Higher derivative Weyl gravity

W. F. Kao*
Institute of Physics, Chiao Tung University, Hsin Chu, Taiwan
(Received 7 September 1999; published 19 January 2000)

A higher derivative Weyl model is analyzed carefully. We show that there is a nontrivial constraint if a symmetry-breaking potential is added to the system. A rigorous proof for the constraint is presented. One, hence, studies the applications of this conformal theory in the inflationary universe.

PACS number(s): 04.50. +h, 04.40.−b, 98.80.Cq

I. INTRODUCTION

Scale-invariant theory is conjectured to be the effective theory of our physical universe for various reasons, as pointed out in Refs. [1,2]. Evidence also indicates that scale symmetry appears to be very important in various branches of physics. For example, QCD [3] is scale invariant and it is also argued that Weyl symmetry may have to do with the missing Higgs problem in electroweak theory [4]. In addition, the Weyl gauge field is speculated to be a candidate for dark matter. It also has many applications in the physics of the early universe [5–9].

It was also shown that if the scale symmetry was broken, the consistent vacuum configuration of the system is not the same as most field theories. Most field theories will admit a vacuum of the form $\partial V(\phi_0)/\partial\phi=0$. Instead, the scalar vacuum of the Weyl model will take $\phi_0[\partial V(\phi_0)/\partial\phi]-4V(\phi_0)=0$, for any kind of symmetry-breaking potential coupled to the Weyl invariant theory [1]. In addition, the scalar vacuum equals the lowest energy state $V(\phi_0)=0$ only if ϕ_0 happens to be the lowest energy state.

In fact, one is able to show that the equation

$$\phi\frac{\partial V(\phi)}{\partial\phi}-4V(\phi)=0 \tag{1}$$

remains valid for all solutions to the equation of motion of any Weyl model coupled to a symmetry-breaking potential $V(\phi)$. This was first done from direct derivation from the field equations [10,11]. A systematic method is also devised later [12]. This constraint is valid for all on-shell scalar fields, not just for the vacuum configuration. In other words, the physical scalar field will in fact be frozen to one of the solutions of the constraint equation (1). Therefore, the scalar field has to be a constant if an effective symmetry-breaking potential develops. One notes in particular that the scale-invariant $c\phi^4$ potential solves the constraint equation (1). Therefore, the scalar field will not be constrained by the constraint equation in the scale-invariant limit.

This phenomenon is certainly very different from the conventional field theories defined with various broken symmetry. This may also be an indication to the resolution why the Higgs field has not been traced so far. In addition, it can be shown that this constraint is related to the special combination of the Weyl connection and its associated conformal fields.

II. WEYL SYMMETRY AND WEYL CONNECTION

Consider the scale-invariant action given by [8]

$$S_{\rm eff}=\int d^4x\sqrt{g}\bigg[-\frac{1}{2}\epsilon\phi^2\tilde{R}-\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi -\frac{1}{4}H_{\mu\nu}H^{\mu\nu}-f\tilde{R}^2-V_{\rm eff}(\phi)\bigg]. \tag{2}$$

Here $\tilde{R}$ is obtained by replacing all $\partial_\alpha g_{\mu\nu}$ in the scalar curvature R by covariant derivatives $\nabla_\alpha g_{\mu\nu}=(\partial_\alpha+2S_\alpha)g_{\mu\nu}$. One can hence show that $\tilde{R}=R+6(D_\mu+S_\mu)S^\mu$ after some algebra. Moreover, the covariant derivative of the scalar field ϕ is defined as $\nabla_\mu\phi=(\partial_\mu-S_\mu)\phi$ while the field tensor for the Weyl vector meson is defined as $H_{\mu\nu}=\partial_\mu S_\nu-\partial_\nu S_\mu$. Note that one can also consider other combinations of higher derivative terms. All of our results still hold in these models. We will ignore these terms since $R^2_{\mu\nu\alpha\beta}$ and $R^2_{\mu\nu}$ are related to R^2 term in four-dimensional Friedmann-Robertson-Walker (FRW) space due to the Euler constraint and the vanishing of the Weyl conformal tensor [13]. One notes, however, that the stability behavior of different curvature squared terms is known to be different with respect to anisotropic perturbations [14].

Note that this action is invariant under the Weyl transformation (WT) $g^\Omega_{\mu\nu}=\Omega^2(x)g_{\mu\nu}$, $\phi^\Omega=\Omega^{-1}(x)\phi$, and $S^\Omega_\mu=S_\mu-\partial_\mu\Omega/\Omega$, provided that $\sqrt{g}V(\phi)$ is scale invariant by itself.

In writing the effective action (2), we have assumed that the only conformal symmetry-breaking term relevant to the low-energy region is due to the effective potential introduced above. In other words, our concern here is the microscopic origin of the hypothesis that matter receives its mass scale from a graviton in contrast to the conventional wisdom of the Brans-Dicke theory [1]. Therefore, one will assume that the measuring field ϕ acquires a vacuum expectation value as a result of instabilities of the full quantum theory. To be more

*Email address: wfgore@cc.nctu.edu.tw

precise, one has assumed that the effective action at low energies is of the form given by Eq. (2) with an effective potential $V_{\rm eff}$ favoring an asymmetric vacuum for the measuring field ϕ. In many circumstances, there will be a symmetry-breaking effect induced by the quantum fluctuation of the scalar field. The one-loop effective potential starting from a scale-invariant potential has been known from derivations based on many different methods [7,15]. Therefore, one may have a very complicated effective potential signifying the effect of dynamical symmetry breaking.

In order to look closely at the physics of the constraint (1), one will derive it in a rigorous way. Indeed, one can show that

$$\delta\mathcal{L}_g=\delta V_g=\frac{\delta\mathcal{L}_g}{\delta g_{\mu\nu}}\delta g_{\mu\nu}+\frac{\delta\mathcal{L}_g}{\delta\nabla_\alpha g_{\mu\nu}}\delta\nabla_\alpha g_{\mu\nu}+\frac{\delta\mathcal{L}_g}{\delta\phi}\delta\phi+\frac{\delta\mathcal{L}_g}{\delta\nabla_\alpha\phi}\delta\nabla_\alpha\phi+\frac{\delta\mathcal{L}_g}{\delta\partial_\alpha S_\nu}\delta\partial_\alpha S_\nu. \qquad (3)$$

Here one writes $\mathcal{L}_g\equiv\sqrt{g}\mathcal{L}$ and $V_g\equiv\sqrt{g}V$ with $V=V(\phi)$ denoting an arbitrary functional of ϕ. Note that all δS_α components have been rearranged into $\delta\nabla_\alpha g_{\mu\nu}$ and $\delta\nabla_\alpha\phi$ according to the prescribed scale symmetry. In addition, the above variation is understood to be done with respect to the Weyl transformation (WT) introduced earlier. Note also that the last term of the above equation vanishes because (i) $\delta\partial_\alpha S_\nu$ is proportional to $\partial_\alpha\partial_\nu\ln\Omega$ which is symmetric with respect to α and ν, while (ii) $\delta\mathcal{L}_g/\delta\partial_\alpha S_\nu$ is skew symmetric with respect to α and ν. It is also straightforward to show that

$$\Omega\frac{\delta V_g}{\delta\Omega}=2\nabla_\alpha\left(\frac{\delta\mathcal{L}_g}{\delta\nabla_\alpha g_{\mu\nu}}g_{\mu\nu}\right)-\nabla_\alpha\left(\frac{\delta\mathcal{L}_g}{\delta\nabla_\alpha\phi}\phi\right), \qquad (4)$$

from Eq. (3) and the scale transformations (WT) introduced earlier. Here we have also applied the equations of motion all together with proper rearrangement. Note also that varying $\mathcal{L}_g$ with respect to $\nabla_\alpha g_{\mu\nu}$ is effectively equivalent to varying $\mathcal{L}_g$ with respect to $2S_\alpha g_{\mu\nu}$ and hence equivalent to the variation with respect to the connection $2S_\alpha$ associated with the metric $g_{\mu\nu}$. Similarly, one can do the same thing for the connection $-S_\alpha$ associated with the scalar field ϕ. Hence, one has $\Omega\,\delta V_g/\delta\Omega=\nabla_\alpha(\delta\mathcal{L}_g/\delta S_\alpha)$. Note that the above equation is nothing but a recombination of various connections all together as specified in Eq. (4). One can thus show that $\Omega\,\delta V_g/\delta\Omega=\nabla_\alpha\nabla_\mu(\delta\mathcal{L}_g/\delta\nabla_\mu S_\alpha)$ with the help of the variational equation of S_α. Finally, one reaches the promised conclusion $\Omega\,\delta V_g/\delta\Omega=0$ due to the skew-symmetric property of the S_μ equation. This gives exactly the constraint equation (1). Note that the constraint can also be shown to be valid with the inclusion of gauge fields and matter fields. It is also valid in D-dimensional conformal theory. The proof can be generalized straightforwardly. Note also that the above argument also applies to the case $\epsilon=-1/6$ where the Weyl gauge field S_μ disappears except in the kinetic term $H^2_{\mu\nu}$. Note that there is another way to derive this constraint [1]. But it was also pointed out by the author [1] that such a derivation is not valid for singular field reparametrization.

III. EFFECTIVE ACTION AND STABILITY OF INFLATION

Note that one can show that the effective action becomes

$$S_{\rm eff}=\int d^4x\sqrt{g}\left[-\frac{1}{2}\epsilon\phi_0^2R-fR^2-V_{\rm eff}(\phi_0)\right] \qquad (5)$$

once the dynamics of the scalar field was frozen by the constraint (1). The Weyl vector meson will acquire a mass of the order of the Planck mass and will be physically decoupled from the effective action in the low-energy limit. In addition, one can also show that the Weyl meson vanishes for the quadratic Lagrangian if it takes the form $S_\mu=(s(t),0,0,0)$ [13] in the background of FRW space. Moreover, one can also show that $s=0$ is a solution to the action (5) provided $R=$const, which is consistent with the de Sitter background in perturbative analysis we will be discussing shortly. Therefore, we will simply turn off the Weyl vector meson for our discussions from now on. Note that similar stability analysis for higher derivative gravity theories can be found in Refs. [16,17].

Note that the Friedmann equation of the system can be shown to be [18]

$$\epsilon\phi_0^2\left(H^2+\frac{k}{a^2}\right)=4f\left[2H\ddot{H}-\dot{H}^2+6H^2\dot{H}-2H^2\frac{k}{a^2}+\left(\frac{k}{a^2}\right)^2\right]+V_0. \qquad (6)$$

Here $H\equiv\dot{a}/a$ with $a(t)$ denoting the scale factor of the FRW space. In addition, $V_0\equiv V_{\rm eff}(\phi_0)$ denotes the effective cosmological constant. It is not easy to solve this equation directly. One can, however, assume first that there is a zeroth-order solution such that $H=H_0$ or effectively $a=a_0\exp H_0t$. One can then set the full solution to the above equation as $H=H_0+\delta H$ and perturb the Friedmann equation accordingly [18,19]. The linear order equation in δH will tell us whether the inflationary solution $H=H_0$ is stable or not. One wishes to obtain an inflationary solution that is stable only for a very brief period of time and the later stage of our universe can exit the inflationary phase afterward. Therefore, one is looking for models such that the perturbed field δH can admit one stable and one unstable solution to the perturbed equation. This will indicate that the universe could possibly start with the inflationary de Sitter phase and exit the inflationary phase due to the unstable perturbation. One is about to show that this effective theory can in fact accommodate one stable mode and one unstable mode in the perturbation introduced here.

Indeed, one can show that the leading order and the consecutive order linear in δH of the Friedmann equation are given by

$$\epsilon \phi_0^2 H_0^2 = V_0, \tag{7}$$

$$\delta \ddot{H} + 3H_0 \delta \dot{H} - \frac{\epsilon \phi_0^2}{4f} \delta H = 0, \tag{8}$$

respectively. Hence, one has $\delta H_{\pm} = \exp[-\frac{3}{2}H_0(1 \pm \sqrt{1+\epsilon\phi_0^2/9fV_0})t]$. Here $H_0^2 = V_0/\epsilon\phi_0^2$. Therefore, one shows that there are one stable mode and one unstable mode if $f>0$. In addition, one can show that the De Sitter universe can exist in the inflationary phase in a duration of the order of $\Delta t \sim 2/3H_0(\sqrt{1+\epsilon\phi_0^2/9fV_0}-1)$. Therefore, one finds that it is possible to induce enough 60 e-fold inflation with properly adjusted parameters.

One can follow the argument of Ref. [16] with the help of the $\dot{H}$-H phase diagram and show that the radiation dominated (RD) solution $a = a_0\sqrt{t}$ does not exist when V_0 is comparably large. In fact, one can show that the higher derivative terms $2H\ddot{H} - \dot{H}^2 + 6H^2\dot{H} = 0$ if $a = a_0\sqrt{t}$. Therefore, RD solution has to be created with the help of the gauge field coupled to the quadratic gravitational field. In fact, one can show that the higher derivative term is negligible at large time $t \to \infty$ for $a \to t^p$ with any positive $p<1$.

In addition, unlike other scalar gravity theory, we do not have a dynamical scalar field that generates a small cosmological constant by slipping down to its minimum potential state after the inflation. Note that the constraint equation (1) indicates that ϕ_0 is not a minimum potential solution unless $V_0 = 0$. Therefore, this theory apparently cannot explain why the cosmological constant is so small, while it was so large in the earlier stage of the inflationary universe. Fortunately, this conformal theory is assumed to be nothing more than an effective theory near some fixed point of the renormalization group [1,2]. Hence one can assume that the effective theory remains effective only during the inflationary scale. Later on, the effective theory no longer holds as an effective theory responsible for the theory for the later evolution of the universe. Hence, if the present universe with small values of the cosmological constant has to do with the similar conformal model proposed here, one will need another form of the symmetry-breaking potential $\tilde{V}_{\rm eff}$ responsible for the physics at a different energy scale. This potential should admit a constraint solution $\phi = \phi_1$ such that $\tilde{V}_{\rm eff}(\phi_1)$ is closed to zero. In addition, the symmetry of this effective symmetry-breaking potential should also reflect the symmetry of the post-inflationary universe. This effective theory would also require a gauge field coupled to the system in order to generate a RD universe.

Note that one can instead start with a model where V_0 is comparably small. In this case, one can show that [16] there exists a solution such that $a = a_0\exp[H_0\exp(\sqrt{\epsilon\phi_0^2/4f}\ \ t)]$ (from $\dot{H} = \sqrt{\epsilon\phi_0^2/4f}H$) in the limit $t \to -\infty$. In addition, this model with a small effective cosmological constant will also admit a soft inflationary solution $a = a_0\exp(\sqrt{V_0/\epsilon\phi_0^2}\,t)$ similar to the case where V_0 is comparably large. The stability analysis shown earlier indicates that this soft inflationary phase can remain stable for a longer duration. Hence, this model represents a strong inflationary phase in the past plus a soft inflationary phase in the future. This could also be a reasonable resolution to the evolution of the universe.

Alternatively, the Bicknell theorem [20] shows that an R^2 term in higher derivative gravity is conformally equivalent to a massive scalar field theory described by the following effective action:

$$S_{\rm eff} = \int \sqrt{\tilde{g}}\left\{-\tilde{R} - \frac{6f^2}{(1-2f\varphi)^2}\left[\tilde{D}_\mu\varphi\tilde{D}^\mu\varphi + \frac{1}{6f}\varphi^2 + \frac{\Lambda}{3f^2}\right]\right\}. \tag{9}$$

Here tilde notation in this section indicates field evaluated in a conformal coordinates [20]. Note that the scalar field is effectively represented by the scalar curvature R which is effectively proportional to H_0^2 in an inflationary background. Therefore, one can assume that the scalar field φ in action (9) is a constant field during the inflation era. Therefore, for simplicity, one will focus on the effective action

$$S_1 = \int d^4x\sqrt{g}\left[-\frac{1}{2}\epsilon\phi_0^2 R - V_1\right] \tag{10}$$

with V_1 denoting the total effective cosmological constant.

Note that if the symmetry-breaking potential has more than one solution to the constraint equation (1), the argument of cosmological wave function may be able to provide a resolution of this problem [21]. For example, if the effective symmetry-breaking potential takes the following form [15]:

$$V_{\rm eff} = \lambda\left[\phi^4\ln\left(\frac{\phi}{\phi_1}\right)^4 + 2(\phi_0^2+\phi_1^2)\phi^2 - \phi_1^2\phi_0^2\right], \tag{11}$$

such that the constraint equation becomes $(\phi^2-\phi_1^2)(\phi^2-\phi_0^2)=0$. Note that the inclusion of terms proportional to ϕ^4 in the effective action will not affect the constraint equation. This homogeneous term can thus be added to adjust the final form of $V_1 = V_{\rm eff}(\phi=\phi_i)$ one wishes without affecting the constraint equation. In addition, one can show that $V_{\rm eff}(\phi_0)/V_{\rm eff}(\phi_1) \to (\phi_0/\phi_1)^2\ln(\phi_0/\phi_1)^4$ if $\phi_0 \gg \phi_1$. Therefore, this kind of potential does give two solutions (or vacuum states): (i) $\phi=\phi_0$ gives a large cosmological constant and (ii) $\phi=\phi_1$ gives a small cosmological constant.

The argument of Hartle-Hawking [21] based on the no-boundary boundary condition shows that the probability of finding a universe in the constant V_1 states is given by $\mathcal{P} \sim \exp(3/8V_1)$. This approach favors a universe with a small cosmological constant [21] which appears to describe the later stage of our physical universe. One may follow the comment of Hawking that we live in the tail of the distribution [22] such that $V_1 = V_{\rm eff}(\phi_0)$ possibly for anthropic reasons. As a result, this conformal model indicates that the ϕ_0

vaccum is chosen by accident and tends to be unstable. Hence the system has to undergo a quantum jump to the ϕ_1 vacuum in the post-inflationary era. This leaves us a universe with a small cosmological constant. Released energy during the quantum jump may have to do with the reheating of the post-inflationary evolution. It is known that the quantum cosmological argument remains controversial partly due to the fact that the analysis can only be done in simple models [21]. It nonetheless indicates that the effect of conformal theory deserves more attention even if the above *speculation* is not rigorous. Hence the above argument may also provide a resolution to the existing problem.

ACKNOWLEDGMENTS

This work is supported in part by Taiwan's National Science Council under Contract No. NSC88-2112-M009-001.

[1] L. Smolin, Nucl. Phys. **B160**, 253 (1979).

[2] K.G. Wilson, Phys. Rev. **84**, 3184 (1971); K.G. Wilson and J. Kogut, Phys. Rep. **12C**, 78 (1974); N.N. Bogolyubov and D. V. Shirkov, *Introduction to Theory of Quantized Fields* (Wiley, New York, 1984); A.B. Zamolodchikov, Yad. Fiz. **46**, 1819 (1987) [Sov. J. Nucl. Phys. **46**, 1090 (1987)].

[3] L.H. Ryder, *Quantum Field Theory* (Cambridge University Press, Cambridge, England, 1985); S. Weinberg, *Quantum Theory of Fields* (Cambridge, London, 1996), Vols. 1 and 2.

[4] H. Cheng, Phys. Rev. Lett. **61**, 2182 (1988); H. Cheng and W.F. Kao, *Consequences of Scale Invariance*, M. I. T. report (1988) (unpublished).

[5] A. Zee, Phys. Rev. Lett. **42**, 417 (1979); **44**, 703 (1980); S.L. Adler, Rev. Mod. Phys. **54**, 729 (1982).

[6] F.S. Accetta, D.J. Zoller, and M.S. Turner, Phys. Rev. D **31**, 3046 (1985); A.S. Goncharov, A.D. Linde, and V.F. Mukhanov, Int. J. Mod. Phys. A **2**, 561 (1987); W.F. Kao, Phys. Rev. D **46**, 5421 (1992); **47**, 3639 (1993).

[7] Lee Smolin, Phys. Lett. **93B**, 95 (1980); J. Polchinski, Nucl. Phys. **B303**, 226 (1988); V.P. Frolov and D.V. Fursaev, Phys. Rev. D **56**, 2212 (1997).

[8] H. Weyl, *Space-Time-Matter* (Dover, New York, 1922); P.A.M. Dirac, Proc. R. Soc. London **A333**, 403 (1973); R. Utiyama, Prog. Theor. Phys. **50**, 2080 (1973); **53**, 565 (1975).

[9] H.T. Nieh and M.L. Yan, Ann. Phys. (N.Y.) **138**, 237 (1982); J.K. Kim and Y. Yoon, Phys. Lett. B **214**, 96 (1988); L.N. Chang and C. Soo, hep-th/9905001.

[10] W.F. Kao, S.Y. Lin, and T.K. Chyi, Phys. Rev. D **53**, 1955 (1996).

[11] Y. Yoon, Phys. Rev. D **59**, 127501 (1999).

[12] W.F. Kao, ''Weyl Symmetry and Nontrivial Constraint,'' CTU report (1996).

[13] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972); N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).

[14] H.-J. Schmidt, Class. Quantum Grav. **5**, 233 (1988).

[15] S. Coleman and E. Weinberg, Phys. Rev. D **7**, 1888 (1973); M. Consoli and A. Ciancitto, Nucl. Phys. **B254**, 653 (1985); N. Tetradis and C. Wetterich, *ibid.* **B383**, 197 (1992); M. Consoli, Phys. Lett. B **305**, 78 (1993); M. Reuter and C. Wetterich, Nucl. Phys. **B391**, 147 (1993); C. Wetterich, Phys. Lett. B **301**, 90 (1993).

[16] V. Müller and H.J. Schmidt, Gen. Relativ. Gravit. **17**, 769 (1985).

[17] H.J. Schmidt and V. Müller, Gen. Relativ. Gravit. **17**, 971 (1985); V. Müller, H.J. Schmidt, and A.A. Starobinsky, Phys. Lett. B **202**, 198 (1988).

[18] W.F. Kao and U.L. Pen, Phys. Rev. D **44**, 3974 (1991); W.F. Kao, U.L. Pen, and P.J. Zhang, gr-qc/9911116.

[19] A. Dobado and A.L. Maroto, Phys. Rev. D **52**, 1895 (1995).

[20] G.V. Bicknell, J. Phys. A **341**, 1061 (1974); B. Whitt, Phys. Lett. **145B**, 176 (1984); S.W. Hawking and J.C. Luttrell, Nucl. Phys. **B247**, 250 (1984); V. Faraoni, E. Gunzig, and P. Nardone, gr-qc/9811047.

[21] J.B. Hartle and S.W. Hawking, Phys. Rev. D **28**, 2960 (1983); A.D. Linde, Nuovo Cimento A **39**, 401 (1984); A. Vilenkin, Phys. Rev. D **30**, 509 (1984); **33**, 3560 (1986); E. Kolb and M.S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, CA, 1990).

[22] S.W. Hawking and Neil Turok, gr-qc/9802062 v2.