Orthogonal ensemble of random matrices and QCD in three dimensions

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We consider the parity-invariant Dirac operator with a mass term in three-dimensional QCD for $N_c=2$ and quarks in the fundamental representation. We show that there exists a basis in which the matrix elements of the Euclidean Dirac operator are real. Assuming that there is spontaneous breaking of flavor and/or parity, we read off from the fermionic action the flavor symmetry-breaking pattern $\mathrm{Sp}(4N_f)\to\mathrm{Sp}(2N_f)\times\mathrm{Sp}(2N_f)$ that might occur in such a theory. We then construct a random matrix theory with the same global symmetries as two-color QCD_3 with fundamental fermions and derive from here the finite-volume partition function for the latter in the static limit. The expected symmetry-breaking pattern is confirmed by the explicit calculation in random matrix theory. We also derive the first Leutwyler-Smilga-like sum rule for the eigenvalues of the Dirac operator.

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I. INTRODUCTION

It has been known for several years that the eigenvalues of the Dirac operator $\gamma_\mu D_\mu$, derived from QCD in a finite volume, are constrained by sum rules [1]. These sum rules, originally derived by Leutwyler and Smilga, can also be obtained from a random matrix theory with the same global symmetries as QCD. This was originally demonstrated by Shuryak and Verbaarschot [2] for QCD in four dimensions with quarks in the fundamental representation and color group SU(3). Subsequently it has also been demonstrated for SU(2) with fundamental fermions and for adjoint fermions ($N_c\geqslant 2$) [3].

The random matrix ensembles corresponding to these various types of theory reflect the respective structures of the matrix elements of the Dirac operator in the three cases and are labeled by a parameter β. For the case of $N_c=2$ (where N_c denotes the number of colors) and fundamental fermions, there exists a basis in which the matrix elements of $\gamma_\mu D_\mu$ are real. The corresponding matrix ensemble has orthogonal symmetry and is labeled $\beta=1$. For $N_c\geqslant 3$ and fundamental fermions, and for arbitrary $N_c\geqslant 2$ and adjoint fermions, the corresponding matrix ensembles have unitary and symplectic symmetry, respectively, and are labeled by $\beta=2,\ 4$.

The sum rules can be expressed using the so-called microscopic spectral density, denoted $\rho_S(\lambda)$, of the distribution of eigenvalues of the random matrix model. It is obtained by magnifying the spectral density in the vicinity of the origin ($\lambda=0$) on the scale of the average eigenvalue spacing, which for interacting quarks and a nonvanishing spectral density at the origin is given by N^{-1} (N here is the size of the random matrices). This microscopic limit is to be contrasted with the large N limit, in which the eigenvalue density smoothes out to some distribution whose macroscopic shape depends on the matrix potential.

Originally, it was conjectured on the basis of the work in [2,4,5] that $\rho_S(\lambda)$ is a universal quantity that depends only on symmetry. The sum rules were determined for a number of cases and further evidence for the proposed scenario was compiled [6,7]. It was also demonstrated, by incorporating a schematic temperature dependence corresponding to the lowest Matsubara frequency into the matrix model, that ρ_S is independent of temperature up to the critical temperature of the model [8,9]. A discussion of universality in the presence of a nonzero chemical potential was given in [10]. The universality was also demonstrated numerically in a number of papers. More recently, it was shown that $\rho_S(\lambda)$ does not depend on the matrix potential chosen for the random matrix theory (RMT) [11]. This comes about because the differential equation determining the orthogonal polynomials corresponding to the matrix model is independent of the choice of (polynomial) potential in the microscopic limit. The orthogonal polynomials in turn completely determine $\rho_S(\lambda)$ and higher-order correlators $\rho_S(\lambda_1,\ldots,\lambda_n)$ in the microscopic limit.

The reason for this universality is that both QCD and the corresponding RMT can be mapped onto the same low-energy, effective partition function. This was first noticed in [2] and further elaborated on in [3,4,6,12]. This partition function expresses the quark mass dependence in the static limit and in a finite volume. The range of volumes considered is the so-called ''mesoscopic range.'' We can think of this as a box of size L such that $L\gg\Lambda_{QCD}^{-1}$, where Λ_{QCD} is the QCD scale paramenter, so that only the low-lying excitations (Goldstone modes) contribute to the partition function, but $L\ll\lambda_\pi^C$ (where λ_π^C is the Compton wavelength of the Goldstone modes), so that we are dealing with the static limit of this partition function (no kinetic terms).

In fact, this effective partition function is a function of one scaling variable $\mathcal{M}V\Sigma$, where $\mathcal{M}$ is a mass matrix, V the space-time volume, and Σ the chiral condensate (assumed to be nonzero). In the RMT the space-time volume corresponds to the size N of the random matrices. The Banks-Casher relation [13]

$$\Sigma=\frac{\pi\rho(0)}{V} \tag{1}$$

relates the density of eigenvalues of the Dirac operator at the origin, $\rho(0)$, to this condensate in the thermodynamic and

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chiral limits (taken in this order). The spacing between eigenvalues is thus $\sim V^{-1}$, as opposed to $\sim V^{-1/4}$ for free quarks in 4D.

The relationship between random matrix theory and finite-volume partition functions has been clarified further by Damgaard, and by Akemann and Damgaard in a series of papers [14].

In this paper we will consider QCD in three Euclidean dimensions and we will take N_c, the number of quark colors, equal to 2. The quarks are in the fundamental representation. We will show that also in three dimensions, there exists a basis in which the Dirac operator for $N_c=2$ has real matrix elements. In the spirit of the universality conjecture, we will construct a random matrix theory with the same global symmetries as the gauge field theory. The average of the fermion determinant over the gluon field configurations is in this approach replaced by a Gaussian average over an ensemble of random Hermitian matrices. From this average we will obtain, using a supersymmetric formalism and through the same kind of steps as in [2,3], and in [8] for QCD in four space-time dimensions, the form of the low-energy QCD partition function. Assuming that spontaneous breaking of global flavor symmetry may occur in such a theory, we will obtain the pattern of such a symmetry breaking.

Except for being of purely theoretical interest, three-dimensional QCD may be relevant for studying the behavior of QCD near the deconfining phase transition and for lattice computations. In Euclidean field theory, at finite temperature the integral over the four-momentum component k_4 is replaced by a sum over Matsubara frequencies and one is left with an effective three-dimensional field theory. On the lattice, it is faster to simulate two colors than three. Therefore the sum rules derivable from $N_c=2$ may be easily checked numerically.

In the next section a basis is constructed in which the Dirac operator for SU(2) color is real. In Sec. III the symmetry-breaking pattern is discussed. In Secs. IV and V, random matrix theory is used as a starting point for deriving the low-energy partition function and the flavor symmetry breaking pattern. In Sec. VI the corresponding sum rules are derived.

II. DIRAC OPERATOR IN 3D

In three-dimensional Minkowski space the QCD Lagrangian is given by

$$\mathcal{L}=-\frac{1}{4}\mathrm{tr}F^2+\sum_{f=1}^{N_f}\bar{\psi}_f(i\not{D}-m_f)\psi_f, \tag{2}$$

where F is the gauge field tensor, $\not{D}\equiv\gamma^\mu D_\mu$, $D_\mu=\partial_\mu+iA_\mu^a\tau_a/2$ is the covariant derivative for SU(2), and m_f is the quark mass corresponding to flavor f. ψ_f are quark spinors in the fundamental representation and f is the flavor index (the indices corresponding to color and spin are suppressed). The lowest-dimensional representation of γ^μ is given by the Pauli matrices $\gamma^0=\sigma_3$, $\gamma^1=i\sigma_1$, $\gamma^2=i\sigma_2$. In this 2D representation, there is no chiral symmetry, since there is no 2×2 matrix that anticommutes with the σ_k.

For all the $m_f=0$, the above Lagrangian is invariant under parity P, but the mass term breaks this P invariance. The parity transformation in 3D is defined by

$$\begin{aligned}
&\psi(t,x_1,x_2)\to\gamma_1\psi(t,-x_1,x_2),\\
&A_0(t,x_1,x_2)\to A_0(t,-x_1,x_2),\\
&A_1(t,x_1,x_2)\to -A_1(t,-x_1,x_2),\\
&A_2(t,x_1,x_2)\to A_2(t,-x_1,x_2).
\end{aligned} \tag{3}$$

We can define a parity-invariant Lagrangian with a nonzero mass term if we take, instead of the σ_k a four-dimensional representation of the γ_μ,

$$\gamma^0=\begin{pmatrix}\sigma_3 & 0\\ 0 & -\sigma_3\end{pmatrix},\quad \gamma^1=\begin{pmatrix}i\sigma_1 & 0\\ 0 & -i\sigma_1\end{pmatrix},$$

$$\gamma^2=\begin{pmatrix}i\sigma_2 & 0\\ 0 & -i\sigma_2\end{pmatrix} \tag{4}$$

and moreover introduce a 4×4 mass matrix corresponding to flavor f:

$$M_f=m_f\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{5}$$

The Dirac operator is then sandwiched between four-spinors $(\phi_f\chi_f)$. In terms of two-spinors, this representation corresponds to N_f two-spinors ϕ_f with mass $+m_f$, and N_f two-spinors χ_f with mass $-m_f$. Under P the mass terms for the two-spinors change sign, so that if the two sets of two-spinors transform into each other in a Z_2 transformation $\phi_f\leftrightarrow\chi_f$, the total Lagrangian is invariant under the combined transformations P and Z_2 [15]. We can use this fact to write down a (P,Z_2)-invariant Lagrangian in the fundamental representation with an appropriate choice of mass term:

$$\mathcal{L}=-\frac{1}{4}\mathrm{tr}F^2+\sum_{f=1}^{2N_f}\bar{\psi}_f i\not{D}\psi_f-\sum_{f=1}^{N_f}m\bar{\psi}_f\psi_f+\sum_{f=N_f+1}^{2N_f}m\bar{\psi}_f\psi_f. \tag{6}$$

(We could also have some components with zero mass, but in the following we will not consider this possibility.)

We now proceed to discuss QCD in three-dimensional Euclidean space. The part of the Lorentz-invariant Lagrangian involving fermion fields is given by $\Sigma_f\bar{\psi}_f(\not{D}+m_f)\psi_f$ where $\bar{\psi}$ denotes the Hermitean conjugate, the masses are chosen in pairs of opposite sign like in Eq. (6), and from now on $\not{D}$ denotes $\gamma_\mu D_\mu$ with the Euclidean gamma matrices $\gamma_0=\sigma_3$, $\gamma_1=\sigma_1$, $\gamma_2=\sigma_2$ satisfying $\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}$.

In four dimensions, we can find a basis such that the Euclidean Dirac operator $i\not{D}$ has real matrix elements. The reason is that this operator possesses (for $N_c=2$) an antiunitary symmetry [3] expressed by

$$[i\not{D}, C\tau_2 K] = 0. \tag{7}$$

Here C is the (Minkowski space) charge conjugation operator, τ_2 is a Pauli matrix in color space, and K denotes complex conjugation. It is easy to show that in 3D, an identical relation (7) holds for the fundamental representation. For Eq. (7) to hold, C should satisfy $C\gamma_\mu^* C^{-1} = -\gamma_\mu$ (where γ_μ are the Pauli matrices, $\gamma_0 = \sigma_3$, $\gamma_1 = \sigma_1$, $\gamma_2 = \sigma_2$) which is precisely the condition for the charge conjugation matrix in Minkowski space. By explicit calculation we find

$$C = i\sigma_2, \tag{8}$$

where C is the 2×2 charge conjugation matrix satisfying $-C = C^T = C^\dagger = C^{-1}$, $C^2 = -1$. As we will now show, the antiunitary symmetry operator $C\tau_2 K$ defines a basis in which the matrix elements of $i\not{D}$ are real. This basis is simply defined by

$$C\tau_2 K\psi_k = \psi_k. \tag{9}$$

Since $(C\tau_2 K)^2 = 1$, such a definition makes sense. (By contrast, in trying to define adjoint fermions in Euclidean space, the square of the corresponding antiunitary operator is -1. The Majorana condition then makes sense only if one introduces conjugation of the second kind, $\psi^{**} = -\psi$.) From the antiunitary condition it follows that

$$\tau_2 C i\not{D} C\tau_2 = -(i\not{D})^*. \tag{10}$$

By using Eqs. (9), (10) and the properties of C it immediately follows that the quantity $\psi_k^\dagger i\not{D}\psi_l$ is real, where ψ_k denotes the basis vectors in Eq. (9). Therefore, the matrix elements $\langle\psi_k|i\not{D}|\psi_l\rangle$ are real in this basis. The fact that the Dirac operator can be real was also used in [16].

III. DISCUSSION OF THE FLAVOR SYMMETRY-BREAKING PATTERN

Before doing the calculation in random matrix theory, we will now discuss the symmetry-breaking pattern we expect to obtain. This can be read off [17] from the form of the fermionic action

$$S_F = \int d^3x \sum_{f=1}^{2N_f} \overline{\psi_f}(\not{D} + m_f)\psi_f, \tag{11}$$

where the γ matrices are $\gamma_0 = \sigma_3$, $\gamma_1 = \sigma_1$, $\gamma_2 = \sigma_2$ and D_μ is the covariant derivative for the SU(2) color group. Now it is easy to verify that $D_\mu^T = -\tau_2 D_\mu \tau_2$ and $\sigma_\mu^T = -\sigma_2\sigma_\mu\sigma_2$ and therefore, keeping in mind that the ψ_f are anticommuting,

$$\overline{\psi_f}\not{D}\psi_f = -\sigma_2\tau_2\psi_f\not{D}\sigma_2\tau_2\overline{\psi_f}, \tag{12}$$

where the τ's are in color space and the σ's in Dirac space. We can then rewrite the fermionic action as

$$S_F = \int d^3x \frac{1}{2}\sum_{f=1}^{2N_f} \begin{pmatrix} \sigma_2\tau_2\psi_f \\ \overline{\psi_f} \end{pmatrix} \begin{pmatrix} 0 & -\not{D} \\ \not{D} & 0 \end{pmatrix} \begin{pmatrix} \psi_f \\ \sigma_2\tau_2\overline{\psi_f} \end{pmatrix} \tag{13}$$

$$= \int d^3x \frac{1}{2}\sum_{f=1}^{2N_f} \begin{pmatrix} \psi_f \\ \sigma_2\tau_2\overline{\psi_f} \end{pmatrix} \begin{pmatrix} 0 & -\sigma_2\tau_2\not{D} \\ \sigma_2\tau_2\not{D} & 0 \end{pmatrix} \times \begin{pmatrix} \psi_f \\ \sigma_2\tau_2\overline{\psi_f} \end{pmatrix}. \tag{14}$$

This expression is invariant under $\mathrm{Sp}(4N_f)$ transformations in flavor space [18]. This is similar to N_f flavors with color symmetry group SU(2) in four dimensions, where the flavor symmetry group for zero mass gets enlarged to $\mathrm{U}(2N_f)$ [17]. The vacuum state will break this symmetry. Assuming the complete axial group is broken (maximal breaking of chiral symmetry), only the symmetry subgroup of $\mathrm{Sp}(4N_f)$ that leaves $\overline{\psi}\psi$ invariant will be unbroken. The chiral condensate for each flavor f has the same sign as the mass m_f. Rewriting the mass term in the form

$$\sum_{f=1}^{2N_f} m_f\overline{\psi_f}\psi_f = \frac{1}{2}\sum_{f=1}^{2N_f} \begin{pmatrix} \psi_f \\ \overline{\psi_f} \end{pmatrix} \begin{pmatrix} 0 & -m_f \\ m_f & 0 \end{pmatrix} \begin{pmatrix} \psi_f \\ \overline{\psi_f} \end{pmatrix} \tag{15}$$

and remembering [cf. Eq. (6)] that the m_f form a diagonal $2N_f\times 2N_f$ matrix in flavor space,

$$\begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}, \tag{16}$$

with N_f of the masses equal to $+m$ and N_f equal to $-m$, one immediately sees that Eq. (15) is invariant under the subgroup $\mathrm{Sp}(2N_f)\times\mathrm{Sp}(2N_f)$. The symmetry-breaking pattern $\mathrm{Sp}(4N_f)\to\mathrm{Sp}(2N_f)\times\mathrm{Sp}(2N_f)$ will be confirmed below by an explicit calculation in random matrix theory.

IV. RANDOM MATRIX THEORY

The Dirac operator $\gamma_\mu D_\mu$ in the 2×2 representation is anti-Hermitian. To construct a random matrix ensemble which is Hermitian and has orthogonal symmetry, we therefore substitute the average over gluon field configurations of the Euclidean fermion determinant

$$Z(M) = \int dA \prod_{f=1}^{2N_f} \det(\not{D} + m_f)\mathrm{e}^{-S[A]} \tag{17}$$

(where $S[A]$ denotes the Yang-Mills action for SU(2) in three Euclidean dimensions, $m_f = m$ for $f = 1,\ldots,N_f$ and $m_f = -m$ for $f = N_f+1,\ldots,2N_f$) in the partition function defining QCD_3 with an average over a real Hermitian random matrix R. We then get a matrix model

$$Z(m) = \int DR\,\mathrm{e}^{-N\Sigma^2\mathrm{tr}(R^2)} \prod_{f=1}^{2N_f} \det(iR + m_f). \tag{18}$$

R is here taken to be a matrix of size $N\times N$, and DR is the invariant (Haar) measure. We take the total density of zero modes (number of small eigenvalues per space-time volume) to be fixed, so we can identify N with the space-time volume [2]. We call the total number of flavors $2N_f$, since we have N_f fermion species with mass m and N_f with mass $-m$. Assuming there is a spontaneous breaking of flavor and/or parity, we will find the pattern of flavor symmetry breaking, while parity will remain unbroken. It was shown in [19] that parity is spontaneously broken by the appearance of an anomalous parity-odd Chern-Simons term at the quantum level in QCD in three dimensions (indeed, in any odd dimension) for an odd number of massless fermion species. For an even number of flavors, the anomaly does not appear, and with our choice of P-invariant masses, parity remains unbroken also at the quantum level.

As we will see, Σ is the value of the order parameter for spontaneous symmetry breaking,

$$\Sigma=-\lim_{m_f\to 0}\lim_{N\to\infty}\frac{1}{N}\frac{\partial}{\partial m_f}\ln Z(m_1,\ldots,m_{2N_f}). \tag{19}$$

Its absolute value will be the same for each flavor [12]. In order to evaluate $Z(m)$ and perform the integration over the random matrices R, we write the product of fermion determinants as an integral over Grassmann fields:

$$\prod_f \det(R-im_f) = \int\prod_f D\phi_f \exp\left[-\sum_f \phi_f^{i*}(R-im_f)_{ij}\phi_f^j\right]. \tag{20}$$

Here the indices i, j run from 1 to N. We will make use of the supersymmetric formalism developed in [20]. We use conjugation of the second kind, $\phi^{**}=-\phi$, for Grassmann variables (see Appendix A of the reference just quoted). This formalism was developed to deal with integrals over both commuting and Grassmann variables, involving graded vectors and matrices. In Eq. (20) we have only the ''fermion-fermion block'' of [20], since our integration variables are pure fermionic. Our integration measure is

$$\prod_f D\phi_f=\prod_{f=1}^{2N_f}\prod_{i=1}^{N} d\phi_f^{i*}d\phi_f^i. \tag{21}$$

To perform the integral over the random matrix R, we complete the square in the exponent of Eq. (18) according to

$$\sqrt{N}\Sigma R_{ij}\to\sqrt{N}\Sigma R_{ij}+\frac{1}{2\sqrt{N}\Sigma}C_{ij}$$

$$C_{ij}\equiv\frac{1}{2}\sum_f\left(\phi_f^{i*}\phi_f^j+\phi_f^{j*}\phi_f^i\right) \tag{22}$$

and perform the Gaussian integral. Here we take care that the matrix C has the same properties as R (real in the extended sense of the supersymmetric formalism and Hermitian). Therefore we have symmetrized the indices i, j and used $R_{ij}=R_{ji}$ in completing the square. Since Grassmann integrals are always convergent, and the integrals in DR are uniformly convergent in the fermionic variables,

$$\int DR\int\prod_f D\phi_f=\int\prod_f D\phi_f\int DR. \tag{23}$$

The substitution (22) yields, after performing the Gaussian integration,

$$Z(m)\sim\int\prod_f D\phi_f\exp\left\{\frac{1}{4N\Sigma^2}\sum_{i,j}\left[\frac{1}{2}\sum_f\begin{pmatrix}\phi_f^i\\ \phi_f^{i*}\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\times\begin{pmatrix}\phi_f^j\\ \phi_f^{j*}\end{pmatrix}\right]^2+im_f\sum_i\sum_f\phi_f^{i*}\phi_f^i\right\}. \tag{24}$$

Now introduce a block-diagonal $4N_f\times 4N_f$ matrix I such that

$$\sum_f^{2N_f}\begin{pmatrix}\phi^i\\ \phi^{i*}\end{pmatrix}_f\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}\phi^j\\ \phi^{j*}\end{pmatrix}_f=\begin{pmatrix}\begin{pmatrix}\phi^i\\ \phi^{i*}\end{pmatrix}_{f=1}\\ \begin{pmatrix}\phi^i\\ \phi^{i*}\end{pmatrix}_{f=2}\\ \vdots\\ \begin{pmatrix}\phi^i\\ \phi^{i*}\end{pmatrix}_{f=2N_f}\end{pmatrix}\begin{pmatrix}0&-1&&&&&&\\1&0&&&&&&\\&&0&-1&&&&\\&&1&0&&&&\\&&&&\cdot&&&\\&&&&&\cdot&&\\&&&&&&\cdot&\\&&&&&&0&-1\\&&&&&&1&0\end{pmatrix}\begin{pmatrix}\begin{pmatrix}\phi^j\\ \phi^{j*}\end{pmatrix}_{f=1}\\ \begin{pmatrix}\phi^j\\ \phi^{j*}\end{pmatrix}_{f=2}\\ \vdots\\ \begin{pmatrix}\phi^j\\ \phi^{j*}\end{pmatrix}_{f=2N_f}\end{pmatrix}$$

$$\equiv\sum_{f,g}^{4N_f}\Phi_f^iI_{fg}\Phi_g^j. \tag{25}$$

In I, each 2×2 block is labeled by a flavor index f, but rearranging the $2N_f$ two-component Grassmann vectors into large vectors of size $4N_f$, we get now a doubling of the indices so that hereafter f, g go from 1 to $4N_f$ and simply label the components in Eq. (25). We now rewrite the square in the exponent as the difference of two terms (while remembering that the ϕ_f^i are anticommuting):

$$\begin{aligned}
&\left[\sum_{f=1}^{2N_f}\begin{pmatrix}\phi^i\\ \phi^{i*}\end{pmatrix}_f\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\phi^j\\ \phi^{j*}\end{pmatrix}_f\right]^2\\
&\quad=\left[\sum_{f,g}^{4N_f}\Phi_f^i I_{fg}\Phi_g^j\right]^2\\
&\quad=-\frac{1}{4}(\Phi_f^i\Phi_g^i+I_{ff'}\Phi_{f'}^j,\Phi_{g'}^j,I_{g'g}^T)^2\\
&\qquad+\frac{1}{4}(\Phi_f^i\Phi_g^i-I_{ff'}\Phi_{f'}^j,\Phi_{g'}^j,I_{g'g}^T)^2\\
&\quad\equiv-\frac{1}{4}F_{fg}^2+\frac{1}{4}\tilde{F}_{fg}^2 \qquad (26)
\end{aligned}$$

(in the last two expressions a sum over repeated flavor indices $f,f',g,g'=1,\ldots,4N_f$ is understood). Performing a Hubbard-Stratonovitch transformation [20]

$$\exp[-\alpha F_{fg}F_{fg}]\sim\int d\sigma_{fg}\exp\left[-\frac{1}{4\alpha}\sigma_{fg}\sigma_{fg}-i\sigma_{fg}F_{fg}\right], \qquad (27)$$

where σ_{fg} is a real variable, the integral in Eq. (24) becomes

$$\begin{aligned}
Z(m)\sim\int\prod_f D\phi_f D\sigma_1 D\sigma_2\exp\Bigg\{&-16N\Sigma^2\mathrm{tr}[(\sigma_1+i\sigma_2)\\
&\times(\sigma_1^T-i\sigma_2^T)]-i\sum_i\sum_{fg}\Phi_f^i(\sigma_1+i\sigma_2)_{fg}\Phi_g^i\\
&+i\sum_j\sum_{f'fg'g}\Phi_{g'}^j,I_{g'g}^T(\sigma_1^T-i\sigma_2^T)_{gf}I_{ff'}\Phi_{f'}^j,\\
&+i\sum_i\sum_{fg}\Phi_f^i\frac{1}{2}\mathcal{M}_{fg}\Phi_g^i\Bigg\} \qquad (28)
\end{aligned}$$

where the masses have been rearranged into an antisymmetric matrix

$$\mathcal{M}=\begin{pmatrix}
0 & m & & & & & & & & \\
-m & 0 & & & & & & & & \\
 & & \ddots & & & & & & & \\
 & & & 0 & m & & & & & \\
 & & & -m & 0 & & & & & \\
 & & & & & 0 & -m & & & \\
 & & & & & m & 0 & & & \\
 & & & & & & & \ddots & & \\
 & & & & & & & & 0 & -m\\
 & & & & & & & & m & 0
\end{pmatrix}. \qquad (29)$$

In Eq. (28), $D\sigma_i$ is the Haar measure for the real antisymmetric matrix σ_i. The σ_i should be chosen antisymmetric in flavor indices since F_{fg} and $\tilde{F}_{fg}$ are antisymmetric, so as to preserve the symmetry of $Z(m)$. Setting $-\sigma_1-i\sigma_2\equiv A$, where A is an antisymmetric complex matrix, we end up with

$$\begin{aligned}
Z(m)\sim\int\prod_f D\phi_f\int DA\exp\Bigg\{&-16N\Sigma^2\,\mathrm{tr}(AA^\dagger)\\
&+i\sum_i\Phi^i\left(A+I^TA^*I+\frac{\mathcal{M}}{2}\right)\Phi^i\Bigg\}. \qquad (30)
\end{aligned}$$

Interchanging again the order of the fermionic integrations and the integration over DA, and subsequently performing the Grassmann integrations, we arrive at

$$Z(m)\sim\int DA\,\exp\{-4N\Sigma^2\mathrm{tr}(AA^\dagger)\}\mathrm{Pf}^N(A+I^TA^*I+\mathcal{M}), \qquad (31)$$

where we have rescaled A by a factor of 2. In Eq. (31) ''Pf'' denotes the Pfaffian (square root of the determinant) of the matrix. Note that the Pfaffian of an antisymmetric matrix is always well defined. This is our expression for the partition function. In the next section we will evaluate it using a saddle point analysis.

V. SADDLE POINT ANALYSIS OF THE PARTITION FUNCTION

To begin evaluating the partition function, we will now decompose the antisymmetric matrix A in (31) into ''polar'' coordinates [21]. This can be achieved for an arbitrary antisymmetric matrix by setting

$$A=U\Lambda U^T, \quad \Lambda=\begin{pmatrix} 0 & \lambda_1 & & & & & \\ -\lambda_1 & 0 & & & & & \\ & & 0 & \lambda_2 & & & \\ & & -\lambda_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \lambda_{2N_f} \\ & & & & & -\lambda_{2N_f} & 0 \end{pmatrix}$$

$$(\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_{2N_f} \geqslant 0). \tag{32}$$

U is unitary. The integration measure *a priori* becomes

$$\int DA = \int_{U \in U(4N_f)/[Sp(2)]^{2N_f}} DUD\Lambda J(\Lambda). \tag{33}$$

The integration over the coset $\mathrm{U}(4N_f)/[\mathrm{Sp}(2)]^{2N_f}$ ensures that there is a one-to-one correspondence between the integration variables in A and those in $U\Lambda U^T$ (cf. [21], Chap. 3). The Jacobian $J(\Lambda)$ was not found in [21]. We calculated it and found that it is indeed a function of Λ only, and that it is of order N_f. Indeed, it could never be of order N; therefore it must drop out at the saddle point in Λ as N gets large. This will always happen in any similar calculation, so that it is not necessary to know the exact form of the Jacobian $J(\Lambda)$, as long as it is a function of λ_f only.

However, the presence of the matrices I limits the U integration to be over the subgroup $\mathrm{Sp}(4N_f)$. This is evident when we consider that the matrix U that block-diagonalizes A also block-diagonalizes $A' \equiv A + I^T A^* I = A - IA^* I$:

$$A=U\Lambda U^T, \quad A'=U\Lambda' U^T, \tag{34}$$

since the values of the matrix elements of A and A' do not enter in the ''angular'' matrices U, but only determine Λ and Λ'. Therefore,

$$\Lambda' = \Lambda - U^\dagger I U^* \, \Lambda \, U^\dagger I U^*. \tag{35}$$

We can choose U such that the eigenvalues of Λ are ordered like in Eq. (32). But both Λ and Λ' have the block-diagonal form appearing in Eq. (32). Therefore also the second term in Eq. (35) has to have this form. Since $I\Lambda I = -\Lambda$ [note that we could have chosen any one of three equivalent forms for I that are all invariant under the symplectic group, by simply rearranging the components in Φ_f^i ([18], paragraph 10-8), and all of these forms satisfy $I\Lambda I = -\Lambda$], that means that in Eq. (35) $U^\dagger I U^* \propto I$. But $UIU^T = I$ is equivalent to $U \in \mathrm{Sp}(4N_f)$. Then $\Lambda' = 2\Lambda$. Like in [2,3] we will determine the saddle point in Λ at $\mathcal{M}=0$ and then expand the Pfaffian at this saddle point to first order in $\mathcal{M}$ to see the symmetry-breaking pattern. At $\mathcal{M}=0$ the integral takes the simple form

$$\begin{aligned} Z(m=0) &\sim \int DUD\Lambda J(\Lambda) \mathrm{e}^{-4N\Sigma^2 \,\mathrm{tr}(\Lambda\Lambda^\dagger)} \det{}^{N/2}(U 2\Lambda U^T) \\ &\sim \int DU \int \prod_{f=1}^{2N_f} d\lambda_f \exp\Big\{ \ln J(\lambda_1, \ldots, \lambda_{2N_f}) \\ &\quad - 4N\Sigma^2 \sum_f (2\lambda_f)^2 + N \sum_f \ln \lambda_f \Big\}, \end{aligned} \tag{36}$$

where we have used that the symplectic matrices are unimodular. The saddle point is at

$$\lambda_f = \pm \frac{1}{4|\Sigma|}. \tag{37}$$

We will now discuss the choice of saddle point manifold. In [20], the saddle point with an equal number of + and − signs was singled out because for the other potential saddle points, the integrand and measure became independent of some Grassmann variables in the supersymmetric Hubbard-Stratonovitch matrices. Integrating over these Grassmann fields then sets $Z(m)$ to zero. Here, we get the same saddle point, but for a different reason, since our σ's have only commuting variables. We can always redefine the angular matrices U so that $\lambda_f \geqslant 0$ [cf. Eq. (32)]. Therefore, we can choose the positive sign in Eq. (37). However, assuming the flavor symmetry is broken spontaneously, the condensate for each flavor has to have the same sign as the mass. This is clear from the proof of the Banks-Casher formula [13]; see also Ref. [12]. Since half of the masses are negative we should choose $|\Sigma| = -\Sigma$ for half, and $|\Sigma| = +\Sigma$ for half of the λ_f at the saddle point.

Therefore, our saddle point should be

$$\Lambda_{sp}=\frac{1}{4\Sigma}\begin{pmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & & & \\ & & \ddots & & & & & \\ & & & 0 & 1 & & & \\ & & & -1 & 0 & & & \\ & & & & & 0 & -1 & \\ & & & & & 1 & 0 & \\ & & & & & & & \ddots \end{pmatrix} \equiv -\frac{1}{4\Sigma}J. \tag{38}$$

We now expand the determinant for small $\mathcal{M}\neq 0$ at $\Lambda=\Lambda_{sp}$. In Eq. (31) the matrix $A+I^TA^*I+\mathcal{M}$ is antisymmetric. This means that the square root of its determinant is a positive real number. This is confirmed by inspection of its explicit form. We can therefore write

$$\det{}^{N/2}(A+I^TA^*I+\mathcal{M})=\det{}^{N/4}(A+I^TA^*I+\mathcal{M})\det{}^{N/4} \times(A^*+I^TAI+\mathcal{M}). \tag{39}$$

This way our final expression for the partition function will be manifestly real after expanding the integrand. We then get

$$\begin{aligned} Z(m)\sim & \int DU\det{}^{N/4}(U2\Lambda_{sp}U^T+\mathcal{M})\det{}^{N/4} \\ & \times(U^*2\Lambda_{sp}U^\dagger+\mathcal{M}) \\ \sim & \int DU\det{}^{N/4}(U2\Lambda_{sp}U^T)\det{}^{N/4}(U^*2\Lambda_{sp}U^\dagger) \\ & \times\exp\left[\frac{N}{4}\mathrm{tr}\ln\left(1+U^*\frac{1}{2}\Lambda_{sp}^{-1}U^\dagger\mathcal{M}\right)\right. \\ & \left.+\frac{N}{4}\mathrm{tr}\ln\left(1+U\frac{1}{2}\Lambda_{sp}^{-1}U^T\mathcal{M}\right)\right] \\ \propto & \int DU\mathrm{e}^{N\Sigma\ \mathrm{Re}\ \mathrm{tr}(UJU^T\mathcal{M})} \end{aligned} \tag{40}$$

to first order in $\mathcal{M}$, where we have used that for a symplectic matrix $\det(U)=1$ [18]. The matrix $\mathcal{M}$ was given in Eq. (29) and J is defined in Eq. (38). In the final expression for $Z(m)$,

$$Z(m)\approx\int_{Sp(4N_f)/[Sp(2N_f)\times Sp(2N_f)]}DU\mathrm{e}^{N\Sigma\ \mathrm{Re}\ \mathrm{tr}(UJU^T\mathcal{M})}, \tag{41}$$

the DU integral goes over the coset space $\mathrm{Sp}(4N_f)/[\mathrm{Sp}(2N_f)\times\mathrm{Sp}(2N_f)]$, due to the structure of the matrix J: it is invariant under the unbroken subgroup $\mathrm{Sp}(2N_f)\times\mathrm{Sp}(2N_f)$. We have thus obtained the flavor symmetry-breaking pattern $\mathrm{Sp}(4N_f)\to\mathrm{Sp}(2N_f)\times\mathrm{Sp}(2N_f)$. The number of broken generators is $4N_f^2$, which is also the number of unbroken generators.

VI. SUM RULES

To derive the first sum rule we will closely follow the method explained in [3]. The sum rules are obtained by expanding the expression for $Z(m)$, Eq. (41), and comparing the coefficients order by order in m^2 to the (normalized) expectation value of the fermion determinant:

$$\left\langle\prod_f\prod_{\lambda_k>0}\left(1+\frac{m^2}{\lambda_k^2}\right)\right\rangle. \tag{42}$$

The expectation value is defined as

$$\langle f(\lambda,m)\rangle=\frac{\int DA\ \mathrm{e}^{-S[A]}\left(\prod_{k,f}\lambda_f^{k2}\right)f(\lambda,m)}{\int DA\ \mathrm{e}^{-S[A]}\left(\prod_{k,f}\lambda_f^{k2}\right)f(\lambda,0)}, \tag{43}$$

where A is the gauge field and $S[A]$ the Euclidean Yang-Mills action. Expanding the integrand in Eq. (41) the $\mathcal{O}(m)$ term is destroyed by the group integration. The surviving group integrals at order m^2 have the form (using the same notation as in [3])

$$\zeta(X)=\int_{U\in G/H}DU\,\mathrm{tr}(UJU^TX)\mathrm{tr}(U^*JU^\dagger X), \tag{44}$$

where G/H is the coset and $X\equiv -N\Sigma mJ$. We now choose generators t_k, $k=1,\ldots,M$, for the U's that are real and antisymmetric. M is the dimension of the coset:

$$M=\frac{4N_f(4N_f+1)}{2}-2\frac{2N_f(2N_f+1)}{2}=4N_f^2. \tag{45}$$

We also wish to choose t_1 such that $\mathcal{M}\equiv -mJ=mt_1$. $\mathcal{M}$ is a $4N_f\times 4N_f$ size matrix. Therefore we normalize the generators so that

$$\mathrm{tr}(t_k t_l) = -4N_f \delta_{kl}. \tag{46}$$

[Note that for anti-Hermitian generators (real and antisymmetric) the minus sign is necessary.] It is then easy to show that for any two matrices A and B,

$$\sum_{k=1}^{M} \mathrm{tr}(At_k)\mathrm{tr}(Bt_k) = -4N_f \mathrm{tr}(AB). \tag{47}$$

It was proved in [3] that $\zeta(t_1)=\zeta(t_2)=\cdots=\zeta(t_M)$ and therefore

$$\zeta(t_1) = \frac{1}{M}\sum_{k=1}^{M} \int DU\, \mathrm{tr}(UJU^T t_k)\, \mathrm{tr}(U^* J U^\dagger t_k). \tag{48}$$

Using Eq. (47) and $\mathrm{tr}(J^2) = -4N_f$ we now immediately see that

$$\zeta(X) = \frac{1}{M}\mathrm{vol}(G/H)(N\Sigma m)^2(4N_f)^2. \tag{49}$$

Inserting this into the expansion we get

$$\frac{Z(m)}{Z(0)} = \left\langle 1 + m^2 2N_f \sum_{\lambda_k>0} \frac{1}{\lambda_k^2} + \cdots \right\rangle$$

$$= 1 + \frac{1}{2}(N\Sigma m)^2 \frac{1}{M}(4N_f)^2 + \cdots, \tag{50}$$

where the volume of the coset cancels in the ratio. Inserting the value of M we therefore arrive at the sum rule

$$\left\langle \sum_{\lambda_k>0} \frac{1}{(N\Sigma\lambda_k)^2} \right\rangle = \frac{2}{2N_f}. \tag{51}$$

Note that the original number of flavors is $2N_f$. This sum rule could be checked numerically. In Refs. [3,6] a universal formula was found describing the first sum rule for the $\beta=1$, 2, and 4 cases in four dimensions. This universal formula relates $\langle \Sigma_k [1/(N\Sigma\lambda_k)^2] \rangle$ to N_f, the dimension of the coset M, and the winding number in four dimensions. We can identify a similar formula

$$\prod_f \left\langle \sum_{\lambda_k>0} \frac{1}{(N\Sigma\lambda_k)^2} \right\rangle = \frac{d^2}{2M} \tag{52}$$

for the three-dimensional cases $\beta=1$, 2 (the case $\beta=4$ remains to be done) by comparing the above result to the sum rule for an even number of flavors given in [12] and noting that this sum rule is given for $\mathrm{SU}(N_f)$ and not for the coset $U(N_f)/[\mathrm{U}(N_f/2)\times\mathrm{U}(N_f/2)]$. In Eq. (52), d is the size of the matrices of the coset. These two factors of d come from the normalization of the generators (46) and from $\mathrm{tr}(J^2)$, and the factor of $\frac{1}{2}$ comes from the expansion of the exponential in Eq. (41). As shown by Eq. (39), $\mathrm{tr}(UJU^T\mathcal{M})$ must be real in our case, so there should not be any additional factors of $\frac{1}{2}$ [as seems to be the case in 4D, probably because the group integrals of the type $\int dV \mathrm{tr}^2(VX)$ vanish (at least for large enough N_f); cf. [3]].

VII. SUMMARY AND OUTLOOK

We have derived the mass dependence of the low-energy effective partition function for parity-invariant QCD in three dimensions with two quark colors using as a starting point a random matrix theory with the global symmetries of this gauge theory. The motivation for this was a universality conjecture according to which the global symmetries of the gauge theory determine the low-lying spectrum of the theory in the microscopic limit. We assumed that flavor symmetry breaking occurs, and saw that in that case the pattern of this symmetry breaking is $\mathrm{Sp}(4N_f)\to\mathrm{Sp}(2N_f)\times\mathrm{Sp}(2N_f)$, while parity is unbroken.

We also indicated how to derive the sum rules constraining the small eigenvalues in the spirit of Leutwyler and Smilga, and obtained the first sum rule. Similar results had previously been obtained by other authors (see the Introduction) for QCD in four space-time dimensions for the ensembles labeled by $\beta=1$, 2, and 4 (orthogonal, unitary, and symplectic ensembles) and in three dimensions for $\beta=2$. Even though these latter ensembles may be more interesting for the real world, the 3D $\beta=1$ case treated here may be one of the easiest to simulate on the lattice. The only case of physical interest remaining is the 3D $\beta=4$ case. A similar treatment of this case requires defining Majorana fermions in Euclidean space. Some work in this direction was performed in [3]. Other interesting directions of work include finite temperature and chemical potential studies (see [8–10] in this context). Another, very ambitious project might be to try similar techniques at the multicritical points of the matrix model where the condensate goes to zero (cf. [22]).

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