Analytic Q ball solutions in a parabolic-type potential

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We introduce a piecewise parabolic potential for a complex scalar field and we show that it admits stable Q ball solutions. These solutions can be found analytically, unlike the case of polynomial potentials. We find stable Q ball solutions, for large enough values of the charge, even when the potential has only one minimum. There can also exist Q balls immersed in a supercooled false vacuum.

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In some four-dimensional field theories where an unbroken continuous global symmetry gives rise to a conserved charge Q, there may appear extended localized solutions with nonzero charge, called Q balls [1]. These are nontopological solitons, and their stability depends on whether their charge can be lost through emission of charged particles. Their energies and volumes grow linearly with Q, they are spherically symmetric in position space, and the corresponding fields are rotating with constant angular speed in internal space. They often have no upper limit on their charge or mass, and they are important in cosmological considerations [2].

A wide variety of Q balls has been studied, the simplest ones being those in theories with one complex scalar field. At large distances from the soliton the field must approach the vacuum solution $\phi=0$, if the charge is to be finite. It turns out that Q balls exist in a potential $V(\phi)$, for large enough values of the charge Q, if the function $2V/|\phi|^2$ has a minimum at some nonzero value of ϕ [1]. Since the potential looks like $|\phi|^2/2$ near $\phi=0$, this means that V must dip below $|\phi|^2/2$ at some point. If the potential is a polynomial of $\phi^*\phi$, then a negative $|\phi|^4$ term is needed to make V dip below $|\phi|^2/2$, and a positive $|\phi|^6$ term to render the potential bounded from below. Such a potential will be nonrenormalizable, but it does have localized classical Q ball solutions.

Unfortunately, the polynomial potentials give nonlinear equations of motion, and solving them is a formidable task. It would be, however, very interesting to obtain analytic expressions describing classical localized solitons. These solitons rely usually on potentials with at least two vacua. The polynomial potentials that would be functions of $\phi^*\phi$, admitting thus a global symmetry, and that would have two minima are necessarily of a degree higher than 2 in $|\phi|$, resulting in highly nonlinear equations. There is however no reason, at least at the classical level, to restrict ourselves to polynomial potentials. In fact, we need not even restrict ourselves to potentials that are continuously differentiable everywhere. We could examine piecewise continuous potentials.

Indeed, this will provide us with a way to make the potential dip below $|\phi|^2/2$, producing thus stable Q balls. This paper will examine a piecewise continuous parabolic potential that does just that, and that is analytically solvable, since all its pieces will be at most quadratic in $|\phi|$.

This potential is

$$V(\phi)=\frac{1}{2l^2}[\rho^2+\epsilon v(v-\rho)-\epsilon v|v-\rho|], \tag{1}$$

with

$$\rho=\sqrt{\phi^*\phi}, \tag{2}$$

where l has dimensions of length and $0<\epsilon$. Piecewise parabolic potentials have also been used in the context of wetting and of oil-water-surfactant mixtures [3].

The resulting equations of motion are linear, albeit inhomogeneous, and as a result one can obtain exact closed form solutions. The qualitative features of these solutions are bound to be similar to the features of the corresponding solutions for polynomial potentials of a similar appearance. In fact, an examination of the analytic vortex solutions admitted by a symmetric piecewise parabolic double well potential showed that even the quantitative details are very similar to those obtained numerically in the case of ϕ^4 vortices [4].

The unusual potential of Eq. (1) is thus a convenient testing ground for calculations of localized solutions. We shall make use of this kinky potential in order to find analytic solutions for Q balls. Note that $2V/|\phi|^2$ is equal to $1/l^2$ for $\rho<v$, while it has a minimum value of $(2-\epsilon)/(2l^2)$ at $\rho=2v$. Thus the necessary condition for the creation of a Q ball is satisfied.

The action for our model is

$$S=\int d^4x\left[\frac{1}{2}\partial_\mu\phi^*\partial^\mu\phi-V\right]. \tag{3}$$

If we measure x, y, z, and t in units of l, having taken c to be 1, ϕ and ρ in units of v, V in units of v^2/l^2, and S in units of v^2l^2, then the above action becomes dimensionless, with

$$V(\rho)=\frac{1}{2}[\rho^2+\epsilon(1-\rho)-\epsilon|1-\rho|]. \tag{4}$$

If $0<\epsilon<1$, this potential will have only one minimum, at $\rho=0$. For $1<\epsilon<2$, the global minimum is at $\rho=0$, but there is a local minimum at $\rho=\epsilon$, the whole potential being positive everywhere. Finally, for $\epsilon>2$ the global minimum is at $\rho=\epsilon$, while there is a local minimum at $\rho=0$. The two minima become degenerate when $\epsilon=2$, and a first order phase transition takes place there.

The field equation that minimizes the action is

$$\frac{1}{2}\partial_\mu\partial^\mu\phi+\frac{\partial V}{\partial\phi^*}=0. \tag{5}$$

The fact that V is a function of $\phi^*\phi$ means that there is a U(1) global continuous symmetry and a corresponding conserved charge

$$Q=\int\frac{1}{2i}\left(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t}\right)d^3x. \tag{6}$$

The corresponding energy is

$$E=\int\left[\frac{1}{2}\left|\frac{\partial\phi}{\partial t}\right|^2+\frac{1}{2}\left|\nabla\phi\right|^2+V\right]d^3x. \tag{7}$$

The Q ball problem consists in minimizing the energy for a given fixed charge Q. We shall use the method of Lagrange multipliers. We shall thus want to minimize the functional

$$\mathcal{E}=E+\omega\left[Q-\frac{1}{2i}\int\left(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t}\right)d^3x\right], \tag{8}$$

with respect to independent variations of $\phi(\vec{x},t)$ and of ω.

We can rewrite $\mathcal{E}$ as

$$\mathcal{E}=Q\omega+\int\left[\frac{1}{2}\left|\frac{\partial\phi}{\partial t}-i\omega\phi\right|^2+\frac{1}{2}\left|\nabla\phi\right|^2+V-\frac{1}{2}\omega^2\phi^*\phi\right]d^3x. \tag{9}$$

The usual assumption leading to Q ball solutions is $\phi=\rho(r)e^{i\omega t}$. It is a consistent solution of the full field equation (5), which reduces to

$$-\rho\omega^2-\nabla^2\rho+\frac{\partial V}{\partial\rho}=0. \tag{10}$$

Then we have to find the extremum of $\mathcal{E}$, where $\mathcal{E}$ has been reduced to

$$\mathcal{E}=Q\omega+\int\left[V+\frac{1}{2}(\nabla\rho)^2-\frac{1}{2}\omega^2\rho^2\right]d^3x. \tag{11}$$

This can be minimized by varying $\rho(r)$ first, with ω kept fixed, and then minimizing with respect to ω. In fact, the variation with respect to $\rho(r)$ yields Eq. (10). It is this equation that we shall solve exactly. Before we undertake this task, though, we can obtain some very useful exact results by scaling the spatial dimensions.

Indeed, let $\rho(r)$ be the field that minimizes exactly the functional $\mathcal{E}$. The corresponding charge of Eq. (6) becomes $Q=\int\omega\rho^2\,d^3x$, with $Q\omega$ always positive. After expressing ω in terms of ρ and Q, we let $\mathcal{E}(\lambda)$ be the value of the functional $\mathcal{E}$ when the field is $\rho(\lambda r)$. Since $\rho(r)$ is the exact extremum of $\mathcal{E}$, $\mathcal{E}(\lambda)$ must have a minimum at $\lambda=1$. Thus

$$(d\mathcal{E}/d\lambda)|_{\lambda=1}=0=-\int\frac{1}{2}(\nabla\rho)^2\,d^3x-3\int V\,d^3x+\frac{3Q^2}{2\int\rho^2\,d^3x}. \tag{12}$$

This virial relation must be satisfied by the exact solution $\rho(r)$. Furthermore, the solution is stable only if $(d^2E/d\lambda^2)|_{\lambda=1}>0$, whence $E<4Q\omega$, a relation that is certainly true when $0<\epsilon<2$.

The real danger for Q balls, however, does not arise from deformations, but from the possible decay into particles. Indeed, the field $\phi=(A/\sqrt{\Omega})e^{-i\vec{k}\cdot\vec{x}+i\omega t}$, where Ω is the volume of space available, is an exact solution of the equation of motion (5) if $\omega=\sqrt{1+k^2}$. This is the free particle solution, with energy $E=Q\sqrt{1+k^2}$. Thus the Q ball will not decay into particles only if $E<Q$.

Let us now proceed to solve the field equation (10). If $\rho<1$, then

$$\frac{d^2\rho}{dr^2}+\frac{2}{r}\frac{d\rho}{dr}=\rho(1-\omega^2). \tag{13}$$

Outside the Q ball is the $\phi=0$ vacuum. Hence $\rho(\infty)=0$. The corresponding solutions of Eq. (13) are $r^{-1}\sin(r\sqrt{\omega^2-1})$ and $r^{-1}\cos(r\sqrt{\omega^2-1})$, if $\omega>1$. But in that case the integral that gives Q will diverge at infinity. So, if Q is to be finite, we must have $\omega\leqslant 1$:

$$1-\omega^2=\nu^2\geqslant 0. \tag{14}$$

Then ρ falls off exponentially at infinity, as it is supposed to. Let us assume that ρ is monotonic, with $\rho(R)=1$. Then the full solution of field equation (10) that is regular at $\rho=0$ and that falls off exponentially at infinity is

$$\begin{aligned}\rho(r)&=\frac{R}{r}e^{\nu(R-r)}\quad\text{if } r>R\ (\rho<1)\\&=\frac{\epsilon}{\nu^2}+\left(1-\frac{\epsilon}{\nu^2}\right)\frac{R}{r}\frac{\sinh\nu r}{\sinh\nu R}\ \text{if } r<R\ (\rho>1).\end{aligned} \tag{15}$$

The field equation (10) indicates that $\nabla^2\rho$ has a finite discontinuity at $r=R$; hence $d\rho/dr$ is continuous there. This continuity implies that

$$\nu^2/\epsilon=f\leqslant 1/2, \tag{16}$$

where

$$x=R\nu \tag{17}$$

and

$$f=\frac{1}{2}-\frac{1}{2x}+e^{-2x}\left(\frac{1}{2}+\frac{1}{2x}\right). \tag{18}$$

The function f varies monotonically between 0 and 1/2.

The mean value $\overline{\rho}$ of $\rho(r)$, defined by $(4\pi R^3/3)\overline{\rho} = \int_0^\infty 4\pi r^2 \rho(r) dr$, can be calculated by integrating directly Eq. (10), without using at all the explicit solution of Eq. (15), and turns out to be $\overline{\rho} = 1/f(x)$.

Equation (15) gives the full solution. The radius R, which gives the size of the Q ball, is determined in terms of ϵ and ω through Eqs. (14), (16), (17) and (18). Examination of the stability shows that the Q ball is stable when $\nu > \gamma$ and unstable when $\gamma > \sqrt{3\omega^2+1}$, where $\gamma R[1+\coth(\gamma R)] = x^2(x+1)^{-1} f^{-1}$.

Let us now embark on the main calculation, the calculation of $\mathcal{E}$ and Q. We have already found the exact solution $\rho(r)$ which satisfies the field equation (10). Use of this equation (10), and of Eq. (15), yields

$$\mathcal{E} = Q\omega + \int_0^R 4\pi r^2 \epsilon(-\rho/2+1) dr = Q\omega + \frac{4\pi\epsilon R^3}{3} - \frac{2\pi\epsilon^2 R^3}{3\nu^2} + \frac{2\pi R^2 \epsilon}{\nu} + \frac{2\pi\epsilon R}{\nu^2}. \quad (19)$$

We can rewrite this as

$$\mathcal{E} = Q\sqrt{1-\epsilon f} + 2\pi\epsilon^{-1/2} f^{-5/2}[2fx^3/3 - x^3/3 + fx^2 + fx]. \quad (20)$$

Thus $\mathcal{E}$ is now a function of Q and x only. Minimization of $\mathcal{E}$ with respect to ω is equivalent then to minimization with respect to x, at fixed Q, and we get thus

$$Q = \frac{4\pi\omega}{\nu^3 f^2}[5x^3/6 - fx^3 - 5fx^2/2 - 5fx/2)]. \quad (21)$$

These last two equations determine E and x, for a given Q and ϵ, and from x we can find ν, ω and R. We have thus solved completely the problem, and we have closed form expressions for all the physical quantities. Note that we could have obtained Q from the virial relation, Eq. (12), without having to do the tedious minimization of the expression of Eq. (20).

Let us now find E/Q in a compact form:

$$\frac{E}{Q} = \omega - \frac{\nu^2}{5\omega} + \frac{8\pi}{15}\frac{\epsilon R^3}{Q}. \quad (22)$$

This quantity must be less than 1 and 4ω, simultaneously, if the Q ball is going to be stable against decay and deformations.

The basic results of this work are Eqs. (22), (21), (18), (17), (16), (15) and (14). Thus, for a given Q, since $\nu = \sqrt{\epsilon f}$ and $\omega = \sqrt{1-\epsilon f}$, we can find x from Eq. (21), using it afterwards for finding ν, ω, $R = x/\nu$, and E/Q.

Let us now examine more closely the behavior of our solution in various limits. We look first at the limit $x \to 0$. In this limit, we can easily show that $f \to x^2/3$, $\nu \to x\sqrt{\epsilon/3}$, $\omega \to 1$, $R \to \sqrt{3/\epsilon}$. Furthermore, $Q \to 6\pi\sqrt{3}\epsilon^{-3/2}/x$ and

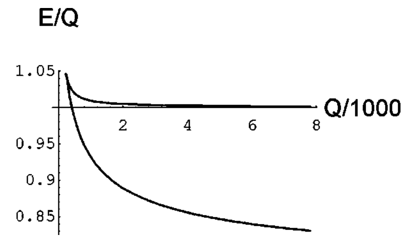

FIG. 1. The energy per unit charge, E/Q, versus $Q/1000$ for $\epsilon = 0.9$. The cusp is at $Q = 239.229$.

$$\frac{E}{Q} \to 1 + \frac{4\epsilon x}{15} - \frac{517\epsilon x^2}{1050}. \quad (23)$$

Thus Q diverges for small x. However, E/Q remains finite. We see in fact that it rises above 1 and then comes back down, having passed through a maximum. Thus E/Q will descend below 1 once x becomes large enough.

Let us now go on to distinguish two cases:

(i) $0 < \epsilon < 2$. In the limit $x \to \infty$ we get $f \to 1/2$, $\omega \to \sqrt{(2-\epsilon)/2}$, $\nu \to \sqrt{\epsilon/2}$, $\gamma \to \sqrt{\epsilon/2}$, $R \to \infty$, $Q \approx 32\pi\epsilon^{-3/2} x^3 \sqrt{2-\epsilon}/3$, $Q/(4\pi R^3/3) \to 4\sqrt{(2-\epsilon)/2}$ and $E/Q \to \sqrt{(2-\epsilon)/2}$.

We see thus that Q diverges when $x \to \infty$, as well as when $x \to 0$. In fact, as shown in Fig. 1, E/Q is a multivalued function of Q, with the cusp corresponding to an extremum of Q with respect to x. The upper branch corresponds to small values of x, while the lower branch corresponds to large values of x. Since E/Q has to be less than 1, it is the branch below the Q axis that represents the Q ball. The Q ball is stable, as long as its charge is above that critical charge where its energy becomes equal to Q.

It is worth noting that the mean value $\overline{\rho}$ of the field tends to 2 for large x, i.e. for large Q, and is thus independent of Q. The energy density and the charge density, as well as the mean value of the field, are all independent of Q in the high Q limit. We have, in other words, a lump of matter.

Q balls then with large enough charge are stable. There is, however, always a classical lower limit on the charge of a stable Q ball, as noted in [1], and at that value of Q we have $E/Q = 1$. The recent claims that there are Q balls with small charge concern polynomial theories, and do not hold for our potential [5]. Note that for $\epsilon < 1$ there is only one minimum, and yet we still get stable Q balls, unlike the case of the ϕ^6 potential.

It is instructive, in fact, to see what happens in the limit $\epsilon \to 0$. In that case E/Q becomes smaller than 1 at $x = 1.234$. Thus, as $\epsilon \to 0$, we have a finite x, but $\nu \to 0$, $\omega \to 1$, and $R \to \infty$. In this region of small ϵ, E becomes equal to Q when $Q \approx 369\epsilon^{-3/2}$. Thus, higher and higher values of Q are required to render the Q ball stable as $\epsilon \to 0$.

(ii) $\epsilon > 2$. When $\epsilon > 2$, the global minimum of the potential is at $\rho = \epsilon$, while there is a local minimum at $\rho = 0$. We thus expect that there will be a first order transition at $\epsilon = 2$. It is nonetheless interesting to examine the possibility of having Q balls in the midst of a supercooled metastable vacuum. We assume, in other words, that $\rho(\infty) = 0$, so that the field is

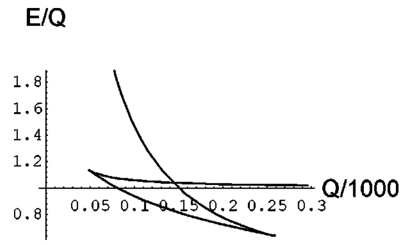

FIG. 2. The energy per unit charge, E/Q, versus $Q/1000$ for $\epsilon=2.5$. The cusps are at $Q=49.4635$, $Q=258.629$.

in the metastable vacuum at infinity. The question is then whether there are Q balls in this case.

For $x \geqslant 2$, Eq. (16) reduces to $\nu^2 \approx \epsilon(x-1)(2x)^{-1}$. However, $\nu \leqslant 1$, from Eq. (14). This can only happen if $x \leqslant \epsilon/(\epsilon-2)$. In fact, as x approaches this asymptotic value, ν tends to 1, ω tends to 0, $Q \rightarrow 0$, while E/Q diverges. Thus E/Q begins at $x=0$ with the value 1, goes to a maximum, then descends below 1, reaches a minimum, and then it comes back up, passes the value 1 and diverges at $x \approx \epsilon/(\epsilon-2)$. Figure 2 shows that E/Q is a multivalued function of Q, with the cusps corresponding to the extrema of Q with respect to x. Here we see that there is a maximum, as well as a minimum, charge if we are to have stable Q balls. Q balls with too great a charge are going to decay if $\epsilon>2$.

Eventually, the equation $E=Q$ stops having two roots. For $\epsilon=3.3$ the two roots coincide. For larger values of ϵ we have $E/Q>1$ for all values of Q.

We conclude then that there are stable Q balls for $\epsilon<2$, with no upper limit for the charge and with a lower limit that becomes progressively higher as ϵ decreases towards 0. There are also Q balls in the midst of a supercooled metastable vacuum, unlike the case of the ϕ^6 potential, but only if the charge Q is within an interval of allowed values, an interval that becomes progressively smaller as ϵ increases, until it disappears completely when ϵ becomes equal to 3.3. Note that at the point $\epsilon=2$, where the phase transition is supposed to occur, the Q ball has a finite size and can thus continue to exist if supercooling occurs. In particular, if $\epsilon=2$, the Q ball with the least possible charge has $R=2.2137$ and $Q=117.39$.

All these results are classical and fully exact. We have thus been able to examine thoroughly the formation of Q balls in a piecewise parabolic potential, obtaining analytic results that are confirmed by the virial relations. Such potentials can therefore be useful in cosmological calculations, in spite of the kinks they involve, because they enable one to avoid the difficult task of solving the nonlinear equations of motion that pertain to polynomial potentials.

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