Derivation of the effective chiral Lagrangian for pseudoscalar mesons from QCD

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We formally derive the chiral Lagrangian for low lying pseudoscalar mesons from the first principles of QCD considering the contributions from the normal part of the theory without taking an approximation. The derivation is based on the standard generating functional of QCD in the path integral formalism. The gluon-field integration is formally carried out by expressing the result in terms of the physical Green's functions of the gluon. To integrate over the quark field, we introduce a bilocal auxiliary field $\Phi(x,y)$ representing the mesons. We then develop a consistent way of extracting the local pseudoscalar degree of freedom $U(x)$ in $\Phi(x,y)$ and integrating out the rest degrees of freedom such that the complete pseudoscalar degree of freedom resides in $U(x)$. With certain techniques, we work out the explicit $U(x)$ dependence of the effective action up to the p^4 terms in the momentum expansion, which leads to the desired chiral Lagrangian in which all the coefficients contributed from the normal part of the theory are expressed in terms of certain quark Green's functions in QCD. Together with the exsisting QCD formulas for the anomaly contributions, the present results lead to the complete effective chiral Lagrangian for pseudoscalar mesons. The final result can be regarded as the fundamental QCD definition of the coefficients in the chiral Lagrangian. The relation between the present QCD definition of the p^2-order coefficient F_0^2 and the well-known appoximate result given by Pagels and Stokar is discussed.

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I. INTRODUCTION

The study of low energy hadron physics in QCD is a longstanding difficult problem due to its nonperturbative nature. For low lying pseudoscalar mesons, a widely used approach is the theory of chiral Lagrangian based on the momentum expansion and the consideration of the global symmetry of the system without dealing with the nonperturbative dynamics of QCD [1,2]. In the chiral Lagrangian approach, the coefficients in the Lagrangian are all unknown phenomenological parameters which are determined by experimental inputs. The number of the unknown parameters increases rapidly with the increase of the precision in the momentum expansion. For example, the chiral Lagrangian for pseudoscaler mesons with three flavors up to the p^4 terms given by Gasser and Leutwyler [2] contains 14 unknown coefficients. When the p^6 terms are taken into account, there are 143 additional unknown coefficients [3].

This kind of approach has also been applied to the electroweak theory [4] for studying the probe of the electroweak symmetry breaking mechanism [5,6]. Since parity and CP are not conserved in the electroweak theory, there are even more unknown coefficients in the electroweak chiral Lagrangian than in the case of QCD [4]. So far, this kind of study is at the point of finding out suitable processes at future high energy colliders to determine the unknown coefficients in the electroweak chiral Lagrangian and investigating to what precision the determination can be made. The relation between the coefficients in the electroweak chiral Lagrangian and the underlying model of the electroweak symmetry breaking mechanism is yet not known except for some very simple models [5].

Further study on understanding the relation between the chiral Lagrangian coefficients and the underlying dynamical theory will be very helpful both in QCD and in the electroweak theory for reducing the number of independent unknown parameters which makes the theory more predictable. There are papers studying approximate formulas for the chiral Lagrangian coefficients based on certain dynamical ansatz [7], but the approach is not completely from the first principles of the underlying theory. Attempts to build closer relations between the chiral Lagrangian and the long distance piece of the underlying theory of QCD by considering the anomaly contributions with certain approximation also exist [8,9]. However, several aspects of it imply that such a kind of approach needs improvement, e.g., (a) the theory does not include spontaneous chiral symmetry breaking, and the chiral symmetry breaking scale is put in by hand; (b) without putting in the chiral symmetry breaking scale, the obtained pion decay constant F_π is proportional to an imposed very low ($\sim$320 MeV) momentum cutoff on the underlying theory of QCD; (c) the positivity of F_π^2 depends on a careful choice of the regularization scheme. The approach in Ref. [10] does not contain the above problems. But in Ref. [10], the approximation of large-N_c limit is taken from the beginning and the approximation of picking up only the local scalar and pseudoscalar pieces of the color-singlet quark-antiquark bilocal operator arising from integrating the gluon-field is taken in the derivation. With the latter approximation, the formula for the p^2-order coefficient F_0^2 in Ref. [10] is expressed in terms of an imposed ultraviolet cutoff, and can hardly be related to the well-known Pagels-Stokar formula

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for F_0^2 [11]. Therefore, further improvement of studying the effective chiral Lagrangian from the fundamental principles of QCD is necessary. Actually, the study can be divided into two steps. The first step is to formally derive the effective chiral Lagrangian from the fundamental principles of QCD and express the coefficients in terms of certain dynamical quantities in QCD, which gives the QCD meanings of the coefficients. The second step is to calculate the related dynamical quantities in QCD to obtain the values of the coefficients. This paper is mainly devoted to the first step.

In this paper, we develop certain techniques with which we are able to formally derive the effective chiral Lagrangian for pseudoscalar moesons from the first principles of QCD without taking approximation, and all the coefficients are expressed in terms of certain Green's functions in QCD. Such expressions can be regarded as the fundamental QCD definitions of the coefficients. As a simple example, we show that, under certain approximations, our p^2-order coefficient F_0^2 reduces to the well-known approximate formula given by Pagels and Stokar [11]. A systematic numerical calculation of the coefficients by solving the related QCD Green's functions in certain approximation (the second step) will be presented in a separate paper [12].

This paper is organized as follows. Section II is on the fundamental generating functional in QCD. We start from it and formally integrate out the heavy-quark and gluon fields to obtain a formal generating functional for the light quark fields. In Sec. III, we introduce a bilocal auxiliary field reflecting the light meson degrees of freedom with which we can integrate out the light quark fields. Then we develop a technique for extracting the degree of freedom of the desired local field $U(x)$ for the pseudoscalar mesons from the bilocal auxiliary field, and formally integrate out the the remaining degrees of freedom of the bilocal auxiliary field to obtain a generating functional for the local field $U(x)$. In Sec. IV, we develop certain techniques to work out the complete $U(x)$ dependence of the effective Lagrangian in the sense of momentum expansion, and obtain the effective chiral Lagranian which is of the form given by Gasser and Leutwyler [2]. In this process we obtain the QCD expressions for all the coefficients in the effective chiral Lagrangian. A discussion on the relation between the present QCD definition of the $O(p^2)$ coefficients F_0^2 and the well-known Pagels-Stokar formula (an approximate result) [11] will be given in Sec. V. Section VI contains concluding remarks.

II. THE GENERATING FUNCTIONAL

Consider a QCD-type gauge theory with SU(N_c) local gauge symmetry. Let $A_\mu^i (i=1,2,\ldots,N_c^2-1)$ be the gauge field, $\psi_\alpha^{a\eta}$ and $\Psi_\alpha^{a\eta}$ be , respectively, light and heavy fermion fields with color index $\alpha(\alpha=1,2,\ldots,N_c)$, Lorentz spinor index η, light flavor index $a(a=1,2,\ldots,N_f)$, and heavy flavor index $\bar{a}(\bar{a}=1,2,\ldots,N_f')$. For convenience, we simply call $\psi_\alpha^{a\eta}$ the ''light quark field,'' $\Psi_\alpha^{a\eta}$ the ''heavy quark field,'' and A_μ^i the ''gluon-field.'' Let us introduce local external sources $J_{\sigma\rho}$ for the composite light quark operators $\bar{\psi}^\sigma\psi^\rho$, where σ and ρ are short notations for the spinor and flavor indices. The external source J can be decomposed into scalar, pseudoscalar, vector, and axial-vector parts

$$J(x)=-s(x)+ip(x)\gamma_5+\not{v}(x)+\not{a}(x)\gamma_5, \tag{1}$$

where $s(x)$, $p(x)$, $v_\mu(x)$, and $a_\mu(x)$ are Hermitian matrices, and the light quark masses have been absorbed into the definition of $s(x)$. The vector and axial-vector sources $\not{v}(x)$ and $\not{a}(x)$ are taken to be traceless.

Since the contributions from the anomaly term to the effective chiral Lagrangian has already been studied in Refs. [9,10], our aim in this paper is to study the complete normal part contributions. So, in this paper, we simply ignore the standard CP-violating term related to the anomaly by taking the θ-vacuum parameter $\theta=0$.

Following Gasser and Leutwyler [2], we start from constructing the following generating functional:

$$\begin{aligned}
Z[J]=&\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\mathcal{D}A_\mu \\
&\times\exp i\int d^4x\{\mathcal{L}(\psi,\bar{\psi},\Psi,\bar{\Psi},A_\mu)+\bar{\psi}J\psi\} \\
=&\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left\{i\int d^4x\bar{\psi}(i\not{\partial}+J)\psi\right\} \\
&\times\int \mathcal{D}\Psi\mathcal{D}\bar{\Psi}\mathcal{D}A_\mu\Delta_F(A_\mu)\exp\left\{i\int d^4x\left[\mathcal{L}_{\rm QCD}(A)\right.\right. \\
&\left.\left.-\frac{1}{2\xi}[F^i(A_\mu)]^2-g\mathcal{I}_i^\mu A_\mu^i+\bar{\Psi}(i\not{\partial}-M-g\not{A})\Psi\right]\right\},
\end{aligned} \tag{2}$$

where $\mathcal{L}_{\rm QCD}(A)=-\frac{1}{4}A_{\mu\nu}^i A^{i\mu\nu}$ is the gluon kinetic energy term, M is the heavy quark mass matrix, $\mathcal{I}_i^\mu\equiv\bar{\psi}\lambda_i/2\gamma^\mu\psi$ are colored currents composed of light quark fields, $-1/2\xi[F^i(A_\mu)]^2$ is the gauge-fixing term and $\Delta_F(A_\mu)$ is the Fadeev-Popov determinant.

Let us first consider the integration over $\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\mathcal{D}A_\mu$ for a given configuration of ψ and $\bar{\psi}$, i.e., the current $\mathcal{I}_i^\mu$ serves as an external source in the integration over $\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\mathcal{D}A_\mu$. The result of such an integration can be formally written as

$$\begin{aligned}
&\int \mathcal{D}\Psi\mathcal{D}\bar{\Psi}\mathcal{D}A_\mu\Delta_F(A_\mu)\exp\left\{i\int d^4x\left[\mathcal{L}_{\rm QCD}(A)\right.\right. \\
&\left.\left.-\frac{1}{2\xi}[F^i(A_\mu)]^2-g\mathcal{I}_i^\mu A_\mu^i+\bar{\Psi}(i\not{\partial}-M-g\not{A})\Psi\right]\right\} \\
&=\exp i\sum_{n=2}^{\infty}\int d^4x_1\cdots d^4x_n\frac{(-i)^n g^n}{n!} \\
&\times G_{\mu_1\cdots\mu_n}^{i_1\cdots i_n}(x_1,\cdots,x_n)\mathcal{I}_{i_1}^{\mu_1}(x_1)\cdots\mathcal{I}_{i_n}^{\mu_n}(x_n),
\end{aligned} \tag{3}$$

where $G_{\mu_1\cdots\mu_n}^{i_1\cdots i_n}$ is the full n-point Green's function of the A_μ^i field containing internal heavy quark lines and with given

sources $\mathcal{I}_i^\mu$. For simplicity, the gluon field integration in this paper is limited to the topologically trivial sector. Inclusion of topologically nontrivial sectors only changes the intermediate results but not the final result (45).

By Fierz reordering, we can further diagonalize the color indices of the light-quark operators, and get

$$
\begin{aligned}
&G^{i_1\cdots i_n}_{\mu_1\cdots\mu_n}(x_1,\cdots,x_n)\left[\overline{\psi}^{a_1}_{\alpha_1}(x_1)\left(\frac{\lambda_{i_1}}{2}\right)_{\alpha_1\beta_1}\gamma^{\mu_1}\psi^{a_1}_{\beta_1}(x_1)\right]\cdots\\
&\quad\times\left[\overline{\psi}^{a_n}_{\alpha_n}(x_n)\left(\frac{\lambda_{i_n}}{2}\right)_{\alpha_n\beta_n}\gamma^{\mu_n}\psi^{a_n}_{\beta_n}(x_n)\right]\\
&\quad=\int d^4x_1'\cdots d^4x_n' g^{n-2}\overline{G}^{\sigma_1\cdots\sigma_n}_{\rho_1\cdots\rho_n}(x_1,x_1',\cdots,x_n,x_n')\\
&\quad\quad\times\overline{\psi}^{\sigma_1}_{\alpha_1}(x_1)\psi^{\rho_1}_{\alpha_1}(x_1')\cdots\overline{\psi}^{\sigma_n}_{\alpha_n}(x_n)\psi^{\rho_n}_{\alpha_n}(x_n'),
\end{aligned}
\tag{4}
$$

where $\overline{G}^{\sigma_1\cdots\sigma_n}_{\rho_1\cdots\rho_n}(x_1,x_1',\cdots,x_n,x_n')$ is a generalized Green's function containing $2n$ space-time points. Then Eq. (2) can be written as

$$
\begin{aligned}
Z[J]=\int\mathcal{D}\psi\mathcal{D}\overline{\psi}\exp i\Bigg\{&\int d^4x\,\overline{\psi}(i\partial\!\!\!/+J)\psi\\
&+\sum_{n=2}^{\infty}\int d^4x_1\cdots d^4x_n d^4x_1'\cdots d^4x_n'\frac{(-i)^n(g^2)^{n-1}}{n!}\\
&\times\overline{G}^{\sigma_1\cdots\sigma_n}_{\rho_1\cdots\rho_n}(x_1,x_1',\cdots,x_n,x_n')\\
&\times\overline{\psi}^{\sigma_1}_{\alpha_1}(x_1)\psi^{\rho_1}_{\alpha_1}(x_1')\cdots\overline{\psi}^{\sigma_n}_{\alpha_n}(x_n)\psi^{\rho_n}_{\alpha_n}(x_n')\Bigg\}.
\end{aligned}
\tag{5}
$$

III. THE AUXILIARY FIELDS

A. The bilocal auxiliary field

For integrating out the light quark fields ψ and $\overline{\psi}$, we introduce a bilocal auxiliary field $\Phi^{(a\eta)(b\zeta)}(x,x')$ by inserting the following constant into Eq. (5):

$$
\int\mathcal{D}\Phi\,\delta(N_c\Phi^{(a\eta)(b\zeta)}(x,x')-\overline{\psi}^{a\eta}_{\alpha}(x)\psi^{b\zeta}_{\alpha}(x')).
\tag{6}
$$

We see from Eq. (6) that the bilocal auxiliary field $\Phi^{(a\eta)(b\zeta)}(x,x')$ embodies the bilocal composite operator $\overline{\psi}^{a\eta}_{\alpha}(x)\psi^{b\zeta}_{\alpha}(x')$ which reflects the meson fields. Inserting Eq. (6) into Eq. (5) we get

$$
\begin{aligned}
Z[J]=\int\mathcal{D}\psi\mathcal{D}\overline{\psi}\mathcal{D}\Phi\,\delta(&N_c\Phi^{(a\eta)(b\zeta)}(x,x')\\
&-\overline{\psi}^{a\eta}_{\alpha}(x)\psi^{b\zeta}_{\alpha}(x'))\exp i\Bigg\{\int d^4x\,\overline{\psi}[i\partial\!\!\!/+J]\psi\\
&+N_c\sum_{n=2}^{\infty}\int d^4x_1\cdots d^4x_n d^4x_1'\cdots d^4x_n'\\
&\times\frac{(-i)^n(N_cg^2)^{n-1}}{n!}\overline{G}^{\sigma_1\cdots\sigma_n}_{\rho_1\cdots\rho_n}(x_1,x_1',\ldots,x_n,x_n')\\
&\times\Phi^{\sigma_1\rho_1}(x_1,x_1')\cdots\Phi^{\sigma_n\rho_n}(x_n,x_n')\Bigg\}.
\end{aligned}
\tag{7}
$$

The δ function in Eq. (7) can be further expressed in the Fourier representation

$$
\begin{aligned}
&\delta(N_c\Phi(x,x')-\overline{\psi}(x)\psi(x'))\\
&\quad\sim\int\mathcal{D}\Pi e^{i\int d^4xd^4x'\Pi(x,x')[N_c\Phi(x,x')-\overline{\psi}(x)\psi(x')]}.
\end{aligned}
$$

With this we can integrate out the ψ and $\overline{\psi}$ fields and get

$$
\begin{aligned}
Z[J]=\int\mathcal{D}\Phi\mathcal{D}\Pi\exp i\Bigg\{&-iN_c\mathrm{Tr}\ln[i\partial\!\!\!/+J-\Pi]\\
&+\int d^4xd^4x'N_c\Phi^{\sigma\rho}(x,x')\Pi^{\sigma\rho}(x,x')\\
&+N_c\sum_{n=2}^{\infty}\int d^4x_1\cdots d^4x_n d^4x_1'\cdots d^4x_n'\\
&\times\frac{(-i)^n(N_cg^2)^{n-1}}{n!}\overline{G}^{\sigma_1\cdots\sigma_n}_{\rho_1\cdots\rho_n}(x_1,x_1',\ldots,x_n,x_n')\\
&\times\Phi^{\sigma_1\rho_1}(x_1,x_1')\cdots\Phi^{\sigma_n\rho_n}(x_n,x_n')\Bigg\},
\end{aligned}
\tag{8}
$$

where Tr is the functional trace with respect to the space-time, spinor and flavor indices.

Let us define the classical field Π_c

$$
\Pi_c\equiv\frac{\int\mathcal{D}\Pi\,\Pi e^{iS}}{\int\mathcal{D}\Pi e^{iS}},
\tag{9}
$$

where S is the argument on the exponential in Eq. (8). Let $\Gamma_0[J,\Phi,\Pi_c]$ be the effective action for Π_c with given J and Φ. Π_c satisfies[1]

$$
\frac{\partial\Gamma_0[J,\Phi,\Pi_c]}{\partial\Pi_c^{\sigma\rho}(x,x')}=0.
\tag{10}
$$

Then $\Gamma_0[J,\Phi,\Pi_c]$ is explicitly

[1]In the conventional approach, one usually introduces an external source $\mathcal{J}$ coupling to the field Π. With this, the right-hand-side of Eq. (10) equals $-\mathcal{J}$ [13]. Equation (10) corresponds to taking $\mathcal{J}=0$. Similarly, when taking $\mathcal{J}=0$, the effective action $\Gamma_0[J,\Phi,\Pi_c]$ equals the generating functional $W_0[J,\Phi,\mathcal{J}]|_{\mathcal{J}=0}$ for the connected Green's fuctions. This leads to the left-hand side of Eq. (11).

$$
\begin{aligned}
& e^{i\Gamma_0[J,\Phi,\Pi_c]} \\
& \equiv \int \mathcal{D}\Pi \exp iN_c \Big\{ -i\mathrm{Tr}\ln[i\partial\!\!\!/ + J - \Pi] \\
& \quad + \int d^4x d^4x' \Phi^{\sigma\rho}(x,x')\Pi^{\sigma\rho}(x,x') \\
& \quad + \sum_{n=2}^{\infty} \int d^4x_1 \cdots d^4x_n d^4x_1' \cdots d^4x_n' \\
& \quad \times \frac{(-i)^n (N_c g^2)^{n-1}}{n!} \overline{G}^{\sigma_1 \cdots \sigma_n}_{\rho_1 \cdots \rho_n}(x_1, x_1', \ldots, x_n, x_n') \\
& \quad \times \Phi^{\sigma_1\rho_1}(x_1,x_1') \cdots \Phi^{\sigma_n\rho_n}(x_n,x_n') \Big\}. \qquad (11)
\end{aligned}
$$

With these symbols, we can formally carry out the integration over the Π field in Eq. (8), and express the result by

$$
Z[J] = \int \mathcal{D}\Phi \exp\{i\Gamma_0[J,\Phi,\Pi_c]\}, \qquad (12)
$$

B. Localization

Since we are aiming at deriving the low energy effective chiral Lagrangian in which the light mesons are approximately described by local fields, we need to *consistently* extract the local field degree of freedom from the bilocal auxiliary field $\Phi^{(a\eta)(b\zeta)}(x,x')$. The extraction should be *consistent* in the sense that the complete degree of freedom of the mesons resides in the local fields without leaving any in the coefficients in the chiral Lagrangian. Otherwise, it will affect the validity of the momentum expansion [2]. In this paper, we propose the following way of extraction, and we shall see in Sec. IV that it is really *consistent*.

The auxiliary field Φ introduced in Eq. (6) has such a property which allows us to define the fields σ and Ω' related to the scalar and pseudoscalar sectors of Φ as

$$
\begin{aligned}
& (\Omega'\sigma\Omega' + \Omega'^{\dagger}\sigma\Omega'^{\dagger})^{ab}(x) = (1)_{\zeta\eta}\Phi^{(b\zeta)(a\eta)}(x,x) \\
& (\Omega'\sigma\Omega' - \Omega'^{\dagger}\sigma\Omega'^{\dagger})^{ab}(x) = (\gamma_5)_{\zeta\eta}\Phi^{(b\zeta)(a\eta)}(x,x). \qquad (13)
\end{aligned}
$$

Here the σ field repesented by a Hermitian matrix describes the modular degree of freedom, and the Ω' field represented by an unitary matrix describes the phase degree of freedom, i.e.,

$$
\sigma^{\dagger}(x) = \sigma(x), \quad \Omega'^{\dagger}(x)\Omega'(x) = 1. \qquad (14)
$$

As usual, we can define $U'(x) \equiv \Omega'^2(x)$ which contains a $U(1)$ factor such that $\det U'(x) = e^{i\vartheta(x)}$, where the determinant is for the flavor matrix. The unitarity property of $U'(x)$ implies that $\vartheta(x)$ is a real field. We can further extract out the $U(1)$ factor and define a field $U(x)$ as $U'(x) \equiv e^{(i/N_f)\vartheta(x)}U(x)$. It is easy to see that $\det U(x) = 1$. Then we can define a new field Ω and decompose U into

$$
U(x) = \Omega^2(x), \qquad (15)
$$

which is the conventional decomposition in the literature. This $U(x)$, as the desired representation of $\mathrm{SU}(N_f)_R \times \mathrm{SU}(N_f)_L$, will be the nonlinear realization of the peudoscalar meson fields in the chiral Lagrangian. Note that the way of introducing the $U(x)$ field is not unique but is up to a chiral rotation which does not affect the final effective chiral Lagrangian since the chiral Lagrangian is chirally invariant. The fields σ and ϑ are intermediate fields which will not appear in the final effective chiral Lagrangian. It is straightforward to subtract σ from the two equations in Eq. (13) and using Eq. (14) to get

$$
\begin{aligned}
& e^{-i[\vartheta(x)/N_f]}\Omega^{\dagger}(x)\mathrm{tr}_l[P_R\Phi^T(x,x)]\Omega^{\dagger}(x) \\
& \quad = e^{i[\vartheta(x)/N_f]}\Omega(x)\mathrm{tr}_l[P_L\Phi^T(x,x)]\Omega(x), \qquad (16)
\end{aligned}
$$

where P_R and P_L are, respectively, the projection operators onto the right-handed and left-handed states, the superscript T stands for the functional transposition (transposition of all indices including the space-time coordinates), and we have expressed the result in terms of Ω. Equation (16) builds up the relation between $\Phi(x,x)$ and $U(x)$ [or $\Omega(x)$]. Taking the determinant of Eq. (16) we can express $\vartheta(x)$ in terms of $\Phi^T(x,x)$ as

$$
e^{2i\vartheta(x)} = \frac{\det[\mathrm{tr}_l[P_R\Phi^T(x,x)]]}{\det[\mathrm{tr}_l[P_L\Phi^T(x,x)]]}, \qquad (17)
$$

where tr_l is the trace with respect to the spinor index.

Equations (13)–(16) describe our idea of localization. To realize this idea in the functional integration formalism, we need a technique to *integrate in* this information to the generating functional (12). For this purpose, we start from the following functional identity for an operator $\mathcal{O}$ satisfying $\det\mathcal{O} = \det\mathcal{O}^{\dagger}$ (see the Appendix for the proof)

$$
\begin{aligned}
& \int \mathcal{D}U\,\delta(U^{\dagger}U - 1)\,\delta(\det U - 1) \\
& \quad \times \mathcal{F}[\mathcal{O}]\,\delta(\Omega\mathcal{O}^{\dagger}\Omega - \Omega^{\dagger}\mathcal{O}\Omega^{\dagger}) = \mathrm{const}, \qquad (18)
\end{aligned}
$$

in which $\int \mathcal{D}U\,\delta(U^{\dagger}U - 1)\,\delta(\det U - 1)$ is an effective invariant integration measure and the function $\mathcal{F}[\mathcal{O}]$ is defined as

$$
\frac{1}{\mathcal{F}[\mathcal{O}]} \equiv \det\mathcal{O} \int \mathcal{D}\sigma\,\delta(\mathcal{O}^{\dagger}\mathcal{O} - \sigma^{\dagger}\sigma)\,\delta(\sigma - \sigma^{\dagger}). \qquad (19)
$$

With the special choice of

$$
\mathcal{O}(x) = e^{-i[\vartheta(x)/N_f]}\mathrm{tr}_l[P_R\Phi^T(x,x)], \qquad (20)
$$

which satisfies $\det\mathcal{O} = \det\mathcal{O}^{\dagger}$, Eq. (18) serves as the functional expression reflecting the relation (16). Inserting Eq. (18) and (20) into the functional (12) and taking the Fourier representation of the δ function

$$
\delta(\Omega\mathcal{O}^{\dagger}\Omega - \Omega^{\dagger}\mathcal{O}\Omega^{\dagger}) \sim \int \mathcal{D}\Xi\, e^{-iN_c\int dx\,\Xi(\Omega\mathcal{O}^{\dagger}\Omega - \Omega^{\dagger}\mathcal{O}\Omega^{\dagger})}, \qquad (21)
$$

we get

$$Z[J]=\int \mathcal{D}\Phi\mathcal{D}U\mathcal{D}\Xi\,\delta(U^\dagger U-1)\,\delta(\det U-1)\times\exp\Big\{i\Gamma_0[J,\Phi,\Pi_c]+i\Gamma_I[\Phi]+iN_c\int d^4x\times\mathrm{tr}_f[\Xi(x)(e^{-i[\vartheta(x)/N_f]}\Omega^\dagger(x)\mathrm{tr}_l[P_R\Phi^T(x,x)]\times\Omega^\dagger(x)-e^{i[\vartheta(x)/N_f]}\Omega(x)\mathrm{tr}_l[P_L\Phi^T(x,x)]\Omega(x))]\Big\},\tag{22}$$

with

$$e^{-i\Gamma_I[\Phi]}\equiv\prod_x\frac{1}{\mathcal{F}[\mathcal{O}(x)]}=\prod_x\Big(\{\det[\mathrm{tr}_lP_R\Phi^T(x,x)]\det[\mathrm{tr}_lP_L\Phi^T(x,x)]\}^{1/2}\times\int\mathcal{D}\sigma\,\delta[(\mathrm{tr}_lP_R\Phi^T)(\mathrm{tr}_lP_L\Phi^T)-\sigma^\dagger\sigma]\times\delta(\sigma-\sigma^\dagger)\Big).\tag{23}$$

In Eq. (22), the information about the relation (16) has been *integrated in*.

Next, we deal with the functional integration over the Φ-field. For this purpose, we define an effective action $\tilde{\Gamma}[\Omega,J,\Xi,\Phi_c,\Pi_c]$ as

$$e^{i\tilde{\Gamma}[\Omega,J,\Xi,\Phi_c,\Pi_c]}=\int\mathcal{D}\Phi\exp\Big\{i\Gamma_0[J,\Phi,\Pi_c]+i\Gamma_I[\Phi]+iN_c\int d^4x\times\mathrm{tr}_f[\Xi(x)(e^{-i[\vartheta(x)/N_f]}\Omega^\dagger(x)\mathrm{tr}_l\times[P_R\Phi^T(x,x)]\Omega^\dagger(x)-e^{i[\vartheta(x)/N_f]}\Omega(x)\mathrm{tr}_l[P_L\Phi^T(x,x)]\Omega(x))]\Big\},\tag{24}$$

in which the classical field $\Phi_c(x,x')$ is defined as

$$\Phi_c=\frac{\int\mathcal{D}\Phi\,\Phi e^{\tilde{S}}}{\int\mathcal{D}\Phi e^{\tilde{S}}},\tag{25}$$

where $\tilde{S}$ stands for the argument in the exponential in Eq. (24). Φ_c satisfies

$$\frac{\partial\tilde{\Gamma}[\Omega,J,\Xi,\Phi_c,\Pi_c]}{\partial\Phi_c^{\sigma\rho}(x,x')}=0.\tag{26}$$

With these symbols, we can formally carry out the $\int\mathcal{D}\Phi$ integration in Eq. (22) and obtain

$$Z[J]=\int\mathcal{D}U\mathcal{D}\Xi\,\delta(U^\dagger U-1)\,\delta(\det U-1)\times\exp\{i\tilde{\Gamma}[\Omega,J,\Xi,\Phi_c,\Pi_c]\}.\tag{27}$$

Here we have formally integrated out all the degrees of freedom in $\Phi(x,x')$ in addition to the extracted local degree of freedom $U(x)$. This localization is different from those in the literature [14].

Similar to the above procedures, we can formally integrate out the Ξ-field by introducing an effective action $S_{\mathrm{eff}}[U,J,\Xi_c,\Phi_c,\Pi_c]$ as follows:

$$e^{iS_{\mathrm{eff}}[U,J,\Xi_c,\Phi_c,\Pi_c]}=\int\mathcal{D}\Xi\,\exp i\tilde{\Gamma}[\Omega,J,\Xi,\Phi_c,\Pi_c],\tag{28}$$

where the classical field Ξ_c is defined as

$$\Xi_c=\frac{\int\mathcal{D}\Xi\,\Xi\,\exp i\tilde{\Gamma}[\Omega,J,\Xi,\Phi_c,\Pi_c]}{\int\mathcal{D}\Xi\,\exp i\tilde{\Gamma}[\Omega,J,\Xi,\Phi_c,\Pi_c]}\tag{29}$$

and satisfies

$$\frac{\partial S_{\mathrm{eff}}[U,J,\Xi_c,\Phi_c,\Pi_c]}{\partial(\Xi_c)^{ab}(x)}=0.\tag{30}$$

Then the Ξ-integration in Eq. (27) can be formally carried out, and we obtain

$$Z[J]=\int\mathcal{D}U\,\delta(U^\dagger U-1)\,\delta(\det U-1)\times\exp\{iS_{\mathrm{eff}}[U,J,\Xi_c,\Phi_c,\Pi_c]\}.\tag{31}$$

We see from Eq. (31) that $S_{\mathrm{eff}}[U,J,\Xi_c,\Phi_c,\Pi_c]$ is just the action for U with a given J.[2]

Using Eqs. (11), (24), (30), (26), (25), and (10), one can further show the following important relation

$$\left.\frac{dS_{\mathrm{eff}}[U,J,\Xi_c,\Phi_c,\Pi_c]}{dJ^{\sigma\rho}(x)}\right|_{U\mathrm{fix}}=N_c\frac{\int\mathcal{D}\Xi\,\Phi_c^{\sigma\rho}(x,x)\exp i\tilde{\Gamma}[\Omega,J,\Xi,\Phi_c,\Pi_c]}{\int\mathcal{D}\Xi\,\exp i\tilde{\Gamma}[\Omega,J,\Xi,\Phi_c,\Pi_c]}\equiv N_c\overline{\Phi}_c^{\sigma\rho}(x,x).\tag{32}$$

[2]Note that Ξ_c, Φ_c, and Π_c are all functionals of U and J through Eqs. (29), (25), and (9).

In Eq. (32), the symbol $\overline{\Phi}_c$ denotes the functional average of Φ_c over the field Ξ weighted by the action $\tilde{\Gamma}[\Omega,J,\Xi,\Phi_c,\Pi_c]$. Equation (32) is crucial in the derivation of the effective chiral Lagrangian.

IV. THE EFFECTIVE LAGRANGIAN

To derive the effective chiral Lagrangian, we need to obtain the $U(x)$ and J dependence of $S_{\rm eff}[U,J,\Xi_c,\Phi_c,\Pi_c]$. Note that $S_{\rm eff}[U,J,\Xi_c,\Phi_c,\Pi_c]$ depends on U and J not only explicitly in Eq. (28) but also implicitly via Ξ_c,Φ_c and Π_c through Eqs. (29), (25), and (9). The remaining task of the derivation of the effective chiral Lagrangian is to work out explicitly the complete U and J dependence of $S_{\rm eff}[U,J,\Xi_c,\Phi_c,\Pi_c]$. The procedure is described as what follows.

First we consider a chiral rotation

$$\begin{aligned} J_\Omega(x)&=[\Omega(x)P_R+\Omega^\dagger(x)P_L][J(x)+i\partial\!\!\!/]\\ &\quad\times[\Omega(x)P_R+\Omega^\dagger(x)P_L],\\ \Phi^T_\Omega(x,y)&=[\Omega^\dagger(x)P_R+\Omega(x)P_L]\Phi^T(x,y)\\ &\quad\times[\Omega^\dagger(y)P_R+\Omega(y)P_L],\\ \Pi_\Omega(x,y)&=[\Omega(x)P_R+\Omega^\dagger(x)P_L]\Pi(x,y)\\ &\quad\times[\Omega(y)P_R+\Omega^\dagger(y)P_L]. \end{aligned} \tag{33}$$

The present theory is symmetric under this transformation. Since the Ξ-field is introduced in Eq. (21) and the operator $\Omega\mathcal{O}^\dagger\Omega-\Omega^\dagger\mathcal{O}\Omega^\dagger$ is invariant under the chiral rotation, there is no need to introduce Ξ_Ω. Furthermore, since $\det\Omega=1$, we can easily see from Eq. (17) that $\vartheta_\Omega(x)=\vartheta(x)$. The explicit dependence of $S_{\rm eff}[U,J,\Xi_c,\Phi_c,\Pi_c]$ on $U(x)$ comes from the explicit $\Omega(x)[\Omega^\dagger(x)]$ dependence of $\tilde{\Gamma}[\Omega,J,\Phi_c,\Pi_c]$ in Eq. (24) [see Eqs. (24) and (28)]. After the chiral rotation, this term becomes

$$\begin{aligned} &+iN_c\int d^4x\,\mathrm{tr}_f(\Xi(x)\{e^{-i[\vartheta(x)/N_f]}\mathrm{tr}_l[P_R\Phi^T_\Omega(x,x)]\\ &\quad-e^{i[\vartheta(x)/N_f]}\mathrm{tr}_l[P_L\Phi^T_\Omega(x,x)]\}), \end{aligned}$$

which no longer depends on $U(x)$ explicitly. Therefore, after the chiral rotation, there is no explicit $U(x)$ dependence of $S_{\rm eff}[U,J,\Xi_c,\Phi_c,\Pi_c]$, i.e., the complete $U(x)$ dependence resides implicitly in the rotated variables with the subscript Ω. For instance, the effective actions $\Gamma_0[J,\Phi_c,\Pi_c]$, $\Gamma_I[\Phi]$, $\tilde{\Gamma}[\Omega,J,\Xi,\Phi_c,\Pi_c]$, and $S_{\rm eff}[U,J,\Xi_c,\Phi_c,\Pi_c]$ can be written as

$$\Gamma_0[J,\Phi,\Pi_c]=\Gamma_0[J_\Omega,\Phi_\Omega,\Pi_{\Omega c}]+\text{anomaly terms}, \tag{34}$$

$$\Gamma_I[\Phi]=\Gamma_I[\Phi_\Omega], \tag{35}$$

$$\tilde{\Gamma}[\Omega,J,\Xi,\Phi_c,\Pi_c]=\tilde{\Gamma}[1,J_\Omega,\Xi,\Phi_{\Omega c},\Pi_{\Omega c}]+\text{anomaly terms}, \tag{36}$$

$$S_{eff}[U,J,\Xi_c,\Phi_c,\Pi_c]=S_{eff}[1,J_\Omega,\Xi_c,\Phi_{\Omega c},\Pi_{\Omega c}]+\text{anomaly terms}. \tag{37}$$

From Eq. (24) we see that

$$\begin{aligned} &e^{i\tilde{\Gamma}[1,J_\Omega,\Xi,\Phi_{\Omega c},\Pi_{\Omega c}]}\\ &\quad=\int\mathcal{D}\Phi_\Omega\exp\Big\{i\Gamma_0[J_\Omega,\Phi_\Omega,\Pi_{\Omega c}]+i\Gamma_I[\Phi_\Omega]\\ &\qquad+iN_c\int d^4x\,\mathrm{tr}_{lf}\Big\{\Xi(x)\Big[-i\sin\frac{\vartheta(x)}{N_f}\\ &\qquad+\gamma_5\cos\frac{\vartheta(x)}{N_f}\Big]\Phi^T_\Omega(x,x)\Big\}\Big\}, \end{aligned} \tag{38}$$

where tr_{lf} denotes the trace with respect to the spinor and flavor indices. The anomaly terms in Eqs. (34), (36), and (37) are all the same arising from the non-invariance of $\mathrm{Tr}\ln[i\partial\!\!\!/+J-\Pi]$ under the chiral rotation. Note that the functional integration measure does not change under the chiral rotation, i.e. $\mathcal{D}\Phi\mathcal{D}\Pi=\mathcal{D}\Phi_\Omega\mathcal{D}\Pi_\Omega$ since the Jaccobians from $\Phi\to\Phi_\Omega$ and $\Pi\to\Pi_\Omega$ cancel each other. We see that the $U(x)$ dependence is simplified after the chiral rotation.

The second approach is the use of Eq. (32). As we have mentioned in Sec. II that we ignore the irrelevant anomaly terms in this study. Then after the chiral rotation, Eq. (32) becomes

$$\left.\frac{dS_{eff}[1,J_\Omega,\Xi_c,\Phi_{\Omega c},\Pi_{\Omega c}]}{dJ^{\sigma\rho}_\Omega(x)}\right|_{U\text{fix, anomaly ignored}}=N_c\overline{\Phi^{\sigma\rho}_{\Omega c}(x,x)}. \tag{39}$$

We see from Eq. (39) that once the J_Ω dependence of $\overline{\Phi_{\Omega c}}$ is explicitly known, one can integrate Eq. (39) over J_Ω and get the $U(x)$ dependence of $S_{\rm eff}[1,J_\Omega,\Xi_c,\Phi_{\Omega c},\Pi_{\Omega c}]$ up to an irrelevant integration constant independent of $U(x)$ and $J(x)$. From that we can derive the effective chiral Lagrangian and the expressions for its coefficients. There can be two ways of figuring out the J_Ω dependence of $\overline{\Phi_{\Omega c}}$. One is to write down the dynamical equations for the intermediate fields Ξ_c, $\Phi_{\Omega c}$, and $\Pi_{\Omega c}$ and solve them (usually this can be done only under certain approximations) to get the J_Ω dependence of these intermediate fields. The other one is to track back to the original QCD expression for the chirally rotated generating functional (2) through (32) and (25) by reverting the procedures in Secs. III and II, which can lead to the fundamental QCD definitions of the chiral Lagrangian coefficients without taking approximation. We take the latter approach in this paper. Because of the δ function $\delta(N_c\Phi^{(a\eta)(b\zeta)}(x,x')-\overline{\psi^{(a\eta0)}_\alpha}(x)\psi^{(b\zeta)}_\alpha(x'))$ in Eq. (7), we can express $\overline{\Phi^{(a\eta)(b\zeta)}_{\Omega c}(x,y)}$ as

$$N_c\overline{\Phi_{\Omega c}^{(a\,\eta)(b\zeta)}(x,y)}=\frac{\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\mathcal{D}A_\mu\mathcal{D}\Xi\,\bar{\psi}_\alpha^a{}^\eta(x)\psi_\alpha^{b\zeta}(y)e^{i\hat{S}[\psi,\bar{\psi},\Psi,\bar{\Psi},A,\Xi]}}{\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\mathcal{D}A_\mu\mathcal{D}\Xi\,e^{i\hat{S}[\psi,\bar{\psi},\Psi,\bar{\Psi},A,\Xi]}},\tag{40}$$

where

$$\begin{aligned}\hat{S}[\psi,\bar{\psi},\Psi,\bar{\Psi},A,\Xi]&\equiv\Gamma_I\left[\frac{1}{N_c}\bar{\psi}\psi\right]+\int d^4x\Big\{\mathcal{L}(\psi,\bar{\psi},\Psi,\bar{\Psi},A_\mu)\\&+\bar{\psi}\Big[\not{v}_\Omega+\not{a}_\Omega\gamma_5-s_\Omega+ip_\Omega\gamma_5\\&+\Xi\left(-i\sin\frac{\vartheta'}{N_f}+\gamma_5\cos\frac{\vartheta'}{N_f}\right)\Big]\psi\Big\}.\end{aligned}\tag{41}$$

In Eq. (40) and all later equations in this paper, the symbol ψ is used as a short notation for the chirally rotated quark field ψ_Ω.[3] In Eq. (41), $\Gamma_I[(1/N_c)\bar{\psi}\psi]$ and ϑ' are the quantities defined in Eqs. (23) and (17) expressed in terms of quark fields, i.e.,

$$\begin{aligned}e^{-i\Gamma_I[(1/N_c)\bar{\psi}\psi]}&=\prod_x\Bigg\{\Bigg[\det\left(\frac{1}{N_c}\mathrm{tr}_{lc}[\psi_R(x)\bar{\psi}_L(x)]\right)\\&\times\det\left(\frac{1}{N_c}\mathrm{tr}_{lc}[\psi_L(x)\bar{\psi}_R(x)]\right)\Bigg]^{1/2}\\&\times\int\mathcal{D}\sigma\,\delta\left(\frac{1}{N_c^2}\mathrm{tr}_{lc}(\psi_R\bar{\psi}_L)\mathrm{tr}_{lc}(\psi_L\bar{\psi}_R)-\sigma^\dagger\sigma\right)\\&\times\delta(\sigma-\sigma^\dagger)\Bigg\},\end{aligned}\tag{42}$$

$$e^{2i\vartheta'(x)}\equiv\frac{\det[\mathrm{tr}_{lc}[\psi_R(x)\bar{\psi}_L(x)]]}{\det[\mathrm{tr}_{lc}[\psi_L(x)\bar{\psi}_R(x)]]},\tag{43}$$

where tr_{lc} is the trace with respect to the spinor and color indices. Equation (43) implies that the range of $\vartheta'(x)$ is $[0,\pi)$.

Note that instantons contribute to both Eqs. (42) and (43) [15]. The $U_A(1)$ violating field-configurations only cause nonvanishing ϑ' but do not contribute to $\Gamma_I[(1/N_c)\bar{\psi}\psi]$.

[3]As an integration variable, whether or not there is a subscript Ω makes no difference. Once the classical equation of motion is concerned, distinguishing the rotated ψ_Ω from the unrotated ψ will be necessary.

With Eqs. (40)–(43), one can integrate Eq. (39) over the rotated souces and obtain

$$\begin{aligned}e^{iS_{eff}[1,J_\Omega,\Xi_c,\Phi_{\Omega c},\Pi_{\Omega c}]}\big|_{\text{anomaly ignored}}&=\int\mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\mathcal{D}A_\mu\mathcal{D}\Xi\,\exp\Big(i\Gamma_I\left[\frac{1}{N_c}\bar{\psi}\psi\right]\\&+i\int d^4x\Big\{\mathcal{L}(\psi,\bar{\psi},\Psi,\bar{\Psi},A_\mu)+\bar{\psi}\Big[\not{v}_\Omega+\not{a}_\Omega\gamma_5-s_\Omega\\&+ip_\Omega\gamma_5+\Xi\left(-i\sin\frac{\vartheta'}{N_f}+\gamma_5\cos\frac{\vartheta'}{N_f}\right)\Big]\psi\Big\}\Big).\end{aligned}\tag{44}$$

For realistic QCD ($N_f=3$), $\cos(\vartheta'/N_f)$ does not vanish. We can then shift the integration variable $\Xi\to\Xi-ip_\Omega/\cos(\vartheta'/N_f)$ to cancel the p_Ω-dependence in the pseudoscalar part of Eq. (44). After carrying out the integration over Ξ, we obtain

$$\begin{aligned}e^{iS_{\mathrm{eff}}[1,J_\Omega,\Xi_c,\Phi_{\Omega c},\Pi_{\Omega c}]}\big|_{\text{anomaly ignored}}&=\int\mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\mathcal{D}A_\mu\\&\times\delta\left(\bar{\psi}^a\left[-i\sin\frac{\vartheta'}{N_f}+\gamma_5\cos\frac{\vartheta'}{N_f}\right]\psi^b\right)\\&\times\exp\Big(i\Gamma_I\left[\frac{1}{N_c}\bar{\psi}\psi\right]+i\int d^4x\Big\{\mathcal{L}(\psi,\bar{\psi},\Psi,\bar{\Psi},A_\mu)\\&+\bar{\psi}\left[\not{v}_\Omega+\not{a}_\Omega\gamma_5-s_\Omega-p_\Omega\tan\frac{\vartheta'}{N_f}\right]\psi\Big\}\Big).\end{aligned}\tag{45}$$

In Eq. (45), there is no p_Ω dependence in the pseudoscalar channel, and the p_Ω dependence appears in the scalar channel as the combination $s_\Omega+p_\Omega\tan(\vartheta'/N_f)$.

Equation (45) shows that $S_{\mathrm{eff}}[1,J_\Omega,,\Xi_c,\Phi_{\Omega c},\Pi_{\Omega c}]$ is the QCD generating functional for the rotated sources $s_\Omega+p_\Omega\tan\vartheta'/N_f$, v_Ω^μ, and a_Ω^μ with a special parity odd degree of freedoms $-i\bar{\psi}^a\psi^b\sin\vartheta'/N_f+\bar{\psi}^a\gamma_5\psi^b\cos\vartheta'/N_f$ frozen. After making a further $U_A(1)$ rotation of the ψ and $\bar{\psi}$ fields, the angle ϑ'/N_f can be rotated away and the frozen degree of freedom becomes just the pesudoscalar degree of freedom $\bar{\psi}^a\gamma_5\psi^b$ as it should be since this degree of freedom is already included in the integrating in of the U field. The automatic occurance of this frozen degree of freedom in the present approach implies that our way of extracting the U-field degree of freedom is really *consistent*, i.e., nothing of the pseudoscalar degree of freedom is left outside U. After the $U_A(1)$ rotation, Γ_I and the Jacobian due to the rotation

will give rise to an extra factor in the integrand, which is the compensation factor for the extraction of the U-field degree of freedom. From the point of view of the auxiliary field Φ, this corresponds to the contributions from integrating out the degrees of freedom other than U, say the σ and η' mesons.

Now we are ready to explicitly work out the effective chiral Lagrangian to the p^2 and p^4 order. As is pointed out in Ref. [2], the vector and axial-vector sources should be regarded as $O(p)$ and the scalar and pseudoscalar sources should be regarded as $O(p^2)$ in the momentum expansion.

A. The p^2 terms

We first consider the p^2-order terms. To this order, the anomaly can be safely ignored. Expanding Eq. (45) up to the order of p^2, we obtain

$$\begin{aligned} S_{\rm eff}[1,J_\Omega,\Xi_c,\Phi_{\Omega c},\Pi_{\Omega c}]|_{p^2\ \rm order} &= \int d^4x \,{\rm tr}_f[F^{ab}(x)s_\Omega^{ab}(x)+F'^{ab}(x)p_\Omega^{ab}(x)] \\ &\quad + \int d^4x d^4z\, G_{\mu\nu}^{abcd}(x,z)a_\Omega^{\mu,ab}(x)a_\Omega^{\nu,cd}(z), \end{aligned} \tag{46}$$

where

$$F^{ab}(x) = -\langle[\bar\psi^a(x)\psi^b(x)]\rangle,$$

$$F'^{ab}(x) = -\left\langle[\bar\psi^a(x)\psi^b(x)]\tan\frac{\vartheta'(x)}{N_f}\right\rangle,$$

$$\begin{aligned} G_{\mu\nu}^{abcd}(x,z) = \frac{i}{2}[&\langle[\bar\psi^a(x)\gamma_\mu\gamma_5\psi^b(x)][\bar\psi^c(z)\gamma_\nu\gamma_5\psi^d(z)]\rangle \\ &-\langle[\bar\psi^a(x)\gamma_\mu\gamma_5\psi^b(x)]\rangle \\ &\times\langle[\bar\psi^c(z)\gamma_\nu\gamma_5\psi^d(z)]\rangle], \end{aligned} \tag{47}$$

and the symbol $\langle O\rangle$ for an operator O appeared in Eq. (47) is defined as

$$\langle O\rangle \equiv \frac{\int \mathcal{D}\mu O}{\int \mathcal{D}\mu}, \tag{48}$$

where

$$\begin{aligned} \mathcal{D}\mu \equiv &\mathcal{D}\psi\mathcal{D}\bar\psi\mathcal{D}\Psi\mathcal{D}\bar\Psi\mathcal{D}A_\mu \\ &\times\delta\left(\bar\psi^a\left[-i\sin\frac{\vartheta'}{N_f}+\gamma_5\cos\frac{\vartheta'}{N_f}\right]\psi^b\right) \\ &\times e^{i\Gamma_I[(1/N_c)\bar\psi\psi]+i\int d^4x\mathcal{L}(\psi,\bar\psi,\Psi,\bar\Psi,A_\mu)}. \end{aligned}$$

For $F^{ab}(x)$, translational invariance and flavor conservation [16] leads to the conclusion that it is simply a space-time independent constant proportional to δ^{ab}. So that it can be written as

$$F^{ab}(x) = F_0^2 B_0 \delta^{ab}, \tag{49}$$

where

$$F_0^2 B_0 \equiv -\frac{1}{N_f}\langle\bar\psi\psi\rangle. \tag{50}$$

For $F'^{ab}(x)$, parity conservation [17] leads to

$$F'^{ab}(x) = 0. \tag{51}$$

For $G_{\mu\nu}^{abcd}(x,z)$, translational invariance leads to the conclusion that it can only depend on $x-z$. We can further expand this dependence in terms of $\delta(x-z)$ and its derivatives. To p^2 order, the derivative terms do not contribute, and the only term left is $\delta(x-z)\int d^4z G_{\mu\nu}^{abcd}(x,z)$. The coefficient $\int d^4z G_{\mu\nu}^{abcd}(x,z)$ is again independent of the space-time coordinates due to translational invariance. Then Lorentz and flavor symmetries imply that $\int d^4x G_{\mu\nu}^{abcd}$ is proportional to $g_{\mu\nu}\delta^{ad}\delta^{bc}$. There cannot be terms of the structure $\delta^{ab}\delta^{cd}$ since this term is to be multiplied by $a_\Omega^{\mu,ab}\ a_\Omega^{\nu,cd}$, and a_Ω^μ is traceless. Therefore the only relevant part of $G_{\mu\nu}^{abcd}(x,z)$ is

$$G_{\mu\nu}^{abcd}(x,z) = \delta(x-z)g_{\mu\nu}\delta^{ad}\delta^{bc}F_0^2 + \text{irrelevant terms}, \tag{52}$$

where

$$\begin{aligned} F_0^2 \equiv &\frac{1}{4(N_f^2-1)}\int d^4x\left[G_{\mu'}^{\mu',abba}(0,x)-\frac{1}{N_f}G_{\mu'}^{\mu',aabb}(0,x)\right] \\ = &\frac{i}{8(N_f^2-1)}\int d^4x\Big[\langle[\bar\psi^a(0)\gamma^\mu\gamma_5\psi^b(0)] \\ &\times[\bar\psi^b(x)\gamma_\mu\gamma_5\psi^a(x)]\rangle-\frac{1}{N_f}\langle[\bar\psi^a(0)\gamma^\mu\gamma_5\psi^a(0)] \\ &\times[\bar\psi^b(x)\gamma_\mu\gamma_5\psi^b(x)]\rangle-\langle[\bar\psi^a(0)\gamma^\mu\gamma_5\psi^b(0)]\rangle \\ &\times\langle[\bar\psi^b(x)\gamma_\mu\gamma_5\psi^a(x)]\rangle+\frac{1}{N_f}\langle[\bar\psi^a(0)\gamma^\mu\gamma_5\psi^a(0)]\rangle \\ &\times\langle[\bar\psi^b(x)\gamma_\mu\gamma_5\psi^b(x)]\rangle\Big]. \end{aligned} \tag{53}$$

Note that there is no term similar to ${\rm tr}_f[v_\Omega^2]$ in Eq. (46). The reason is that there exists a hidden symmetry $s_\Omega\to h^\dagger s_\Omega h$, $p_\Omega\to h^\dagger p_\Omega h$, $a_\Omega^\mu\to h^\dagger a_\Omega^\mu h$, and $v_\Omega\to h^\dagger v_\Omega^\mu h + h^\dagger i\partial^\mu h$ in which the vector source transforms inhomogeneously. So that the vector source can only appear together with the derivative $i\partial^\mu$ to form a covariant derivative, and a hidden symmetry covariant quadratic form of the covariant derivative can only be an antisymmetric tensor [see Eq. (56)]

which does not contribute when multiplied by a symmetric coefficient of the type of Eq. (52).

With Eqs. (50), (51), and (52) the effective action (46) is then

$$S_{\rm eff}[1,J_\Omega,\Xi_c,\Phi_{\Omega c},\Pi_{\Omega c}]|_{p^2\ \rm order}$$
$$=F_0^2\int d^4x\,{\rm tr}_f[a_\Omega^2+B_0 s_\Omega]$$
$$=F_0^2\int d^4x\,{\rm tr}_f\left[\frac{1}{4}[\nabla^\mu U^\dagger][\nabla_\mu U]+\frac{1}{2}B_0[U(s-ip)+U^\dagger(s+ip)]\right], \tag{54}$$

where ∇_μ is the covariant derivative related to the external sources defined in Ref. [2]. The integrand is just the p^2 order chiral Lagrangian given by Gasser and Leutwyler in Ref. [2]. Now the coefficients F_0^2 and B_0 are defined in Eqs. (53) and (50) and are expressed in terms of certain Green's functions of the quark fields. These can be regarded as the fundamental QCD definitions of F_0^2 and B_0.

B. The p^4 terms

The p^4-order terms can be worked out along the same line. The relevant terms for the normal part contributions (ignoring anomaly contributions) are

$$S_{\rm eff}[1,J_\Omega,\Xi_c,\Phi_{\Omega c},\Pi_{\Omega c}]|_{p^4\ \rm order,\ normal}$$
$$=\int d^4x\,{\rm tr}_f[-\mathcal{K}_1[d_\mu a_\Omega^\mu]^2-\mathcal{K}_2(d^\mu a_\Omega^\nu-d^\nu a_\Omega^\mu)\times(d_\mu a_{\Omega,\nu}-d_\nu a_{\Omega,\mu})+\mathcal{K}_3[a_\Omega^2]^2+\mathcal{K}_4 a_\Omega^\mu a_\Omega^\nu a_{\Omega,\mu}a_{\Omega,\nu}$$
$$+\mathcal{K}_5 a_\Omega^2{\rm tr}_f[a_\Omega^2]+\mathcal{K}_6 a_\Omega^\mu a_\Omega^\nu{\rm tr}_f[a_{\Omega,\mu}a_{\Omega,\nu}]+\mathcal{K}_7 s_\Omega^2+\mathcal{K}_8 s_\Omega{\rm tr}_f[s_\Omega]+\mathcal{K}_9 p_\Omega^2+\mathcal{K}_{10}p_\Omega{\rm tr}_f[p_\Omega]+\mathcal{K}_{11}s_\Omega a_\Omega^2$$
$$+\mathcal{K}_{12}s_\Omega{\rm tr}_f[a_\Omega^2]-\mathcal{K}_{13}V_\Omega^{\mu\nu}V_{\Omega,\mu\nu}+i\mathcal{K}_{14}V_\Omega^{\mu\nu}a_{\Omega,\mu}a_{\Omega,\nu}+\mathcal{K}_{15}p_\Omega d^\mu a_{\Omega,\mu}], \tag{55}$$

where the covariant derivative d_μ and the antisymmetric tensor $V_\Omega^{\mu\nu}$ are defined as

$$d^\mu a_\Omega^\nu\equiv\partial^\mu a_\Omega^\nu-iv_\Omega^\mu a_\Omega^\nu+ia_\Omega^\nu v_\Omega^\mu,$$
$$V_\Omega^{\mu\nu}=\partial^\mu v_\Omega^\nu-\partial^\nu v_\Omega^\mu-iv_\Omega^\mu v_\Omega^\nu+iv_\Omega^\nu v_\Omega^\mu, \tag{56}$$

and the fifteen coefficients $\mathcal{K}_1\ldots\mathcal{K}_{15}$ are determined by the following integrations of the Green's functions:

$$\frac{i}{4}\int d^4x x^{\mu'}x^{\nu'}\langle[\bar\psi^a(0)\gamma^\mu\gamma_5\psi^b(0)][\bar\psi^c(x)\gamma^\nu\gamma_5\psi^d(x)]\rangle_C$$
$$=\left[\left(\frac{1}{2}\mathcal{K}_1-\mathcal{K}_2\right)(g^{\mu\mu'}g^{\nu\nu'}+g^{\mu\nu'}g^{\nu\mu'})+2\mathcal{K}_2 g^{\mu\nu}g^{\mu'\nu'}\right]\delta^{ad}\delta^{bc}+\text{irrelevant terms},$$
$$-\frac{i}{24}\int d^4x d^4y d^4z\langle[\bar\psi^{a_1}(0)\gamma^{\mu_1}\gamma_5\psi^{b_1}(0)][\bar\psi^{a_2}(x)\gamma^{\mu_2}\gamma_5\psi^{b_2}(x)][\bar\psi^{a_3}(y)\gamma^{\mu_3}\gamma_5\psi^{b_3}(y)][\bar\psi^{a_4}(z)\gamma^{\mu_4}\gamma_5\psi^{b_4}(z)]\rangle_C$$
$$=\frac{1}{6}\Bigg\{\delta^{a_1b_2}\Bigg[\delta^{a_2b_3}\delta^{a_3b_4}\delta^{a_4b_1}\left(\frac{1}{2}(g^{\mu_1\mu_2}g^{\mu_3\mu_4}+g^{\mu_1\mu_4}g^{\mu_2\mu_3})\mathcal{K}_3+g^{\mu_1\mu_3}g^{\mu_2\mu_4}\mathcal{K}_4\right)+\delta^{a_2b_4}\delta^{a_4b_3}\delta^{a_3b_1}\left(\frac{1}{2}(g^{\mu_1\mu_2}g^{\mu_3\mu_4}$$
$$+g^{\mu_1\mu_3}g^{\mu_2\mu_4})\mathcal{K}_3+g^{\mu_1\mu_4}g^{\mu_2\mu_3}\mathcal{K}_4\right)\Bigg]+\delta^{a_1b_3}\Bigg[\delta^{a_3b_2}\delta^{a_2b_4}\delta^{a_4b_1}\left(\frac{1}{2}(g^{\mu_1\mu_3}g^{\mu_2\mu_4}+g^{\mu_1\mu_4}g^{\mu_2\mu_3})\mathcal{K}_3+g^{\mu_1\mu_2}g^{\mu_3\mu_4}\mathcal{K}_4\right)$$
$$+\delta^{a_3b_4}\delta^{a_4b_2}\delta^{a_2b_1}\left(\frac{1}{2}(g^{\mu_1\mu_3}g^{\mu_2\mu_4}+g^{\mu_1\mu_2}g^{\mu_3\mu_4})\mathcal{K}_3+g^{\mu_1\mu_4}g^{\mu_2\mu_3}\mathcal{K}_4\right)\Bigg]+\delta^{a_1b_4}\Bigg[\delta^{a_4b_2}\delta^{a_2b_3}\delta^{a_3b_1}\left(\frac{1}{2}(g^{\mu_1\mu_4}g^{\mu_2\mu_3}$$
$$+g^{\mu_1\mu_3}g^{\mu_2\mu_4})\mathcal{K}_3+g^{\mu_1\mu_2}g^{\mu_3\mu_4}\mathcal{K}_4\right)+\delta^{a_4b_3}\delta^{a_3b_2}\delta^{a_2b_1}\left(\frac{1}{2}(g^{\mu_1\mu_4}g^{\mu_2\mu_3}+g^{\mu_1\mu_2}g^{\mu_3\mu_4})\mathcal{K}_3+g^{\mu_1\mu_3}g^{\mu_2\mu_4}\mathcal{K}_4\right)\Bigg]$$
$$+\delta^{a_1b_2}\delta^{a_2b_1}\delta^{a_3b_4}\delta^{a_4b_3}[g^{\mu_1\mu_2}g^{\mu_3\mu_4}2\mathcal{K}_5+(g^{\mu_1\mu_3}g^{\mu_2\mu_4}+g^{\mu_1\mu_4}g^{\mu_2\mu_3})\mathcal{K}_6]+\delta^{a_1b_3}\delta^{a_3b_1}\delta^{a_2b_4}\delta^{a_4b_2}[g^{\mu_1\mu_3}g^{\mu_2\mu_4}2\mathcal{K}_5$$
$$+(g^{\mu_1\mu_2}g^{\mu_3\mu_4}+g^{\mu_1\mu_4}g^{\mu_2\mu_3})\mathcal{K}_6]+\delta^{a_1b_4}\delta^{a_4b_1}\delta^{a_2b_3}\delta^{a_3b_2}[g^{\mu_1\mu_4}g^{\mu_2\mu_3}2\mathcal{K}_5+(g^{\mu_1\mu_2}g^{\mu_3\mu_4}+g^{\mu_1\mu_3}g^{\mu_2\mu_4})\mathcal{K}_6]\Bigg\}$$
$$+\text{irrelevant terms},$$

$$\frac{i}{2}\int d^4x\langle[\bar{\psi}^a(0)\psi^b(0)][\bar{\psi}^c(x)\psi^d(x)]\rangle_C=\mathcal{K}_7\delta^{ad}\delta^{bc}+\mathcal{K}_8\delta^{ab}\delta^{cd},$$

$$\frac{i}{2}\int d^4x\left\langle[\bar{\psi}^a(0)\psi^b(0)]\tan\frac{\vartheta'(0)}{N_f}[\bar{\psi}^c(x)\psi^d(x)]\tan\frac{\vartheta'(x)}{N_f}\right\rangle_C=\mathcal{K}_9\delta^{ad}\delta^{bc}+\mathcal{K}_{10}\delta^{ab}\delta^{cd},$$

$$\frac{1}{8}\int d^4xd^4y\langle[\bar{\psi}^{a_1}(0)\psi^{b_1}(0)][\bar{\psi}^{a_2}(x)\gamma^\mu\gamma_5\psi^{b_2}(x)][\bar{\psi}^{a_3}(y)\gamma_\mu\gamma_5\psi^{b_3}(y)]\rangle_C$$
$$=\frac{1}{2}\mathcal{K}_{11}(\delta^{a_1b_2}\delta^{a_2b_3}\delta^{a_3b_1}+\delta^{a_1b_3}\delta^{a_3b_2}\delta^{a_2b_1})+\mathcal{K}_{12}\delta^{a_1b_1}\delta^{a_2b_3}\delta^{a_3b_2}+\text{irrelavent terms},$$

$$\mathcal{K}_{13}=\frac{i}{576(N_f^2-1)}\int d^4x\Big[(5g_{\mu\nu}g_{\mu'\nu'}-2g_{\mu\mu'}g_{\nu\nu'})x^{\mu'}x^{\nu'}$$
$$\times\Big(\langle\bar{\psi}^a(0)\gamma^\mu\psi^b(0)\bar{\psi}^b(x)\gamma^\nu\psi^a(x)\rangle_C-\frac{1}{N_f}\langle\bar{\psi}^a(0)\gamma^\mu\psi^a(0)\bar{\psi}^b(x)\gamma^\nu\psi^b(x)\rangle_C\Big)\Big],$$

$$\mathcal{K}_{14}=\frac{i}{36}[2g_{\mu_1\mu_4}g_{\mu_2\mu_3}T_A^{\mu_1\mu_2\mu_3\mu_4}+2g_{\mu_1\mu_2}g_{\mu_3\mu_4}T_A^{\mu_1\mu_2\mu_3\mu_4}-g_{\mu_1\mu_3}g_{\mu_2\mu_4}T_A^{\mu_1\mu_2\mu_3\mu_4}$$
$$-2g_{\mu_1\mu_4}g_{\mu_2\mu_3}T_B^{\mu_1\mu_2\mu_3\mu_4}-2g_{\mu_1\mu_2}g_{\mu_3\mu_4}T_B^{\mu_1\mu_2\mu_3\mu_4}+g_{\mu_1\mu_3}g_{\mu_2\mu_4}T_B^{\mu_1\mu_2\mu_3\mu_4}],$$

$$\mathcal{K}_{15}=\frac{i}{4(N_f^2-1)}\int d^4xx^\mu\left[\left\langle\bar{\psi}^a(0)\psi^b(0)\tan\frac{\vartheta'(0)}{N_f}\bar{\psi}^b(x)\gamma_\mu\gamma_5\psi^a(x)\right\rangle_C-\frac{1}{N_f}\left\langle\bar{\psi}^a(0)\psi^a(0)\tan\frac{\vartheta'(0)}{N_f}\bar{\psi}^b(x)\gamma_\mu\gamma_5\psi^b(x)\right\rangle_C\right] \tag{57}$$

with T_A, T_B defined as

$$-\frac{1}{2}\int d^4xd^4yx^{\mu_4}\langle\bar{\psi}^{a_1}(0)\gamma^{\mu_1}\psi^{b_1}(0)$$
$$\times\bar{\psi}^{a_2}(x)\gamma^{\mu_2}\gamma_5\psi^{b_2}(x)\bar{\psi}^{a_3}(y)\gamma^{\mu_3}\gamma_5\psi^{b_3}(y)\rangle_C$$
$$=\delta^{a_1b_2}\delta^{a_2b_3}\delta^{a_3b_1}T_A^{\mu_1\mu_2\mu_3\mu_4}$$
$$+\delta^{a_1b_3}\delta^{a_3b_2}\delta^{a_2b_1}T_B^{\mu_1\mu_2\mu_3\mu_4}+\text{irrelevant terms}. \tag{58}$$

In Eqs. (57) and (58), $\langle\cdots\rangle_C$ denotes the connected part of $\langle\cdots\rangle$, and the irrelavent terms are those leading to $\mathrm{tr}_f[a_\Omega^\mu]$ or $\mathrm{tr}_f[v_\Omega^\mu]$ after multiplied by the corresponding sources.

To further evaluate the rotated source parts in Eq. (55), we make use of the p^2-order equation of motion

$$d_\mu a_\Omega^\mu=-B_0\left[p_\Omega-\frac{1}{N_f}\mathrm{tr}_f(p_\Omega)\right] \tag{59}$$

and the following identities:

$$d^\mu a_\Omega^\nu-d^\nu a_\Omega^\mu=\frac{1}{2}[\Omega^\dagger F_R^{\mu\nu}\Omega-\Omega F_L^{\mu\nu}\Omega^\dagger],$$

$$a_\Omega^\mu=\frac{i}{2}\Omega^\dagger[\nabla^\mu U]\Omega^\dagger(x),$$

$$s_\Omega=\frac{1}{2}[\Omega(s-ip)\Omega+\Omega^\dagger(s+ip)\Omega^\dagger],$$

$$p_\Omega=\frac{i}{2}[\Omega(s-ip)\Omega-\Omega^\dagger(s+ip)\Omega^\dagger],$$

$$V_\Omega^{\mu\nu}=\frac{1}{2}[\Omega^\dagger F_R^{\mu\nu}\Omega+\Omega F_L^{\mu\nu}\Omega^\dagger]$$
$$+\frac{i}{4}\Omega^\dagger[-(\nabla^\mu U)U^\dagger(\nabla^\nu U)$$
$$+(\nabla^\nu U)U^\dagger(\nabla^\mu U)]\Omega^\dagger, \tag{60}$$

where $F_R^{\mu\nu}$ and $F_L^{\mu\nu}$ are, respectively, the field-strength tensors of the right-handed and left-handed souces defined in Ref. [2]. With Eqs. (59) and (60) and taking $N_f=3$, Eq. (55) becomes

$$
\begin{aligned}
&-S_{\rm eff}[1,J_\Omega,\Xi_c,\Phi_{\Omega c},\Pi_{\Omega c}]\big|_{p^4\ \text{order, normal}} \\
&=\int d^4x[L_1^{(\rm norm)}[{\rm tr}_f(\nabla^\mu U^\dagger\nabla_\mu U)]^2+L_2^{(\rm norm)}{\rm tr}_f[\nabla_\mu U^\dagger\nabla_\nu U]{\rm tr}_f[\nabla^\mu U^\dagger\nabla^\nu U]+L_3^{(\rm norm)}{\rm tr}_f[(\nabla^\mu U^\dagger\nabla_\mu U)^2] \\
&\quad+L_4^{(\rm norm)}{\rm tr}_f[\nabla^\mu U^\dagger\nabla_\mu U]{\rm tr}_f[\chi^\dagger U+\chi U^\dagger]+L_5^{(\rm norm)}{\rm tr}_f[\nabla^\mu U^\dagger\nabla_\mu U(\chi^\dagger U+U^\dagger\chi)]+L_6^{(\rm norm)}[{\rm tr}_f(\chi^\dagger U+\chi U^\dagger)]^2 \\
&\quad+L_7^{(\rm norm)}[{\rm tr}_f(\chi^\dagger U-\chi U^\dagger)]^2+L_8^{(\rm norm)}{\rm tr}_f[\chi^\dagger U\chi^\dagger U+\chi U^\dagger\chi U^\dagger]-iL_9^{(\rm norm)}{\rm tr}_f[F_{\mu\nu}^R\nabla^\mu U\nabla^\nu U^\dagger+F_{\mu\nu}^L\nabla^\mu U^\dagger\nabla^\nu U] \\
&\quad+L_{10}^{(\rm norm)}{\rm tr}_f[U^\dagger F_{\mu\nu}^R UF^{L,\mu\nu}]+H_1^{(\rm norm)}{\rm tr}_f[F_{\mu\nu}^R F^{R,\mu\nu}+F_{\mu\nu}^L F^{L,\mu\nu}]+H_2^{(\rm norm)}{\rm tr}_f[\chi^\dagger\chi]], \qquad (61)
\end{aligned}
$$

where $\chi\equiv 2B_0(s+ip)$.[4] The integrand in Eq. (61) is just the normal part contributions to the p^4-order terms in the chiral Lagrangian in Ref. [2], and the coefficients are now defined by

$$
\begin{aligned}
L_1^{(\rm norm)}&=\frac{1}{32}\mathcal{K}_4+\frac{1}{16}\mathcal{K}_5+\frac{1}{16}\mathcal{K}_{13}-\frac{1}{32}\mathcal{K}_{14}, \\
L_2^{(\rm norm)}&=\frac{1}{16}(\mathcal{K}_4+\mathcal{K}_6)+\frac{1}{8}\mathcal{K}_{13}-\frac{1}{16}\mathcal{K}_{14}, \\
L_3^{(\rm norm)}&=\frac{1}{16}(\mathcal{K}_3-2\mathcal{K}_4-6\mathcal{K}_{13}+3\mathcal{K}_{14}), \\
L_4^{(\rm norm)}&=\frac{\mathcal{K}_{12}}{16B_0}, \\
L_5^{(\rm norm)}&=\frac{\mathcal{K}_{11}}{16B_0}, \\
L_6^{(\rm norm)}&=\frac{\mathcal{K}_8}{16B_0^2}, \\
L_7^{(\rm norm)}&=-\frac{\mathcal{K}_1}{16N_f}-\frac{\mathcal{K}_{10}}{16B_0^2}-\frac{\mathcal{K}_{15}}{16B_0N_f}, \\
L_8^{(\rm norm)}&=\frac{1}{16}\left[\mathcal{K}_1+\frac{1}{B_0^2}\mathcal{K}_7-\frac{1}{B_0^2}\mathcal{K}_9+\frac{1}{B_0}\mathcal{K}_{15}\right], \\
L_9^{(\rm norm)}&=\frac{1}{8}(4\mathcal{K}_{13}-\mathcal{K}_{14}), \\
L_{10}^{(\rm norm)}&=\frac{1}{2}(\mathcal{K}_2-\mathcal{K}_{13}), \\
H_1^{(\rm norm)}&=-\frac{1}{4}(\mathcal{K}_2+\mathcal{K}_{13}), \\
H_2^{(\rm norm)}&=\frac{1}{8}\left[-\mathcal{K}_1+\frac{1}{B_0^2}\mathcal{K}_7+\frac{1}{B_0^2}\mathcal{K}_9-\frac{1}{B_0}\mathcal{K}_{15}\right]. \qquad (62)
\end{aligned}
$$

The twelve standard coefficients $L_1^{(\rm norm)}$, $L_2^{(\rm norm)},\ldots,L_{10}^{(\rm norm)}$, $H_1^{(\rm norm)}$, $H_2^{(\rm norm)}$ are expressed in terms of twelve independent p^4-order coefficients $\mathcal{K}_2$, $\mathcal{K}_{3,4}\equiv\mathcal{K}_3-2\mathcal{K}_4$, $\mathcal{K}_{4,5}\equiv\mathcal{K}_4+2\mathcal{K}_5$, $\mathcal{K}_{4,6}\equiv\mathcal{K}_4+\mathcal{K}_6$, $\mathcal{K}_7$, $\mathcal{K}_8$, $\mathcal{K}_{1,9,15}\equiv\mathcal{K}_1-(1/B_0^2)\mathcal{K}_9+(1/B_0)\mathcal{K}_{15}$, $\mathcal{K}_{1,10,15}\equiv\mathcal{K}_1+(N_f/B_0^2)\mathcal{K}_{10}+(1/B_0)\mathcal{K}_{15}$, $\mathcal{K}_{11}$, $\mathcal{K}_{12}$, $\mathcal{K}_{13}$, and $\mathcal{K}_{14}$.

The total coefficients are then

$$
\begin{aligned}
L_i&=L_i^{(\rm norm)}+L_i^{(\rm anom)}, \quad i=1,\ldots,10, \\
H_j&=H_j^{(\rm norm)}+H_j^{(\rm anom)}, \quad j=1,2, \qquad (63)
\end{aligned}
$$

where $L_i^{(\rm anom)}$ and $H_j^{(\rm anom)}$ are the anomaly contributions to the coefficients given in Refs. [9,10].

So we have formally derived the p^4-order terms of the chiral Lagrangian from the fundamental principles of QCD without taking approximations and have expressed all the coefficients in terms of the integrations of certain Green's functions in QCD. Equations (57), (58), and (62) give the fundamental QCD definitions of the the twelve coefficients $L_1^{(\rm norm)}\cdots L_{10}^{(\rm norm)}$, $H_1^{(\rm norm)}$, and $H_2^{(\rm norm)}$. The procedure can be carried on order by order in the momentum expansion. The expressions (50), (53), (57), (58), and (62) are convenient for lattice QCD calculations of the fifteen coefficients.

V. ON THE COEFFICIENTS F_0^2 AND B_0

So far, we have given the formal QCD definitions of the fourteen coefficients of the chiral Lagrangian up to p^4 order. To get the values of the coefficients, we need to solve the relevant Green's functions which is a hard task, and we shall present the calculations in a separate paper [12]. To have an idea of how our present formulas are related to other known approximate results, we take the p^2-order coefficients F_0^2 and B_0 [Eqs. (50) and (53)] as examples and make the following simple discussion.

[4] In Ref. [2], χ is defined as $\chi\equiv 2B_0(s+ip)e^{(i/3)\theta}$. In this paper we have taken $\theta=0$.

As we have mentioned in Sec. IV, there can be two ways of figuring out the explicit J_Ω dependence of $\overline{\Phi_{\Omega c}}$ for evaluating $S_{\rm eff}$ from Eq. (39). One of them is to solve the dynamical equations for the intermediate fields $\Phi_{\Omega c}$, $\Pi_{\Omega c}$, and Ξ_c, and the other is to track back to the original QCD generating functional without the intermediate fields. For convenience, we took the latter way in the above derivation of the chiral Lagrangian. To compare our results with the known approximate results, we are going to take certain approximations, say the large N_c limit, with which the calculation of the intermediate fields becomes even more convenient. Thus we take the former way in the following discussion.

First we take the large N_c limit. It can be easily checked that, in this limit, the functional integrations in Eqs. (11), (24), and (28) can be simply carried out by the saddle point approximation (taking the classical orbit in the semiclassical approximation). The saddle point equations (10), (26), and (30) are just the dynamical equations determining $\Pi_{\Omega c}, \Phi_{\Omega c}, \Xi_c$ as functions of J_Ω, which are

$$\Phi_{\Omega c}^{(a\eta)(b\zeta)}(x,y) = -i[(i\not\partial + J_\Omega - \Pi_{\Omega c})^{-1}]^{(b\zeta)(a\eta)}(y,x),$$

$$\widetilde{\Xi}^{\sigma\rho}(x)\,\delta(x-y) + \Pi_{\Omega c}^{\sigma\rho}(x,y) + \sum_{n=1}^{\infty} \int d^4x_1 \cdots d^4x_n d^4x_1' \cdots d^4x_n' \frac{(-i)^{n+1}(N_c g^2)^n}{n!} \times \overline{G}^{\sigma\sigma_1\cdots\sigma_n}_{\rho\rho_1\cdots\rho_n}(x,y,x_1,x_1',\ldots,x_n,x_n') \Phi_{\Omega c}^{\sigma_1\rho_1}(x_1,x_1')\cdots \times \Phi_{\Omega c}^{\sigma_n\rho_n}(x_n,x_n') = 0,$$

$$\mathrm{tr}_l\left[\left(-i\sin\frac{\vartheta(x)}{N_f} + \gamma_5\cos\frac{\vartheta(x)}{N_f}\right)\Phi_{\Omega c}^T(x,x)\right] = 0, \quad (64)$$

where $\widetilde{\Xi}$ is a short notation for the following quantity:

$$\widetilde{\Xi}^{\sigma\rho}(x) \equiv \frac{\partial}{\partial \Phi_c^{\sigma\rho}(x,x)} \int d^4y\, \mathrm{tr}_{lf}\left\{\Xi_c(y)\left[-i\sin\frac{\vartheta_c(y)}{N_f} + \gamma_5\cos\frac{\vartheta_c(y)}{N_f}\right]\Phi_{\Omega,c}^T(y,y)\right\}\Big|_{\Xi_c\,\text{fixed}}. \quad (65)$$

In Eqs. (64) and (65), $\vartheta_c(x)$ depends on $\Phi(x,x)$ through Eq. (17). Note that the effective action $\Gamma_l[\Phi_\Omega]$ belongs to $O(1/N_c)$, so that it does not contribute in the present approximation. In (64), the field $\Pi_{\Omega c}$ can be easily eliminated and the resulting equation is

$$[i\not\partial + i\Phi_{\Omega c}^{T,-1} + \not v_\Omega + \not a_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5 + \widetilde{\Xi}]^{\sigma\rho}(x,y) + \sum_{n=1}^{\infty}\int d^4x_1\cdots d^4x_n d^4x_1'\cdots d^4x_n' \times \frac{(-i)^{n+1}(N_c g^2)^n}{n!}\overline{G}^{\sigma\sigma_1\cdots\sigma_n}_{\rho\rho_1\cdots\rho_n}(x,y,x_1,x_1',\cdots,x_n,x_n') \times \Phi_{\Omega c}^{\sigma_1\rho_1}(x_1,x_1')\cdots\Phi_{\Omega c}^{\sigma_n\rho_n}(x_n,x_n') = 0. \quad (66)$$

In order to compare our results with the usual dynamical equations in the ladder approximation, we further take the ladder approximation which, in the present case, corresponds to ignoring all the $n>1$ terms in Eq. (66) and with

$$\overline{G}^{\sigma_1\sigma_2}_{\rho_1\rho_2}(x_1,x_1',x_2,x_2') = -\frac{1}{2}G_{\mu_1\mu_2}(x_1,x_2)(\gamma^{\mu_1})_{\sigma_1\rho_2}(\gamma^{\mu_2})_{\sigma_2\rho_1} \times \delta(x_1'-x_2)\,\delta(x_2'-x_1) + O\left(\frac{1}{N_c}\right)\text{term},$$

where $G_{\mu\nu}(x,y)$ is the gluon propagator without internal light-quark lines. Then, in the ladder approximation, Eq. (66) becomes

$$[i\not\partial + i\Phi_{\Omega c}^{T,-1} + \not v_\Omega + \not a_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5 + \widetilde{\Xi}](x,y) + \frac{1}{2}g^2 N_c G_{\mu\nu}(x,y)\,\gamma^\mu\Phi_{\Omega c}^T(x,y)\,\gamma^\nu = 0. \quad (67)$$

On the other hand, in the large N_c limit, $\overline{\Phi_{\Omega c}}$ is just $\Phi_{\Omega c}$ which is the full physical propagator of the quark with the rotated sources. When the sources are turned off, $\Phi_{\Omega c}$ can be expressed in terms of the quark self-energy $\Sigma(-p^2)$ and the wave function renormalization $Z(-p^2)$ by the standard expression

$$\Phi_0^{T(a\eta)(b\zeta)}(x,y) \equiv [\Phi_{\Omega c}^T]^{(a\eta)(b\zeta)}(x,y)\big|_{s_\Omega = p_\Omega = v_\Omega^\mu = a_\Omega^\mu = 0} = \delta^{ab}\int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\left[\frac{-i}{Z(-p^2)\not p - \Sigma(-p^2)}\right]^{\eta\zeta}, \quad (68)$$

in which translational invariance and the flavor and parity conservations have been considered. Plugging Eq. (68) into Eq. (67) we have

$$-[Z(-p^2)-1]\not{p}+\Sigma(-p^2)+\frac{1}{2}iN_c g^2\int\frac{d^4q}{(2\pi)^4}$$

$$\times G_{\mu\nu}(p-q)\gamma^\mu\frac{1}{Z(-q^2)\not{q}-\Sigma(-q^2)}\gamma^\nu=0, \quad (69)$$

where $G_{\mu\nu}(p)$ is the gluon propagator in the momentum representation, and the fact $\tilde{\Xi}|_{\text{sources}=0}=0$ is taken into account. Equation (69) is just the usual Schwinger-Dyson equation in the ladder approximation.

With the solution of the Schwinger-Dyson equation, the formula (50) for $F_0^2B_0$ can be expressed as

$$F_0^2B_0=4iN_c\int\frac{d^4p}{(2\pi)^4}\frac{\Sigma(-p^2)}{Z^2(-p^2)p^2-\Sigma^2(-p^2)}. \quad (70)$$

By definition [see Eqs. (46) and (52)], F_0^2 is related to the coefficient of the term linear in a_Ω in the expansion of $\bar{\Phi}_{\Omega c}$. We denote

$$\int d^4z\Phi_{1,\mu}^{T\eta\zeta}\left(\frac{x+y}{2}-z,x-y\right)a_\Omega^{ab\mu}(z)$$

$$\equiv[\Phi_{\Omega c}^T]^{(a\eta)(b\zeta)}(x,y)|_{\text{linear in } a_\Omega^\mu}$$

$$=\int d^4z\frac{d^4pd^4q}{(2\pi)^8}e^{-ip[(x+y)/2-z]-iq(x-y)}$$

$$\times\Phi_{1,\mu}^{T\eta\zeta}(p,q)a_\Omega^{ab\mu}(z). \quad (71)$$

Then F_0^2 is determined by[5]

$$F_0^2=\frac{N_c}{8}(\gamma^\mu\gamma_5)^{\eta\zeta}\int\frac{d^4q}{(2\pi)^4}\Phi_{1,\mu}^{T\zeta\eta}(0,q). \quad (72)$$

To this order, we still have $\tilde{\Xi}|_{\text{linear in } a_\Omega^\mu}=0$, and Eq. (67) reduces to

$$-i\left[\Phi_0^T\left(q+\frac{p}{2}\right)\right]^{-1}\Phi_{1,\mu}^T(p,q)\left[\Phi_0^T\left(q-\frac{p}{2}\right)\right]^{-1}+\gamma_\mu\gamma_5$$

$$+\frac{1}{2}g^2N_c\int\frac{d^4k}{(2\pi)^4}G_{\mu'\nu'}(k-q)\gamma^{\mu'}\Phi_{1,\mu}^T(p,k)\gamma^{\nu'}=0. \quad (73)$$

In the literature, a further approximation of dropping the last term in Eq. (73) is usually taken [18] (It can be shown that to leading order in dynamical perturbation [11], this term can be reasonably ignored [12].) Moreover, to leading order in dynamical perturbation or in the Landau gauge $Z(-p^2)=1$. Then F_0^2 becomes

$$F_0^2=\frac{iN_c}{8}\int\frac{d^4q}{(2\pi)^4}\text{tr}_l\left[\gamma^\mu\gamma_5\frac{1}{\not{q}-\Sigma(-q^2)}\gamma_\mu\gamma_5\frac{1}{\not{q}-\Sigma(-q^2)}\right]$$

$$=-4iN_c\int\frac{d^4q}{(2\pi)^4}\left[\frac{1}{8}\frac{\partial}{\partial q^\mu}\left(\frac{q^\mu}{q^2-\Sigma^2(-q^2)}\right)\right.$$

$$\left.+\frac{\left[\Sigma(-q^2)+\frac{1}{2}q^2\Sigma'(-q^2)\right]\Sigma(-q^2)}{[q^2-\Sigma^2(-q^2)]^2}\right]. \quad (74)$$

The anomaly contribution to F_0^2 calculated in Ref. [10] is of the same form as the first term in Eq. (74) but with an opposite sign, so that it just cancel the first term in Eq. (74). Then Eq. (74) is just the well-known Pagels-Stokar formula for F_0^2 [11]. Thus the Pagels-Stokar formula is an approximate result of our formula by taking the approximations of the large N_c limit, the ladder approximation and dropping the last term in Eq. (73) (or to leading order in dynamical perturbation).

VI. CONCLUSIONS

In this paper, we have derived the normal part contributions to the chiral Lagrangian for pseudoscalar mesons up to the p^4 terms from the fundamental principles of QCD without taking approximations. Together with the anomaly part contributions given in Refs. [9,10], it leads to the complete QCD theory of the chiral Lagrangian.

We started, in Sec. II, from the fundamental generating functional (2) in QCD, and formally expressed the integration over the gluon-field in terms of physical gluon Green's functions. Then we integrated out the quark fields by introducing a bilocal auxiliary field $\Phi(x,y)$ [see Eq. (6)]. To extract the degree of freedom of the local pseudoscalar-meson field $U(x)$, we developed, in Sec. III, a technique for *extracting* it from the bilocal auxiliary field $\Phi(x,y)$ [see Eqs. (13) and (15)], and *integrating in* the extraction constraint into the generating functional [see Eqs. (18), (20), and (22)]. This procedure is *consistent* in the sense that the complete pseudoscalar meson degree of freedom is converted into the $U(x)$ field such that the pseudoscalar degree of freedom in the quark sector is automatically frozen in the path-integral formulation of the effective action S_{eff}.

We then developed two techniques for working out the explicit $U(x)$ dependence of S_{eff} in Sec. IV. The first one is to introduce a chiral rotation (33) which simplifies the $U(x)$ dependence in such a way that the $U(x)$ dependence resides only implicitly in the rotated sources and some rotated intermediate fields, and the second one is to implement Eq. (39) to obtain S_{eff} from the averaged field $\bar{\Phi}_{\Omega c}$. To avoid dealing

[5]This is an alternative expression for F_0^2 equivalent to Eq. (53).

with the intermediate fields, we tracked back to the original QCD generating functional with which the implicit $U(x)$ dependence only resides in the rotated sources. With all these, we expanded $S_{\rm eff}$ in power series of the rotated sources and explicitly derived the p^2 terms and p^4 terms of the chiral Lagrangian for pseudoscalar mesons [2]. In this formulation, all the fifteen coefficients in the chiral Lagrangian are expressed in terms of certain Green's functions in QCD [see Eqs. (50), (53), (62), and (57)]. These formulas can be regarded as the fundamental QCD definitions of the fifteen coefficients in the chiral Lagrangian. These expressions are convenient for lattice QCD calculation of the fifteen coefficients.

To see the relation between our QCD definition and the well-known approximate results in the literature, we took the p^2-order coefficients F_0^2 and B_0 as examples in Sec. V. With the approximations of large N_c limit, ladder approximation, and dropping the momentum-integration term in the approximate Bethe-Salpeter equation (73), our formula reduces to the well-known Pagels-Stokar formula for the pion decay constant F_π [11]. The complete calculation of the fifteen coefficients will be presented in a separate paper [12]. The derivation of the effective chiral Lagrangian further including the η' meson or the ρ meson, and the application of the present approach to the electroweak chiral Lagrangian are all in preparation.

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APPENDIX

Here we give the proof of Eqs. (18) and (19) in the text. Consider a matrix operator $\mathcal{O}$ satisfying $\det\mathcal{O}=\det\mathcal{O}^\dagger$, we calculate the following functional integration:

$$I\equiv\int \mathcal{D}U\,\delta(U^\dagger U-1)\,\delta(\det U-1)\times\mathcal{F}[\mathcal{O}]\,\delta(\Omega\mathcal{O}^\dagger\Omega-\Omega^\dagger\mathcal{O}\Omega^\dagger),$$

where $\mathcal{D}U\,\delta(U^\dagger U-1)\,\delta(\det U-1)$ serves as the invariant integration measure at the present case. The two delta-functions $\delta(U^\dagger U-1)$, $\delta(\det U-1)$ constrain the integration to the subspace with unitary and unity determinant of the U field.

We can rewrite $\delta(\det U-1)\,\delta(U^\dagger U-1)$ as

$$\begin{aligned}&\delta(\det U-1)\,\delta(U^\dagger U-1)\\&=\frac{1}{2}\,\delta([\det U]^2-1)\,\theta(\det U)\,\delta(U^\dagger U-1)\\&=\frac{1}{2}\,\delta(\det U[\det U-\det U^\dagger])\,\theta(\det U)\,\delta(U^\dagger U-1)\\&=\frac{1}{2}\,\delta(\det U-\det U^\dagger)\frac{\theta(\det U)}{\det U}\,\delta(U^\dagger U-1),\end{aligned}$$

so that

$$I=\frac{1}{2}\int \mathcal{D}U\,\delta(U^\dagger U-1)\,\delta(\det U-\det U^\dagger)\times\frac{\theta(\det U)}{\det U}\mathcal{F}[\mathcal{O}]\,\delta(\Omega\mathcal{O}^\dagger\Omega-\Omega^\dagger\mathcal{O}\Omega^\dagger).$$

Next, we introduce two auxiliary fields Σ and $\tilde{\Sigma}$ and write I as

$$\begin{aligned}I=&\frac{1}{2}\int \mathcal{D}U\mathcal{D}\Sigma\mathcal{D}\tilde{\Sigma}\,\delta(U^\dagger U-1)\,\delta(\det U-\det U^\dagger)\\&\times\frac{\theta(\det U)}{\det U}\,\delta(\tilde{\Sigma}-\Sigma)\mathcal{F}[\mathcal{O}]\,\delta(\Sigma-\Omega^\dagger\mathcal{O}\Omega^\dagger)\\&\times\delta(\tilde{\Sigma}-\Omega\mathcal{O}^\dagger\Omega)\\=&\frac{1}{2}\int \mathcal{D}U\mathcal{D}[\Omega^\dagger\Sigma\Omega]\mathcal{D}[\Omega^\dagger\tilde{\Sigma}\Omega]\,\delta(U^\dagger U-1)\\&\times\delta(\det U-\det U^\dagger)\frac{\theta(\det U)}{\det U}\mathcal{F}[\mathcal{O}]\\&\times\delta(\Omega^\dagger[\tilde{\Sigma}-\Sigma]\Omega)\,\delta(\Omega^\dagger\Sigma\Omega-U^\dagger\mathcal{O})\\&\times\delta(\Omega^\dagger\tilde{\Sigma}\Omega-\mathcal{O}^\dagger U).\end{aligned}$$

We then change the integration variables Σ and $\tilde{\Sigma}$ into

$$\Sigma\to\Sigma'=\Omega^\dagger\Sigma\Omega\tilde{\Sigma}\to\tilde{\Sigma}'=\Omega^\dagger\tilde{\Sigma}\Omega \tag{A1}$$

and get

$$
\begin{aligned}
I &= \frac{1}{2}\int \mathcal{D}U\mathcal{D}\Sigma'\mathcal{D}\tilde{\Sigma}'\,\delta(U^\dagger U-1)\,\delta(\det U-\det U^\dagger)\frac{\theta(\det U)}{\det U}\,\delta(\tilde{\Sigma}'-\Sigma')\,\mathcal{F}[\mathcal{O}]\,\delta(\Sigma'-U^\dagger\mathcal{O})\,\delta(\tilde{\Sigma}'-\mathcal{O}^\dagger U)\\
&= \frac{1}{2}\int \mathcal{D}U\mathcal{D}\Sigma'\mathcal{D}\tilde{\Sigma}'\,\delta(U^\dagger U-1)\,\delta(\det U-\det U^\dagger)\frac{\theta(\det U)}{[\det U]^2}\,\delta(\tilde{\Sigma}'-\Sigma')\,\mathcal{F}[\mathcal{O}]\,\delta(U\Sigma'-\mathcal{O})\,\delta(\tilde{\Sigma}'-\Sigma'^\dagger)\\
&= \frac{1}{2}\int \mathcal{D}U\mathcal{D}\Sigma'\mathcal{D}\tilde{\Sigma}'\det(\Sigma'^5)\,\delta(\tilde{\Sigma}'[U^\dagger U-1]\Sigma')\,\delta(\det[\tilde{\Sigma}'U]-\det[\tilde{\Sigma}'U^\dagger])\frac{\theta(\det U)}{[\det(U\Sigma')]^2}\,\delta(\tilde{\Sigma}'-\Sigma')\\
&\quad\times\mathcal{F}[\mathcal{O}]\,\delta(U\Sigma'-\mathcal{O})\,\delta(\tilde{\Sigma}'-\Sigma'^\dagger)\\
&= \frac{1}{2}\int \mathcal{D}[U\Sigma']\mathcal{D}\Sigma'\mathcal{D}\tilde{\Sigma}'\det(\Sigma'^4)\,\delta(\Sigma'^\dagger U^\dagger U\Sigma'-\Sigma'^\dagger\Sigma')\times\delta(\det[U\Sigma']-\det[\Sigma'^\dagger U^\dagger])\frac{\theta(\det U)}{[\det(U\Sigma')]^2}\\
&\quad\times\delta(\tilde{\Sigma}'-\Sigma')\,\mathcal{F}[\mathcal{O}]\,\delta(U\Sigma'-\mathcal{O})\,\delta(\tilde{\Sigma}'-\Sigma'^\dagger).
\end{aligned}
$$

We further change the integration variables U into

$$U\to U'=U\Sigma', \tag{A2}$$

and get

$$
\begin{aligned}
I &= \frac{1}{2}\int \mathcal{D}U'\mathcal{D}\Sigma'\mathcal{D}\tilde{\Sigma}'\,\delta(U'^\dagger U'-\Sigma'^\dagger\Sigma')\,\delta(\det U'-\det U'^\dagger)\frac{\det(\Sigma'^4)\,\theta(\det U'/\det\Sigma')}{[\det(U')]^2}\\
&\quad\times\delta(\tilde{\Sigma}'-\Sigma')\,\mathcal{F}[\mathcal{O}]\,\delta(U'-\mathcal{O})\,\delta(\tilde{\Sigma}'-\Sigma'^\dagger).
\end{aligned}
$$

The U' and $\tilde{\Sigma}'$ integrations can be carried out and we obtain

$$
\begin{aligned}
I &= \frac{1}{2[\det\mathcal{O}]^2}\,\delta(\det\mathcal{O}-\det\mathcal{O}^\dagger)\int \mathcal{D}\Sigma'\,\delta(\mathcal{O}^\dagger\mathcal{O}-\Sigma'^\dagger\Sigma')\det(\Sigma'^4)\,\theta\left(\frac{\det\mathcal{O}}{\det\Sigma'}\right)\mathcal{F}[\mathcal{O}]\,\delta(\Sigma'-\Sigma'^\dagger)\\
&= \frac{1}{2[\det\mathcal{O}]^3}\,\delta\left(1-\frac{\det\mathcal{O}^\dagger}{\det\mathcal{O}}\right)\int \mathcal{D}\Sigma'\,\delta(\mathcal{O}^\dagger\mathcal{O}-\Sigma'^\dagger\Sigma')[\det(\Sigma'^\dagger\Sigma')]^2\,\theta\left(\frac{\det\mathcal{O}}{\sqrt{\det[\Sigma'^\dagger\Sigma']}}\right)\mathcal{F}[\mathcal{O}]\,\delta(\Sigma'-\Sigma'^\dagger)\\
&= \frac{1}{2[\det\mathcal{O}]^2}\,\delta\left(1-\frac{\det\mathcal{O}^\dagger}{\det\mathcal{O}}\right)\int \mathcal{D}\Sigma'\,\delta(\mathcal{O}^\dagger\mathcal{O}-\Sigma'^\dagger\Sigma')[\det(\mathcal{O}^\dagger\mathcal{O})]^2\,\theta\left(\frac{\det\mathcal{O}}{\sqrt{\det[\mathcal{O}^\dagger\mathcal{O}]}}\right)\mathcal{F}[\mathcal{O}]\,\delta(\Sigma'-\Sigma'^\dagger)\\
&= \frac{1}{2}\det\mathcal{O}\,\delta\left(1-\frac{\det\mathcal{O}^\dagger}{\det\mathcal{O}}\right)\mathcal{F}[\mathcal{O}]\int \mathcal{D}\Sigma'\,\delta(\mathcal{O}^\dagger\mathcal{O}-\Sigma'^\dagger\Sigma')\,\delta(\Sigma'-\Sigma'^\dagger)\\
&= \frac{1}{2}\det\mathcal{O}\,\delta(0)\,\mathcal{F}[\mathcal{O}]\int \mathcal{D}\Sigma'\,\delta(\mathcal{O}^\dagger\mathcal{O}-\Sigma'^\dagger\Sigma')\,\delta(\Sigma'-\Sigma'^\dagger).
\end{aligned}
\tag{A3}
$$

In the last step, we have used the property $\det\mathcal{O}=\det\mathcal{O}^\dagger$. Taking $\mathcal{F}[\mathcal{O}]$ to be

$$\frac{1}{\mathcal{F}[\mathcal{O}]}\equiv\det\mathcal{O}\int \mathcal{D}\Sigma'\,\delta(\mathcal{O}^\dagger\mathcal{O}-\Sigma'^\dagger\Sigma')\,\delta(\Sigma'-\Sigma'^\dagger), \tag{A4}$$

Eq. (A3) becomes

$$\int \mathcal{D}U\,\delta(U^\dagger U-1)\,\delta(\det U-1)\times\mathcal{F}[\mathcal{O}]\,\delta(\Omega\mathcal{O}^\dagger\Omega-\Omega^\dagger\mathcal{O}\Omega^\dagger)=\text{const}, \tag{A5}$$

which is of the form of Eq. (18) in the text.

Next we look at the meaning of the variable Σ' in Eq. (A4). The constraints on Σ' in Eq. (A4) are

$$\Sigma'^{\dagger}=\Sigma', \quad \Sigma'^{2}=\mathcal{O}^{\dagger}\mathcal{O}. \tag{A6}$$

On the other hand, Eqs. (13), (14), and (20) in the text show that the σ field is constrained as

$$\sigma^{\dagger}=\sigma, \quad (\Omega^{\dagger}\sigma\Omega)^{2}=\mathcal{O}^{\dagger}\mathcal{O}. \tag{A7}$$

Comparing Eq. (A6) with Eq. (A7), we find $\Sigma' \sim \Omega^{\dagger}\sigma\Omega$. We know that the definition of σ is not unique. It is up to a hidden symmetry transformation $\sigma \rightarrow h^{\dagger}\sigma h$. Therefore Σ' can be regarded as an equivalent definition of σ, and thus Eq. (A4) can be written as

$$\frac{1}{\mathcal{F}[\mathcal{O}]}=\det\mathcal{O}\int \mathcal{D}\sigma\,\delta(\mathcal{O}^{\dagger}\mathcal{O}-\sigma^{\dagger}\sigma)\,\delta(\sigma-\sigma^{\dagger}), \tag{A8}$$

which is just Eq. (19) in the text.

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