is diagonal in color, at one loop order the color trace is trivial:

$$\mathcal{S}^1 = \sum_{j=1}^{N} \mathcal{S}_j = -\sum_{j=1}^{N} \log[i\not\partial - i\mu_j \gamma_0]. \tag{6}$$

We compute the $\mathcal{S}_j$ in two different ways, first by using the μ_j's and second by using the C_j's. The first is careful and certainly correct, but awkward. The trace in $\mathcal{S}_j$ involves an integral over the loop momentum, $K^\mu = (k^0, \vec{k})$, where at $T \neq 0$, $k_0 = (2l+1)\pi T$ for integral $l = \pm 1, \pm 3 \ldots$. Performing the sum over l by contour integration,

$$\mathcal{S}_j = -\beta V 2T^3 \int \frac{d^3k}{(2\pi)^3} \{\log[1 + e^{-(k-\mu_j)/T}] + \log[1 + e^{-(k+\mu_j)/T}]\}; \tag{7}$$

V is the volume of space, so βV is the volume of space-time. To do the integration over spatial momentum, we use zeta function regularization; this was used at $T \neq 0$ by Actor [12,13] and by Weldon [14]. Expanding the logarithm and integrating over $\vec{k}$ gives

$$\mathcal{S}_j = -\beta V \frac{4T^4}{\pi^2} \sum_{m=1}^{\infty} (-1)^{m+1} \frac{1}{m^4} \cosh\left(\frac{m\mu_j}{T}\right). \tag{8}$$

In each cosh function we expand as a power series, interchange the order of the two series, and so obtain a sum over η functions,

$$\mathcal{S}_j = -\beta V \frac{4T^4}{\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(\frac{\mu_j}{T}\right)^{2n} \eta(4-2n), \tag{9}$$

where the η function is defined as

$$\eta(s) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{1}{i^s} = (1 - 2^{1-s})\zeta(s), \quad \mathrm{Re}(s) > 1. \tag{10}$$

Zeta function regularization is used to define the sum over negative s. Then a wonderful simplication enters: $\eta(-2m) = 0$ for any integral m, so only three terms contribute to $\mathcal{S}_j$,

$$\mathcal{S}_j = -\beta V \frac{4T^4}{\pi^2} \left\{ \eta(4) + \frac{\eta(2)}{2} \left(\frac{\mu_j}{T}\right)^2 + \frac{\eta(0)}{4!} \left(\frac{\mu_j}{T}\right)^4 \right\}. \tag{11}$$

For each μ_j, this expression is periodic in q_j modulo 1 over the interval $\in (-1/2, 1/2)$. Using the explicit values of the η function,

$$\mathcal{S}^1 = -\beta V \left\{ \frac{7\pi^2}{180} N T^4 + \frac{T^2}{6} \sum_{j=1}^{N} \mu_j^2 + \frac{1}{12\pi^2} \sum_{j=1}^{N} \mu_j^4 \right\}. \tag{12}$$

The real part of $\mathcal{S}^1$ agrees with Actor [12]; he neglected the imaginary part.

The second way to compute $\mathcal{S}^1$ is to start with the known result for the potential at zero chemical potential, as function of the q_j, and then substitute the C_j's for the q_j's:

$$\mathcal{S}^1 = -\beta V \frac{4}{3} \pi^2 T^4 \sum_{j=1}^{N} \left(B_4(C_j) - \frac{1}{30} \right). \tag{13}$$

Here $B_4(x)$ is the Bernoulli function,

$$B_4(x) = x^2 [1 - x\, \mathrm{sgn}(x)]^2. \tag{14}$$

For real values of x, $B_4(x)$ is defined to be a function of x modulo 1. We analytically continue to complex values of x in the simplest possible way, requiring that the real part of x be defined modulo 1, and that the sign function be that for the real part:

$$x \equiv \mathrm{Re}[x]_{mod\ 1} + \mathrm{Im}[x], \quad \mathrm{sgn}(x) \equiv \mathrm{sgn}(\mathrm{Re}[x]). \tag{15}$$

With these definitions,

$$B_4(C_j) = \left[\frac{1}{4} + \left(\frac{\mu_j}{2\pi T} \right)^2 \right]^2, \tag{16}$$

and it is easy to see that Eqs. (12) and (13) agree. Notice that we added a constant to the potential in Eq. (14) so that one obtains the usual free energy for $q_j = 0$.

While the first derivation was cumbersome, it has the virtue that it is really no different at $\mu \neq 0$ than that at $\mu = 0$; all we need in Eq. (9) is the $\eta(s)$ function for negative values of s. In contrast, with the second derivation it is necessary to analytically continue the Bernoulli function $B_4(x)$ to complex values of x. Both Eqs. (12) and (13) are analytic continuations of the same function, Eq. (7), so this continuation is unique.

Once we are certain about how to analytically continue a Bernoulli function, it is then immediate to continue on to two loop order. We simply start with the known result for the two loop potential at $q_j \neq 0,\ \mu = 0$ [4,7], and substitute the C_j's for the q_j's, to obtain

$$\mathcal{S}^2 = -\beta V \frac{g^2}{4} T^4 \Bigg[\sum_{j \neq k=1}^{N} B_2(C_{jk})[B_2(C_j)+B_2(C_k)] - B_2(C_j)B_2(C_k) + \frac{N-1}{N} \sum_{j=1}^{N} \left(\frac{1}{3} - B_2(C_j) \right) B_2(C_j) - \frac{4}{3} \sum_{j,k=1}^{N} [B_3(C_j) - B_3(C_k)] B_1(C_{jk}) + \frac{5}{144}(N^2-1) \Bigg], \tag{17}$$

where we introduce the variable

$$C_{jk} = C_j - C_k \tag{18}$$

and the Bernoulli functions

$$B_1(x) = x - \frac{1}{2}\,\mathrm{sgn}(x), \quad B_2(x) = x^2 - x\,\mathrm{sgn}(x) + \frac{1}{6},$$

$$B_3(x) = x^3 - \frac{3}{2}x^2\,\mathrm{sgn}(x) + \frac{1}{2}x. \tag{19}$$

The B_1, B_2 and B_3 are also Bernoulli functions; up to a numerical constant, they are the third, second and first derivatives of B_4, respectively. We analytically continue to complex x as before, using the definitions of Eqs. (15). We only need $B_1(x)$ for real values of x; in this case, it obeys $B_1(-x) = -B_1(x)$. When $q_j = 0$, Eq. (17) reduces to the $\sim g^2$ free energy for $\mu, T \neq 0$ [15].

It is useful to understand where the terms in Eq. (17) arise from. At two loop order, quarks contribute to the effective potential from a single diagram in which a gluon crosses a quark loop. The first term in Eq. (17), involving $B_2(C_{jk})$, represents the case in which the two quark lines in the loop carry different C_j; then the exchanged gluon is off diagonal, and $C_{jk} = C_j - C_k$ enters. (Notice that the quark chemical potential drops out of C_{jk}.) In contrast, the terms in Eq. (17) involving $B_2(C_j)$ arise when both quark lines in the loop have the same C_j, so that the exchanged gluon is neutral with respect to the background field. This is true for the gluon in Feynman gauge.

The origin of the terms $\sim B_1$ and B_3 in Eq. (17) is due to two sources. The first is the longitudinal, and so gauge dependent, part of the gluon propagator, which contributes when the phase of the Wilson line is nonzero.

The second source was noted by Belyaev [5]: what is physically relevant is not the effective potential for constant A_0, but that for the phase of the Wilson line. This is most efficiently computed using a constrained functional integral approach, as discussed by Korthals Altes [7]. The effective potentials coincide at one loop order, but at two loop order, a finite renormalization of the Wilson line must be taken into account. This introduces again B_1 and B_3: B_3, for example, is the derivative of the one loop potential. The gauge dependence from this source cancels precisely against that of the longitudinal part of the gluon propagator.

We draw two conclusions from our results. Consider first the limit of zero temperature. The parameters for the phase of the Wilson line, the q_j, enter into the $\mu_j \sim q_j T$, Eq. (4). From the effective potential in terms of the μ_j's, Eq. (12), at one loop order the effective potential only depends upon the q_j through terms which are at least *quadratic* in the temperature, $\sim \Sigma_j \mu_j^3 \sim \Sigma_j q_j^2 \mu^2 T^2$ [terms linear in T cancel because $\Sigma_j q_j = 0$, Eq. (2)]. This remains true at two loop order, although it is less obvious for a potential written in terms of the C_j's, Eq. (17), than in terms of the μ_j's.

That the effective potential for the phase of the Wilson line is flat at zero temperature is reasonable. A phase for the Wilson line is parametrized by a constant A_0 field; at nonzero temperature this cannot be undone by a gauge transformation, because it alters the boundary conditions at $T \neq 0$. At zero temperature, however, the gauge transformation required to undo a constant A_0 field is at infinite time and so inconsequential.

We stress that our result implies that only the *phase* of the Wilson line is independent of μ at $T = 0$. For a theory with dynamical quarks, the Wilson line has a nonzero vacuum expectation value at $\mu = T = 0$. If the quarks have a small but nonzero mass, this mass will ensure that the vacuum expectation value of the phase vanishes, since that is the state with even parity. At zero temperature, as the quark density increases, the vacuum expectation value of the Wilson line will change as well, as the effects of baryon, or equivalently quark, loops enter. We presume that at $T = 0$, the vacuum expectation value of the Wilson line increases as μ does; this change can only be computed by techniques rather different from those herein. What we can show is that nothing interesting happens with the phase of the Wilson line. Remember that at $T \neq 0$ and $\mu = 0$, there are metastable states associated with the phase of the Wilson line [8]. Our results show that there is nothing analogous when $T = 0$ and $\mu \neq 0$.

Our second conclusion is when $T \neq 0$, any point at which a $q_j = \pm 1/2$ is special. From the definition of μ_j, Eq. (4), for such a q_j the energy k^0 changes from an odd multiple of πT (plus the chemical potential), into an even multiple. The corresponding part of the free energy, $\mathcal{S}_j$, is then that of a free boson at $\mu, T \neq 0$. Thus whenever $q_j = \pm 1/2$, the quarks ''Bose condense'': the true ground state is not a filled Fermi sea, as for any other value of q_j, but a state in which there is a macroscopic density of states with zero momentum. Depending upon the q_j's, several colors may condense at the same time. For two colors, for example, one can have all colors Bose condense at the same point. When $q_j = \pm 1/2$, the two loop terms in the effective potential are just some perturbative corrections.

This does not violate the spin-statistics theorem, because a system with $q_j \neq 0$ does not satisfy standard thermodynamic properties. For example, from Eq. (7), the statistical distribution function is $n(k)=1/\{\exp[(k-\mu_j)/T]+1\}$; when $q_j=\pm 1/2$, $\mu_j=\mu+i\pi T$, and $n(k)$ is *minus* the statistical distribution function for bosons. Because of behavior such as this, some authors have argued that $q_j\neq 0$ is an artifact of the imaginary time approach [9]. We do not think so (see, e.g., Holland and Wiese, and Bronoff and Korthals Altes [3]), but do not presume to have assuaged the doubts of others.

While we have only computed at one and two loop order, we believe that our two principal results — that the potential is flat for $\mu\neq 0$ and $T=0$, and that there is ''Bose condensation'' for $q_j=\pm 1/2$ — are true order by order in the loop expansion. In particular, in going from one to two loop order, there is no sign that the effective potential develops any singularity as $T\rightarrow 0$.

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