Entropic *C* theorems in free and interacting two-dimensional field theories

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The relative entropy in two-dimensional field theory is studied on a cylinder geometry, interpreted as finite-temperature field theory. The width of the cylinder provides an infrared scale that allows us to define a dimensionless relative entropy analogous to Zamolodchikov's c function. The one-dimensional quantum thermodynamic entropy gives rise to another monotonic dimensionless quantity. I illustrate these monotonicity theorems with examples ranging from free field theories to interacting models soluble with the thermodynamic Bethe ansatz. Both dimensionless entropies are explicitly shown to be monotonic in the examples that we analyze.

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I. INTRODUCTION

It has been shown that the irreversible character of the renormalization group (RG) can be cast in a sort of H theorem analogous to Boltzmann's, thus generalizing this theorem from ordinary time evolution to the evolution with the RG parameter [1]. The irreversible quantity, the field theory entropy relative to a fixed point of the RG, is a monotonic function of the coupling constants and increases in the crossover from one fixed point to another less stable. However, the Wilson RG picture considered in [1], wherein one has to deal with all the couplings generated by the RG action, relevant and irrelevant alike, turns out to be too complex and was indeed assimilated to a non-equilibrium thermodynamics setting. One can start with only the relevant couplings but then one must utilize a different RG which changes some infrared (IR) scale. A possibility is to define the field theory on a finite geometry characterized by some parameter, loosely associated with its size, which plays the role of the IR scale. Then the monotonicity theorem for the relative entropy can be cast as a RG theorem similar to the celebrated Zamolodchikov c theorem [2].

Among the various geometries we could consider, the cylinder stands out for its simplicity. It is defined by only one scale, the length of the compact dimension, and the finite-size corrections to the partition function turn out to be computable. Moreover, on a cylinder of circumference β the monotonicity theorem adopts a form with a thermodynamic interpretation, the temperature being $T=1/\beta$ [2]. Thus the inverse temperature is used as RG parameter, providing a thermodynamic interpretation of the RG, as in Ref. [3]. The connection with concepts of $1+1$ quantum field theory (QFT) at finite temperature is intellectually appealing and useful for computational purposes. For example, finite-size corrections are calculated in terms of the properties of one-dimensional (1D) quantum gases. In addition to the relative entropy, the 1D quantum entropy provides another monotonic quantity with a different interpretation. We must remark that the definition and monotonicity of the relative entropy, as exposed in Ref. [1], already have a thermodynamical motivation in the 2D context, independently of the type of geometry. However, in field theory we prefer to dissociate the coupling from a thermal interpretation and we reserve the concept of temperature for its role in the 1D quantum picture. Nevertheless, it shall be evident that the proofs of the monotonicity theorems for the 2D relative entropy or for the 1D quantum entropy are essentially the same.

In Ref. [2] these ideas were illustrated only with free field models and the calculations of the corresponding finite-size corrections were presented very concisely. We shall begin here with a more detailed analysis of the properties of both types of entropy, in particular, considering whether they are universal quantities. Next, we proceed to the explicit calculation of the finite-size corrections for soluble models corresponding to free-field theories, including thermodynamic quantities as well as the expectation values of the stress tensor, and hence of the entropic monotonic quantities. The properties of these quantities will be displayed in the corresponding plottings. Further to free-field models, it will be demonstrated that interacting models are also suitable for calculation of their finite-size corrections and monotonic quantities with powerful methods. In particular, integrable models on the cylinder are appropriate for application of the thermodynamic Bethe ansatz (TBA). Plots of the monotonic quantities obtained with this method display similar behavior to those of free-field models.

The paper is divided in three parts. The first part is devoted to formulating the monotonicity theorems for 2D field theory and to giving its thermodynamic interpretation on the cylinder. The second part applies these theorems to the relatively simple cases of the Gaussian and Ising models. They allow an explicit calculation of thermodynamic quantities and their connection with the components of the stress tensor. Section III is devoted to interacting models which lend themselves to computation of thermodynamic quantities. The essential tool is the thermodynamic Bethe ansatz, which is first applied to models with purely statistical interaction, resulting again in explicit expressions for the relevant quantities, and in second place to models in which the TBA equations have to be solved numerically. Afterwards, there comes a discussion of the results obtained and, finally, two appendices, the first one on the method for the computation of finite-size corrections based on the Euler-MacLaurin formula and the second one on the computation of the expectation value of the stress tensor on the cylinder for free theories.

II. ENTROPIC *C* THEOREMS

A. General properties of the relative entropy in two-dimensional field theory

Let us briefly recall some concepts already introduced in Ref. [1]. The field theory probability distribution associated to some statistical system is given by

$$\mathcal{P}[\phi,\{\lambda\}]=e^{-I[\phi,\{\lambda\}]+W[\{\lambda\}]}, \tag{1}$$

and depends on some stochastic field ϕ and a set of coupling constants $\{\lambda\}$. The quantity $W[\{\lambda\}]$ is needed for normalization and is of course minus the logarithm of the partition function. A composite field is defined as the derivative of the action with respect to some coupling constant:

$$f_\lambda=\frac{\partial I}{\partial\lambda}. \tag{2}$$

For example, if we consider the thermal coupling, the coupling constant is the inverse temperature and the composite field represents the energy. As is usual, we assume for simplicity that $I[\phi,\{\lambda\}]$ is linear in the coupling constants.

The relative entropy, a concept borrowed from probability theory, turns out to be the Legendre transform of $W(\lambda)-W(0)$ with respect to λ [1]:

$$S_{\rm rel}(\lambda)=W(\lambda)-W(0)-\lambda\frac{dW}{d\lambda}=W-W_0-\lambda\,\langle f_\lambda\rangle. \tag{3}$$

Obviously, $S_{\rm rel}(0)=0$. Furthermore, as a straightforward consequence of its definition,

$$\begin{aligned}\lambda\frac{dS_{\rm rel}}{d\lambda}&=\lambda\frac{dW}{d\lambda}-\lambda\frac{d}{d\lambda}\left(\lambda\frac{dW}{d\lambda}\right)\\&=-\lambda^2\frac{d^2W}{d\lambda^2}\\&=-\lambda^2\frac{d}{d\lambda}\langle f_\lambda\rangle\\&=\lambda^2\langle(f_\lambda-\langle f_\lambda\rangle)^2\rangle\geqslant 0.\end{aligned} \tag{4}$$

For the thermal coupling, $S_{\rm rel}$ has indeed the interpretation of a real thermodynamic entropy which increases with temperature. In other cases, it may or may not have a thermodynamic interpretation but its properties hold nonetheless.

Some qualifications are in order. In field theory we deal with local fields, so $f_\lambda=\int\Phi_\lambda$, where $\Phi_\lambda(z)$ is a local composite field, function of the 2D coordinates $z=x_1+i\,x_2$. We must remark that, although these fields are usually constructed as actual composites of the basic field ϕ, the existence of this field needs not be assumed, as in some modern formulations where it is replaced by the *action principle* [4]. This remark is important when we start from a 2D conformal field theory. To prevent the appearance of ultraviolet (UV) divergences it is convenient to define all the quantities with a UV cutoff Λ—for example, $W[\lambda,\Lambda]$,— which must be eventually removed to define universal quantities. Even though W is nonuniversal, we expect $S_{\rm rel}$ to be [1]. In order to have universality, we consider RG relevant or marginal couplings: In two dimensions the scaling dimension of the field Φ must be such that $0\leqslant d_\Phi\leqslant 2$. This condition may not be sufficient and shall be made more precise shortly. W and $S_{\rm rel}$ are extensive and it is convenient to define the associated specific quantities dividing by the total volume—or area in two dimensions. Henceforth, we use specific quantities but keep the same notation for simplicity. We are interested in an entropy relative to a RG fixed point, so we must substract from the coupling constants their values at that point. (The fixed-point coupling constants may be null in some cases.) Finally, there is an assumption of positivity of the probability distribution implied in the inequality (4), like in Zamolodchikov's theorem.

To derive a universal expression for the specific $S_{\rm rel}$ we must analyze its dependence on the UV cutoff. We can use the scaling form of the specific W,

$$W(\lambda,\Lambda)=\Lambda^2\,\mathcal{F}\left(\frac{\lambda^{2/y}}{\Lambda^2}\right), \tag{5}$$

where $y=2-d_\Phi>0$ is the dimension of the coupling λ. For the thermal field, the local energy density, y is the inverse of the critical exponent ν. If the scaling function is continuously differentiable around zero (class C^1), and we denote $F_0=\mathcal{F}(0)$, $F_1=\mathcal{F}'(0)$, W can be expanded as

$$W(\lambda,\Lambda)=\Lambda^2F_0+F_1\lambda^{2/y}+\Lambda^2o(\Lambda^{-2}), \tag{6}$$

with $o(\Lambda^{-2})$ asymptotically smaller than Λ^{-2}, hence resulting in a vanishing term as $\Lambda\to\infty$. Given that the UV divergent term of this expansion cancels in $W(\lambda)-W(0)$, the relative entropy yields a finite result in the infinite cutoff limit, namely,

$$S_{\rm rel}(\lambda)=W(\lambda)-W(0)-\lambda\frac{dW}{d\lambda}=F_1\frac{y-2}{y}\lambda^{2/y}. \tag{7}$$

Thus the significance of the assumed regularity condition on the scaling function is that it is sufficient to endow the monotonicity theorem with universality. One can certainly think of simple functions that are not class C^1. For example, the function $\mathcal{F}(x)=F_0-x\ln x+o(x)$, which will appear in some of the models studied later.

We now examine the question of universality in terms of local fields. This method will lead us to a more concrete formulation. Let us begin by writing the monotonicity theorem (4) as

$$\lambda\frac{\partial S_{\rm rel}}{\partial\lambda}=\lambda^2\int d^2z\langle:\Phi(z)::\Phi(0):\rangle\geqslant 0, \tag{8}$$

with the use of the definition of normal-ordered composite fields, $:\Phi:=\Phi-\langle\Phi\rangle$. We can study the UV convergence of this integral. As a prerequisite, note that possible UV divergences in the definition of the composite field Φ are removed by the substraction of $\langle\Phi\rangle$. The most singular part of the correlation function for short distance is given by

$$\langle:\Phi(z)::\Phi(0):\rangle \sim |z|^{-2d_\Phi}. \tag{9}$$

Hence, the integral converges if $0 \leq d_\Phi < 1$, that is, $1 < y \leq 2$. Then the derivative of the relative entropy, $dS_{\rm rel}/d\lambda$, is a universal quantity and so is $S_{\rm rel}$, because the integration constant is fixed by the condition $S_{\rm rel}(0)=0$. For dimensional reasons, it must adopt a form like that in Eq. (7):

$$S_{\rm rel}(\lambda)=B\lambda^{2/y}, \tag{10}$$

where B is a constant. In fact, upon inversion of the Legendre transform, this form implies that W has the previous first-order expansion (6), except in the case of $y=2$.

Fields Φ satisfying $0 \leq d_\Phi < 1$ are called strongly relevant [5,6]. They include the thermal coupling of the unitary minimal models of conformal field theory (CFT), except the Ising model, wherein the local energy density has $d_\Phi = 1$. We shall find that the relative entropy of the Ising model is indeed nonuniversal. In principle, fields with $1 \leq d_\Phi \leq 2$ give rise to a nonuniversal relative entropy, $S_{\rm rel}(\lambda,\Lambda)$. It is monotonic and essentially independent of Λ as long as $\lambda^{2/y} \ll \Lambda^2$, which is the condition necessary for the continuum field theory of the statistical system to be meaningful. In this sense, one may consider this non-universal relative entropy within the philosophy of *effective field theories*, a term which refers to theories that are not renormalizable but suitable for calculation of many physical quantities for scales much lower than the cutoff.

We must remark that the simple power-law forms of the relative entropy (10) and the monotonicity theorem are not very informative, in the sense that, once we know that $S_{\rm rel}$ is finite, they follow from dimensional analysis. We thus see the necessity of introducing a new parameter, for example, through a finite geometry. We will indeed obtain a richer and more illuminating version of the relative entropy and the monotonicity theorem when we introduce a finite geometry.

Let us introduce the stress tensor trace, $\Theta := T^a_a$. Since Θ gives the response to a change of scale and the only scale is in the coupling constant, it is in general proportional to the relevant field Φ:

$$\Theta = y\lambda\Phi.$$

Hence, we can put the monotonicity theorem for the specific relative entropy in an interesting form:

$$\lambda\frac{\partial S_{\rm rel}}{\partial\lambda}(\lambda,\Lambda)=\frac{1}{y^2}\int d^2z\langle:\Theta(z)::\Theta(0):\rangle \geq 0. \tag{11}$$

We will have the occasion to comment on this form in what follows.

B. Finite-size corrections. The cylinder and one-dimensional thermodynamics

So far, the relative entropy has been proved to be monotonic with respect to the coupling constants. Now we would like to reformulate the monotonicity theorem for the relative entropy as showing irreversibility under the RG. We need to substitute the coupling constant λ by some quantity which can be interpreted as a RG parameter. A common way to introduce a RG parameter is through some IR scale. For example, we may consider a finite size system with a characteristic length, such as a strip or cylinder of width L. According to finite-size scaling ideas, the free energy can be split into a *bulk part* and a *universal finite-size correction*. The latter constitutes a suitable function to derive a nontrivial relative entropy. Moreover, one can take advantage of the fact that the classical partition function on a cylinder of width β is equivalent to the one-dimensional quantum partition function at temperature $T=1/\beta$ to give the RG a thermodynamic interpretation [3]. Indeed, relevant thermodynamic functions of this quantum system are given by derivatives with respect to β. The first one is the energy, which has one part independent of the temperature and another that vanishes at $T=0$, corresponding to the bulk part and the finite size correction, respectively. The part independent of the temperature, which is nonuniversal, represents the ground-state energy. A more interesting quantity is the specific one-dimensional quantum entropy, which turns out to be universal and will prove to be the right quantity for a thermodynamic monotonicity theorem.

Let us then consider the system on a cylinder, equivalent to finite temperature field theory. The partition function is $Z=\mathrm{Tr}\, e^{-\beta H}$, which can be represented as a functional integral on $S^1\times\mathbb{R}$ with $\beta=1/T$ the length of the compact dimension. We assume that the specific logarithm of the partition function on a cylinder of width β and length L as $L\to\infty$ can be split into a bulk part and a finite-size correction,[1]

$$\frac{-\ln Z}{L}=\beta\frac{F}{L}=e_0(\Lambda,\lambda)\beta+\frac{C(\beta,\lambda)}{\beta}, \tag{12}$$

where $C(\beta,\lambda)$ is a universal dimensionless function having a finite limit as $\beta\to\infty$. Hence, defining $x=\beta\lambda^{1/y}$ we write $C(\beta,\lambda)$ as a single-variable function, $C(x)$. At a RG fixed point it is proportional to the CFT central charge, $C(0)=-\pi c/6$ [7,8].

One can readily calculate the 1D energy

$$\frac{E}{L}=-\frac{\partial\ln Z/L}{\partial\beta}=e_0+\frac{1}{\beta^2}\left(\beta\frac{\partial C}{\partial\beta}-C\right)=e_0-\frac{1}{\beta^2}\left(C-x\frac{dC}{dx}\right). \tag{13}$$

At zero temperature ($\beta\to\infty$) the system is on its ground state and therefore e_0 represents the specific ground state

[1]This formula has already been proposed for generic dimension d, on the grounds of dimensional analysis [3].

energy whereas the C part is a finite-size effect. From the energy, Eq. (13), we can compute the thermodynamic entropy

$$\frac{S}{L}=\beta\frac{E-F}{L}=-\frac{2}{\beta}\left(C-\frac{x}{2}\frac{dC}{dx}\right), \tag{14}$$

which is universal, since it contains no contribution from e_0. Moreover, the entropy vanishes at zero temperature, in accord with the third law of thermodynamics. The relation between S and C in Eq. (14) implies the proportionality between S and c at the critical point (CP), namely, $S/L=\pi c/(3\beta)$. This is reminiscent of the relation between geometric entropy for a CFT and central charge found in [9].

The theorem of increase of the relative entropy (4) holds on a finite geometry and guarantees that $S_{\rm rel}(\lambda,\beta,\Lambda)$ increases with λ. We calculate the relative entropy substituting $W=-\ln Z/(\beta L)=F/L$ according to Eq. (12):

$$\begin{aligned}S_{\rm rel}(\lambda,\beta,\Lambda)&=W(\lambda,\beta,\Lambda)-W(0,\beta,\Lambda)-\lambda\frac{\partial W(\lambda,\beta,\Lambda)}{\partial\lambda}\\&=S_{\rm rel}(\lambda,\Lambda)+\frac{1}{\beta^2}\left(C(x)-C(0)-\lambda\frac{\partial C}{\partial\lambda}\right)\\&=S_{\rm rel}(\lambda,\Lambda)+\frac{\pi c}{6\beta^2}+\frac{1}{\beta^2}\left(C-\frac{x}{y}\frac{dC}{dx}\right),\end{aligned} \tag{15}$$

where $S_{\rm rel}(\lambda,\Lambda)=\lim_{\beta\to\infty}S_{\rm rel}(\lambda,\beta,\Lambda)$ is the bulk relative entropy. If this entropy is universal we have shown that it takes the form $S_{\rm rel}(\lambda)=B\lambda^{2/y}$. Then the presence of the scale β allows us to define a dimensionless relative entropy,

$$\mathcal{C}(x)=\beta^2S_{\rm rel}(\lambda,\beta)=\frac{\pi c}{6}+Bx^2+\left(C-\frac{x}{y}\frac{dC}{dx}\right). \tag{16}$$

In terms of the monotonicity theorem adopts a dimensionless form,

$$x\frac{d\mathcal{C}}{dx}=\frac{\beta^2}{y}\int d^2z\langle:\Theta(z)::\Theta(0):\rangle. \tag{17}$$

Since derivatives with respect to x are equivalent to derivatives with respect to β, $\mathcal{C}$ embodies RG irreversibility, in the manner of Zamolodchikov's theorem [10]. Although $\mathcal{C}(0)=0$, we can redefine it such that it is proportional to the central charge c at the CP by substracting the constant term $\pi c/6$ from both sides of Eq. (16), enhancing the similarity with Zamolodchikov's c function. We could say that it also plays the role of an off-critical "central charge." From Eq. (16) it is clear that $\mathcal{C}(x)$ has a bulk part proportional to x^2 and a finite-size correction, expressed in terms of $C(x)$. As $x\to\infty$, $C(x)$ tends to a finite limit and so does the finite-size part of $\mathcal{C}(x)$. Hence, in the low-temperature limit $x\to\infty$ the bulk part dominates, $\mathcal{C}(x)\approx Bx^2$, so that $\mathcal{C}(x)$ diverges, unless $B=0$.

If the relative entropy is not universal, we can nevertheless define a dimensionless relative entropy but then as a function of two variables, namely, $\mathcal{C}(x,\beta\Lambda)$. Since we must have that $\beta\Lambda\gg 1$, monotonicity still holds for moderate values of x.

In parallel with the relative entropy, now it is natural to consider the behavior of the absolute 1D quantum entropy S with respect to β:

$$\frac{\partial S}{\partial\beta}=\frac{\partial}{\partial\beta}(\beta E-\beta F)=\beta\frac{\partial E}{\partial\beta}=\beta\frac{\partial^2(\beta F)}{\partial\beta^2}. \tag{18}$$

We have again monotonicity, for βF is a convex function of β, as deduced from the expression of its second derivative as the average $-\langle(H-\langle H\rangle)^2\rangle$. Unlike the monotonicity of the 2D relative entropy, Eq. (4), here H is the *total* Hamiltonian, that is, including the critical part H^* [e.g., the kinetic term $H^*=\int(\partial\phi)^2/2$]. This monotonicity is in principle unrelated to the monotonicity of $S_{\rm rel}$ with respect to the coupling constant. Thus it allows us to define a different monotonic dimensionless function,

$$\tilde{\mathcal{C}}(x)=\frac{S}{L\lambda^{1/y}}=-\frac{2C}{x}+\frac{dC}{dx}. \tag{19}$$

At the critical point $S/L=\pi c/(3\beta)$, implying that $\tilde{\mathcal{C}}(x)$ diverges linearly at $x=0$, whereas $\mathcal{C}(0)=0$. On the other hand, as the temperature is lowered $(x\to\infty)$ $\tilde{\mathcal{C}}(x)$ vanishes.

We see that there are several quantities that can be related at a RG fixed point but have a different physical origin and clearly differ away from it. The quantity which has been more prominent in the literature is the finite-size correction to the free energy $C(x)$. It was proposed as a monotonic function in Refs. [11,3]. It has sometimes been related to the dimensionless quantity $3\beta^2\langle T\rangle/\pi$, which gives the central charge c at the fixed point. To clarify this question, we prove here that this expectation value is instead related to the 1D quantum entropy S, showing on the way the general relation of expectation values of stress tensor components with thermodynamic quantities. Let us consider the expectation values of the complex components of the stress tensor, $\Theta:=T_a^a$ and $T:=T_{11}-T_{22}-2iT_{12}$, on the cylinder geometry. We have the equalities

$$E/L=\langle T_{11}\rangle,\quad F/L=\langle T_{22}\rangle,$$

which come from the definition of the stress tensor and are completely general. One deduces that

$$S/(L\beta)=\langle T_{11}-T_{22}\rangle=\langle T\rangle, \tag{20}$$

which generalizes the standard relation $F/L=(-1/2)\langle T\rangle$ [7,8], actually only valid at the fixed point. Therefore, the monotonic function $\tilde{\mathcal{C}}=(\beta/\lambda^{1/y})\langle T\rangle$ is the one related with the expectation value $\langle T\rangle$. In fact,

$$\langle T\rangle=-\frac{2}{\beta^2}\left(C-\frac{x}{2}\frac{dC}{dx}\right), \tag{21}$$

containing the term $x\,dC/dx$, which vanishes at the fixed point.

The coupling may have been understood in all the above as taking the statistical system off criticality. However, nothing in the arguments above requires that for $\lambda \neq 0$ the correlation length be finite. Actually, we can well envisage the situation in which a coupling of a system at a multicritical point is such that the coupled system is still critical. This situation is described in field theory as a *massless flow*, which causes the system to undergo a crossover ending at another non-trivial fixed point of the RG. However, we will only study here massive flows, with a finite correlation length and hence a mass parameter m. In free theories, as considered in [2], m is the mass of the particles, bosons or fermions. In interacting theories there is a mass spectrum, which can be deduced from the long distance behavior of the two-point correlation function. We will be considering theories soluble with the TBA, which directly renders the mass spectrum. One may then select the lowest mass of the TBA spectrum and define the dimensionless variable as $x=\beta m$. In massive theories the function $C(x)$ vanishes exponentially as $x\to\infty$, and so do the entropic functions $\mathcal{C}(x)$ and $\tilde{\mathcal{C}}(x)$.

III. FINITE SIZE THERMODYNAMICS FOR FREE FIELD MODELS

A. The continuum limit of the lattice Gaussian and Ising models

The 2D Gaussian model on a square lattice with thermal coupling constant ß is exactly soluble,[2] yielding

$$W(ß)=\frac{1}{2}\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\ln[1-2\,ß(\cos k_x+\cos k_y)] \quad (22)$$

per site [12]. It has a CP for $ß_c=1/4$. The continuum limit is performed by redefining wave vectors as $k=ap$, a being the lattice spacing, and considering W per unit area. Although k belongs to a Brillouin zone, in the continuum limit p runs over the domain, $-\Lambda<p_x,p_y<\Lambda$ $(\Lambda\sim\pi/a)$, which becomes the entire plane as $\Lambda\to\infty$. In the continuum limit we have the field theory of free bosonic particles of mass m such that

$$m^2a^2=16(ß_c-ß), \quad (23)$$

so that $y=2$ and the coupling is $r=m^2$, omitting an irrelevant proportionality constant.

The relative entropy per unit area of the Gaussian model is best calculated with field theory methods, for example, using dimensional regularization [1]. It can be expressed as

$$S_{\rm rel}=\frac{\Gamma[(4-d)/2]}{(4\pi)^{d/2}d}r^{d/2},$$

which in $d=2$ yields

$$S_{\rm rel}=\frac{r}{8\pi}. \quad (24)$$

However, it is more illustrative to start with the expression of the cutoff logarithm of the partition function per unit area

$$W[r,\Lambda]\equiv-\ln Z[r,\Lambda]=\frac{1}{2}\int_0^\Lambda\frac{d^2p}{(2\pi)^2}\ln\frac{p^2+r}{\Lambda^2}, \quad (25)$$

which can be integrated exactly and yields

$$W[r,\Lambda]=\frac{1}{2\pi}\frac{\Lambda^2}{4}\left\{-1-\frac{r}{\Lambda^2}\ln\frac{r}{\Lambda^2}+\left(1+\frac{r}{\Lambda^2}\right)\times\ln\left(1+\frac{r}{\Lambda^2}\right)\right\}. \quad (26)$$

Naturally, it is UV divergent. For large Λ it becomes

$$W[r,\Lambda]=\frac{1}{8\pi}\left\{-\Lambda^2+r\ln\frac{\Lambda^2}{r}+r+\mathrm{O}(\Lambda^{-2})\right\}, \quad (27)$$

exhibiting a quadratic and a logarithmic divergence.

Recalling the discussion on the general structure of W in the previous section, we see that we are in the case of logarithmic corrections to a pure scaling form. Nevertheless, it is easily derived that in the present case all the divergences in Λ cancel in the relative entropy, yielding in the infinite cutoff limit

$$S_{\rm rel}=W(r)-W(0)-r\frac{dW}{dr}=\frac{r}{8\pi}, \quad (28)$$

in accord with the dimensional regularization result. To be precise, in this cutoff regularization the quadratic divergence cancels by the substraction of $W(0)$ and the logarithmic divergence by the Legendre transform, while in dimensional (or analytic) regularization the quadratic divergence does not appear but there is a pole in W, equivalent to the logarithmic term in Λ, that cancels in $S_{\rm rel}$.

Another interesting and exactly soluble example is the 2D Ising model on a square lattice, with

$$W(ß)=-\frac{1}{2}\int_{-\pi}^{\pi}\frac{d^2k}{(2\pi)^2}\ln[\cosh^2(2\,ß)-\sinh(2\,ß)\times(\cos k_x+\cos k_y)]-\ln 2 \quad (29)$$

per lattice site [13]. The critical point occurs for the value of ß such that the argument of the logarithm vanishes when $k_x,k_y\to 0$, namely, when

$$f(ß):=\cosh^2(2\,ß)-2\,\sinh(2\,ß)=0, \quad (30)$$

[2]The thermal coupling constant of a 2D lattice model is of course the inverse 2D *temperature*. Since we shall be using throughout the corresponding 1D temperature, $\beta=1/T$, we avoid mentioning a 2D temperature and use the notation ß for the 2D coupling constant.

with solution $ß_c = \mathrm{arcsinh}(1)/2 = \ln(\sqrt{2}+1)/2 \approx 0.440687$. The expansion of $f(ß)$ near $ß_c$ yields

$$f(ß) = 8(ß-ß_c)^2 + 8\sqrt{2}(ß-ß_c)^3 + \mathrm{O}(ß-ß_c)^4.$$

If we define m by

$$m^2 a^2 = 16(ß-ß_c)^2 \tag{31}$$

and redefine the momentum k as $k = ap$, with a the lattice spacing, we obtain near the critical point that

$$W(ß) = -\frac{a^2}{2}\int_{-\pi/a}^{\pi/a} \frac{d^2p}{(2\pi)^2} \ln[(p^2+m^2)a^2/2]. \tag{32}$$

In other words, the corresponding field theory is described by a W per unit area given by minus that in Eq. (25). It agrees with the well known description of this model in terms of a free Majorana fermion theory. However, the relative entropy is not minus that of the Gaussian model, since now the coupling constant is proportional to m instead of being $r = m^2$, since Eq. (31) implies that $y = 1$. One obtains

$$S_{\mathrm{rel}}(r) = W(r) - W(0) - m\frac{dW}{dm} = -\frac{m^2}{8\pi}\left(1 + \ln\frac{m^2}{\Lambda^2}\right). \tag{33}$$

It diverges in the limit of infinite cutoff, which cannot be removed to obtain a universal value. Nevertheless, for $m \ll \Lambda$, where the field theory makes sense, the relative entropy in Eq. (33) is monotonic. This is not surprising because it coincides near the CP with the exact relative entropy of the square-lattice Ising model, represented in [1].

B. Derivation of finite-size quantities

The expression of W on a lattice of finite size $L_1 \times L_2$ is obtained by replacing the integrals in Eq. (22) or Eq. (29) with sums over discrete momenta with step $2\pi/L_1$ and $2\pi/L_2$. When $L_1, L_2 \gg a$ we approach the thermodynamic limit and the sums become integrals plus some finite-size corrections. However, the double limit $L_1, L_2 \to \infty$ is complicated to study, and it is better to consider finite-size effects only in one direction. Alternatively, it is sometimes convenient to consider a non-symmetrical lattice with different coupling constants in the horizontal and vertical directions. In particular, the quantum 1D Gaussian or Ising models on a chain of sites can be obtained as the extreme anisotropic limit of the 2D Gaussian or Ising models [14,15]. The CP is still where the correlation length diverges but now correlations are calculated only between horizontal spins. Now the partition function is $Z = \mathrm{Tr}\, e^{-\beta H}$, which can be represented in the continuum limit as a functional integral on $\mathbb{R}\times S^1$ with $\beta = 1/T$ the length of the compact dimension. It may be good to recall that here β has no relation with the coupling constant, unlike $ß$ in the classical 2D models above, and plays instead the role of RG parameter.

Let us first consider the specific ground-state energy of the 1D lattice system. For the Gaussian model in the continuum limit it is given by

$$e_0 = \int_0^\Lambda \frac{dp}{2\pi}\sqrt{p^2+m^2}$$
$$= \frac{1}{4\pi}[\Lambda\sqrt{\Lambda^2+m^2} - m^2\log m + m^2\log(\Lambda + \sqrt{\Lambda^2+m^2})]. \tag{34}$$

When $\Lambda \to \infty$ the leading terms are

$$e_0 = \frac{1}{8\pi}\left\{2\Lambda^2 + 2m^2\ln\frac{2\Lambda}{m} + m^2 + \mathrm{O}(\Lambda^{-2})\right\}. \tag{35}$$

It is quadratically divergent. The logarithmic divergence is universal, that is, independent of the regularization method, and corresponds to the logarithmic divergence of $W[r]$, Eq. (27). However, note that the Λ^2 and m^2 terms are nonuniversal and their coefficients change from -1 and 1 in Eq. (27) to 2 and $2\ln 2 + 1$, respectively. We shall show below that the specific ground-state energy of the Ising model is given by the same formula, except for an overall minus sign, in agreement with its free energy in Eq. (32).

In order to compute finite-size effects we first consider the behavior of the ground state energy on a segment of length L at zero temperature, connected with the well-known Casimir effect. It provides the finite-size correction $C(x)$ that we need. To see this, let us take the specific W, according to Eq. (12),

$$\frac{-\ln Z}{L\beta} = e_0(\Lambda, m) + \frac{C(\beta m)}{\beta^2}, \tag{36}$$

and interchange the roles of L and β: we have that at low-temperature

$$\frac{-\ln Z}{L\beta} = e_0(\Lambda, m) + \frac{C(Lm)}{L^2}. \tag{37}$$

Since $E = -\partial \ln Z/\partial\beta$, this formula also gives the specific ground-state energy on a segment of length L at zero temperature. The term proportional to L is the bulk ground-state energy considered above and the finite-size correction is the Casimir energy. In other words, the Casimir energy provides the universal function $C(m\beta)$.

1. Direct calculation of the Casimir energy

The Gaussian model with periodic boundary condition has a ground state energy

$$E_0 = \frac{1}{2}\sum_{n=-\infty}^{\infty}\sqrt{\left(\frac{2\pi n}{L}\right)^2 + m^2} = \frac{m}{2} + \sum_{n=1}^{\infty}\sqrt{\left(\frac{2\pi n}{L}\right)^2 + m^2}. \tag{38}$$

When $L \to \infty$ one recovers the continuum integral of Eq. (34). However, if we are interested in the vicinity of the critical theory we may consider the limit $L \to \infty$ but with mL small. This limit is known to provide a method to calculate the CFT central charge. Then the series can be evaluated by expanding the square root in powers of mL and interchanging the sums. We obtain

$$E_0=\frac{m}{2}+\frac{2\pi}{L}\sum_{l=0}^{\infty}\binom{1/2}{l}\left(\frac{mL}{2\pi}\right)^{2l}\zeta(2l-1)$$
$$=\frac{m}{2}+\frac{2\pi}{L}\left[\zeta(-1)+\frac{1}{2}\left(\frac{mL}{2\pi}\right)^{2}\zeta(1)-\frac{1}{8}\left(\frac{mL}{2\pi}\right)^{4}\zeta(3)+\cdots\right]. \quad (39)$$

Since E_0 is divergent, the result amounts to a *zeta-function* regularization of it. The first term, with $\zeta(-1)=-1/12$, yields $c=1$. The next term, proportional to L, accounts for the bulk term e_0. Despite the regularization, it is still divergent, since $\zeta(z)$ has a simple pole at $z=1$. This pole is equivalent to a logarithmic divergence in regularizations with a UV cutoff, as generally happens when comparing analytic with cutoff regularizations. The way to realize it for this case is to restrict the sum $\zeta(1)=\Sigma_1^{\infty}(1/n)$ up to some large number N. Then

$$\zeta(1)\simeq\sum_{1}^{N}\frac{1}{n}=\log N+\gamma+O\left(\frac{1}{N}\right). \quad (40)$$

The connection with the regularization provided by considering the system on a discrete chain of spacing a can be made taking $N=L/a$, the number of sites. An alternative procedure of regularization is first to segregate the divergent bulk part, with the form (34), from the finite-size corrections by using the Euler-MacLaurin formula (Appendix A).

The Ising model on a closed chain is amenable to an analogous treatment. Its ground state energy for $T>T_c$ is like Eq. (38) but with negative sign and with wave numbers that are odd powers of π/L [16]

$$E_0=-\frac{1}{2}\sum_{n=-\infty}^{\infty}\sqrt{\left(\frac{(2n+1)\pi}{L}\right)^2+m^2}$$
$$=-\sum_{n=0}^{\infty}\sqrt{\left(\frac{(2n+1)\pi}{L}\right)^2+m^2}. \quad (41)$$

The expansion in powers of mL yields

$$E_0=\frac{2\pi}{L}\sum_{l=0}^{\infty}\binom{1/2}{l}\left(\frac{mL}{2\pi}\right)^{2l}(1-2^{2l-1})\zeta(2l-1)$$
$$=\frac{2\pi}{L}\left[\frac{1}{2}\zeta(-1)-\frac{1}{2}\left(\frac{mL}{2\pi}\right)^{2}\zeta(1)+\frac{7}{8}\left(\frac{mL}{2\pi}\right)^{4}\zeta(3)+\cdots\right]. \quad (42)$$

The central charge is $c=1/2$ and the bulk term is minus that of the Gaussian model.

2. Thermodynamic calculation of finite-size effects

Now we concern ourselves with the deviation of the energy at non-zero temperature from the ground-state energy or, in other words, the finite-size β correction to the free energy. For the Gaussian model it can be expressed as the free energy of the ideal Bose gas constituted by the elementary excitations,

$$\beta\frac{F}{L}=e_0\beta+\int_{-\infty}^{\infty}\frac{dp}{2\pi}\ln(1-e^{-\beta\epsilon(p)}), \quad (43)$$

where the one-particle energy is $\epsilon(p)=\sqrt{p^2+m^2}$. This formula can also be obtained by an explicit calculation of the finite-size corrections in the 2D lattice model [17]. When $m=0$ it can be used to calculate the central charge [8]. Nevertheless, an expansion in powers of m^2 is not advisable: The ensuing integral at the next order is IR divergent; that is to say, the expression (43) is nonanalytic at $m^2=0$. Fortunately, the integral can be computed by changing the integration variable to ϵ and expanding the logarithm in powers of $e^{-\beta\epsilon}$. We obtain

$$\beta\frac{F}{L}=e_0\beta-\frac{m}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}K_1(nm\beta), \quad (44)$$

where $K_1(x)$ is a modified Bessel function of the second kind.

We now define the dimensionless quantity $x=m\beta$ and perform a small-x expansion, which yields

$$\beta\frac{F}{L}=e_0\beta-\frac{1}{\pi\beta}\left\{\zeta(2)-\frac{\pi}{2}x-\frac{x^2}{4}\left(\ln\frac{x}{4\pi}+\gamma-\frac{1}{2}\right)+O(x^4)\right\} \quad (45)$$

$$=e_0\beta-\frac{\zeta(2)}{\pi\beta}+\frac{m}{2}+\beta\frac{m^2}{4\pi}\left(\ln\frac{m\beta}{4\pi}+\gamma-\frac{1}{2}\right)+O(m^4), \quad (46)$$

where $O(x^4)$ denotes an analytic remainder of fourth order. The term with $\zeta(2)=\pi^2/6$ gives the usual $m=0$ part and central charge $c=1$. The non-analyticity in m^2 of the integral for F (43) manifests itself in the appearance of the $m/2$ and logarithmic terms. The former also appears as a zero mode contribution in the Casimir energy (38). The full x-power series is obtained as follows: The series of Bessel functions (44) is slowly convergent and one may apply to it a Mellin transform to convert it into a rapidly convergent one [18]. Fortunately, its Mellin transform is a power series of x. Furthermore, after replacing β with L it coincides with Eq. (39) from $l=2$ onwards. Of course, this should be expected on the grounds of symmetry on a torus under interchange of its sides L and β, that is, modular symmetry. Incidentally, the exact free energy on the torus can be computed with some more sophisticated mathematics [19,17]. Its two cylinder limits yield Eq. (39) or Eq. (46). Nevertheless, to show the modular invariance of the exact expression on the torus is not easy: It can be done performing its expansion in powers of the dimensionless modular invariant parameter m^2A, with A the area of the torus, but it is very laborious.

It is interesting to relate the logarithmic term in the expansion of F (46) with the Casimir energy calculated in the previous subsection. It was remarked there that the $\zeta(1)$ divergence can be interpreted as a logarithmic divergence in the cutoff. Adding the logarithmic terms in e_0 and $C(x)$ one obtains

$$\frac{x^2}{4\pi}\left(\ln\frac{\Lambda\beta}{2\pi}+\gamma\right)\approx\frac{x^2}{4\pi}\log N, \tag{47}$$

where N is the number of lattice sites in the time direction. We see that it is equivalent to the $\zeta(1)$ divergence (40) under the interchange $\beta\leftrightarrow L$.

Now we can calculate the specific entropy

$$\frac{S}{L}=+\frac{\pi}{3\beta}-\frac{1}{2}m+\beta\frac{m^2}{4\pi}+\mathrm{O}(m^4). \tag{48}$$

It has no IR singularity at $m\to 0$ as opposed to the free energy or the energy. The last term is just twice the relative entropy of a box of size $L\beta$, Eq. (28), times β. In more generality, for the Gaussian model there is a relation between both types of entropy, namely,

$$\begin{aligned}S_{\mathrm{rel}}(r,\beta)&=W(r,\beta)-W(0,\beta)-r\frac{\partial W(r,\beta)}{\partial r}\\&=S_{\mathrm{rel}}(r)+\frac{1}{\beta^2}\left(C(x)-C(0)-r\frac{\partial C}{\partial r}\right)\\&=\frac{r}{8\pi}-\frac{S}{2L\beta}+\frac{\pi}{6\beta^2}.\end{aligned} \tag{49}$$

For other models there is no direct relation between the 1D entropy and the 2D relative entropy.

The derivation of thermodynamic quantities for the Ising model is analogous. The free energy is that of an ideal Fermi gas

$$\begin{aligned}\beta\frac{F}{L}&=e_0\beta-\int_{-\infty}^{\infty}\frac{dp}{2\pi}\ln(1+e^{-\beta\epsilon(p)})\\&=e_0\beta+\frac{m}{\pi}\sum_{n=1}^{\infty}\frac{(-)^n}{n}K_1(nm\beta)\end{aligned} \tag{50}$$

where the one-particle spectrum close to the critical point is again $\epsilon(p)=\sqrt{p^2+m^2}$. This integral is computed like the bosonic one. The small-m expansion yields

$$\begin{aligned}\beta\frac{F}{L}&=e_0\beta-\frac{1}{\pi\beta}\left\{\frac{1}{2}\zeta(2)+\frac{x^2}{4}\left(\ln\frac{x}{4\pi}+\gamma-\frac{1}{2}\right)+\mathrm{O}(x^4)\right\}\\&=e_0\beta-\frac{\zeta(2)}{2\pi\beta}-\beta\frac{m^2}{4\pi}\left(\ln\frac{m\beta}{4\pi}+\gamma-\frac{1}{2}\right)+\mathrm{O}(m^4).\end{aligned} \tag{51}$$

As well as for the Gaussian model, it is possible to obtain the x-power series by the Mellin transform of the series of Bessel functions (50). Similarly, after replacing β with L it coincides with Eq. (42) from $l=2$ onwards.

In this case the specific entropy is

$$\frac{S}{L}=+\frac{\pi}{6\beta}-\beta\frac{m^2}{4\pi}+\mathrm{O}(m^4). \tag{52}$$

For the Ising model the relative entropy is related to the energy, instead of S:

$$\begin{aligned}S_{\mathrm{rel}}(m,\beta)&=W(m,\beta)-W(0,\beta)-m\frac{\partial W(m,\beta)}{\partial m}\\&=S_{\mathrm{rel}}(m)+\frac{1}{\beta^2}\left(C(x)-C(0)-m\frac{\partial C}{\partial m}\right)\\&=-\frac{m^2}{8\pi}\left(1+\ln\frac{m^2}{\Lambda^2}\right)-\left(\frac{E}{L}-e_0\right)+\frac{\pi}{12\beta^2}.\end{aligned} \tag{53}$$

We see that for free theories we can derive explicit formulas for the free energy—and hence for the entropy—as well as perturbative expansions. Moreover, both the 1D entropy and the 2D relative entropy give rise to monotonic central charges, as we proceed to study, introducing before for convenience the stress tensor.

C. Expectation values of the stress tensor and entropic C theorems

The previous section has shown the calculation of finite-size corrections for various quantities of free models with concepts pertaining to the 1D quantum theories, namely, the lattice Casimir energy or the statistics of quantum gases. The same results can be attained with the use of 2D Green function techniques, through the calculation of expectation values of the complex components of the stress tensor, taking into account their relation with thermodynamic quantities already remarked. We shall rewrite the monotonic functions in a suitable way to confirm these relations, hence explaining the structure of those functions. We thus start with the expressions of the expectation values, $\Theta:=T_a^a$ and $T:=T_{11}-T_{22}-2iT_{12}$, in the cylinder geometry, as derived by 2D Green function techniques (Appendix B):

$$\langle\Theta\rangle=\pm\frac{m^2}{2\pi}\left(K_0(0)+2\sum_{n=1}^{\infty}(\pm)^nK_0(nm\beta)\right), \tag{54}$$

$$\langle T\rangle=\pm\frac{m^2}{2\pi}\left(K_2(0)+2\sum_{n=1}^{\infty}(\pm)^nK_2(nm\beta)\right), \tag{55}$$

with the same sign convention as before. The modified Bessel functions are divergent at zero, namely, $K_0(0)$ is logarithmic divergent and $K_2(0)$ is quadratically divergent. These are UV divergences, like those already considered for W, which can be removed by normal order.

Using the recursion relations satisfied by the Bessel functions we can write the free energy (44) or (50) as

$$\beta\frac{F}{L}=e_0\beta\mp\frac{m^2\beta}{2\pi}\sum_{n=1}^{\infty}(\pm)^n[K_2(nm\beta)-K_0(nm\beta)]$$

$$=-\frac{\beta}{2}\langle T-\Theta\rangle=\beta\langle T_{22}\rangle, \tag{56}$$

showing its relation with the expectation values of the components of the stress tensor, an example of the relations obtained at the end of Sec. I. Notice that it implies a definite form for e_0, to be compared with Eq. (27) or Eq. (35). (See Appendix B.)

Similarly, we can calculate

$$\frac{\partial W}{\partial r}=\frac{\partial e_0}{\partial r}\pm\frac{1}{2\pi}\sum_{n=1}^{\infty}(\pm)^nK_0(nm\beta)=\frac{1}{2r}\langle\Theta\rangle, \tag{57}$$

$$\frac{E}{L}=e_0\pm\frac{m^2}{2\pi}\sum_{n=1}^{\infty}(\pm)^n[K_2(nm\beta)+K_0(nm\beta)]$$

$$=\frac{1}{2}\langle T+\Theta\rangle=\langle T_{11}\rangle. \tag{58}$$

The first equation is just a particular case of the expression of the derivative of W with respect to r as the expectation value of the ''crossover part'' of the action [1], since Θ is proportional to it. Having the values of W and its derivative available we further obtain for the Gaussian model that

$$S_{\rm rel}(r,\beta)=S_{\rm rel}(r)+\frac{\pi}{6\beta^2}-\frac{r}{2\pi}\sum_{n=1}^{\infty}K_2(nm\beta)$$

$$=S_{\rm rel}(r)+\frac{\pi}{6\beta^2}-\frac{1}{2}\langle:T:\rangle \tag{59}$$

and for the Ising model that

$$S_{\rm rel}(r,\beta)=S_{\rm rel}(r)+\frac{\pi}{12\beta^2}+\frac{r}{2\pi}\sum_{n=1}^{\infty}(-)^n$$

$$\times[K_2(nm\beta)+K_0(nm\beta)]$$

$$=S_{\rm rel}(r)+\frac{\pi}{12\beta^2}-\frac{1}{2}\langle:T+\Theta:\rangle. \tag{60}$$

We have substituted $S_{\rm rel}(r)$ for $e_0(r)-e_0(0)-r\partial e_0(r)/\partial r$ and $e_0(r)-e_0(0)-m\partial e_0/\partial m$, respectively. One obtains the finite expectation values of normal-ordered stress tensor components owing to the subtraction of $W(0,\beta)$.[3] Using the connection between the 1D entropy and the stress tensor, $S/(L\beta)=\langle:T:\rangle$, pointed out at the end of Sec. II, one can directly obtain S.

Thus the dimensionless relative entropies for the Gaussian or Ising models, respectively, are

$$\mathcal{C}(x)=\frac{\pi}{6}+\frac{x^2}{8\pi}-\frac{x^2}{2\pi}\sum_{n=1}^{\infty}K_2(nx), \tag{61}$$

$$\mathcal{C}(x)=\frac{\pi}{12}-\frac{x^2}{8\pi}\left[1+\ln\frac{x^2}{(\Lambda\beta)^2}\right]+\frac{x^2}{2\pi}\sum_{n=1}^{\infty}(-)^n$$

$$\times[K_2(nx)+K_0(nx)]. \tag{62}$$

The other monotonic quantity, $\tilde{\mathcal{C}}=S/(Lm)$, is essentially common and can be written as

$$\tilde{\mathcal{C}}(x)=\frac{x}{\pi}\sum_{n=1}^{\infty}(\pm)^{n+1}K_2(nx), \tag{63}$$

which is, of course, $\tilde{\mathcal{C}}=(\beta/m)\langle:T:\rangle$, according to the expression of $\langle T\rangle$ (55). Series expansions of C, $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are derived from Eq. (46) or Eq. (51). Both $\mathcal{C}$ and $\tilde{\mathcal{C}}$ are plotted in Fig. 1. The Ising-model $\mathcal{C}$ is for the value $(\Lambda\beta)^2=10000$. It is useful to recall that in general $(\Lambda\beta)^2\sim N$, the number of 2D lattice sites in a box of side β.

IV. INTERACTING MODELS

For interacting models, the free energy is in principle not available in closed form. Nevertheless, one can perform its perturbative expansion. In two dimensions one can take advantage of the information provided by the methods of conformal field theory (CFT), namely, the non-perturbative dimensions and correlations of fields at the critical point; then one speaks of *deformed CFT's*. For example, one can perform a perturbative expansion around the critical point. This approach, called *conformal perturbation theory*, is well suited for the calculation of an entropy relative to the critical point; the perturbation parameter is $\lambda\beta^y$. The expansions in Eqs. (46) and (51) are instances of it and can be obtained from the respective $c=1$ or $c=1/2$ CFT [19]. The logarithmic term in them is due to UV divergencies. The analysis in UV divergences is done by examining the behavior of integrals of correlators for coincident points. If $1<y<2$ there are no UV divergences in the perturbative expansion and, furthermore, this expansion is arguably convergent [5,6]. In contrast, when $y<1$ a finite number of terms will diverge. The condition $1<y<2$ agrees with the non-perturbative regularity condition for the relative entropy found before. Conformal perturbation theory is considerably powerful but, at any rate, the perturbative expansion only converges for a limited range of $\lambda\beta^y$, while we are interested in the behavior of thermodynamical quantities over the entire range of the coupling constant.

Some 2D models are partially soluble with the thermodynamic Bethe ansatz (TBA) [20]. In particular, models for which the interaction is of purely statistical nature lend themselves to a derivation of closed expressions for the free energy and the entropy similar to the ones for free field models, albeit more complicated. Hence, the entropic *C* theorems can

[3] One must be careful when evaluating $\partial e_0(r)/\partial r$. Since $e_0(r)=\mp(m^2/2\pi)[K_2(0)-K_0(0)]$ (Appendix B), it may seem that $e_0(r)-r[\partial e_0(r)/\partial r]\equiv 0$. However, $K_2(0)$ and $K_0(0)$ contain an m dependence, because of regularization.

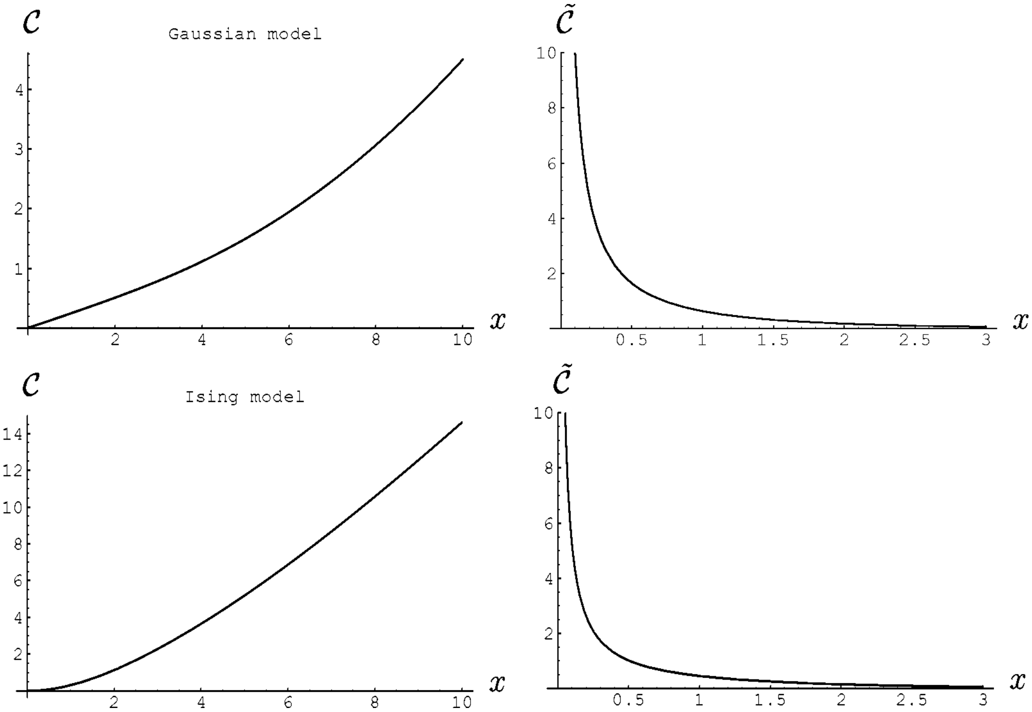

FIG. 1. $\mathcal{C}$ and $\tilde{\mathcal{C}}$ for the Gaussian and Ising models. In the latter model $\mathcal{C}$ is for $\Lambda\beta=100$.

be explicitly verified for them. The Bethe ansatz assumes a factorized form for the wave functions and hence an expression for the energy as a sum of contributions of independent quasi-particle levels, though these quasi-particles have nontrivial exchange properties. To determine the structure of these levels is a complicated business but it dramatically simplifies in the thermodynamic limit, constituting the basis of the TBA. This method is nonperturbative in nature and provides thermodynamical quantities over the entire range of the coupling constant. In principle, the TBA yields, further to the universal finite-size correction function $C(x)$, a contribution proportional to x^2, which is interpreted as a UV-finite bulk term and therefore has been called *universal bulk term* [21,5]. Therefore, in this section we redefine $C(x)$ to include this universal bulk term.

In spite of the virtues of the TBA approach, the TBA equations themselves are by no means easy to solve and it is customary to resort to numerical calculation to obtain the coefficients of the series expansion in βm. In this sense the TBA approach is not superior to perturbation theory in its region of convergence, with which one obtains analytic expressions for these coefficients. It is only in the case of purely statistical interaction where the TBA approach is definitely superior, for one can then solve the TBA equations algebraically. Thus we treat this case first. It applies to models of Calogero-Sutherland type, which represent the dynamics of spinons or other non-interacting particles with fractional statistics. We can calculate the entropy and check that is monotonic with respect to β. However, it is beyond our means to calculate expectation values of the stress tensor, since the expression of the stress tensor is not available. In second place, we shall treat the general interacting integrable case by the numerical solution of the TBA equations and compare with the results of conformal perturbation theory.

Last, there is an alternative to conformal perturbation theory or the TBA that can be applied to any model, namely, numerical finite-size scaling on a chain [22]. Like the TBA, by its own nature it is not limited to a restricted range of βm. However, the numerical calculations required to obtain similar accuracy to that of the TBA are prohibitive in practice. Thus this brute-force method is not actually effective to compute *off-critical* quantities and no use will be made of it here.

A. Models with purely statistical interaction

The role of fractional statistics in condensed matter physics, as a generalization of the regular bosonic or fermionic symmetry properties under particle exchange, has been long recognized [23]. Its modern version has given rise to the concept of anyons. As it happens, this type of statistics leads to highly non-trivial correlations between particles which are difficult to disentangle and indeed constitute what has been called statistical interaction. The form of this interaction can be best realized by transforming the particles to standard fermions or bosons with a peculiar interaction. Models with purely statistical interaction are usually referred to as generalized ideal gases [24,25]. It is customary to consider the free particles as fermions and parametrize the statistics by a number g, such that the maximum number of particles that can fit in a single fermion momemtum level is $1/g$. For no statistical interaction $g=1$. If $g=1/n$ the single-fermion levels can accomodate n particles and in the limit $n\to\infty$ the statistics becomes bosonic. Some models with apparently complex interactions can be transformed into generalized ideal gases, as occurs for the Calogero-Sutherland models [26] or their lattice version [27].

Therefore, models with purely statistical interaction are interesting systems and, in addition, sufficiently complex to be a suitable benchmark for our irreversibility theorem. We consider the spin-$SU(2)$ level-one Wess-Zumino-Witten Lagrangian in the bosonic representation,

$$H=\int dx[(\partial_t\phi)^2+(\partial_x\phi)^2]. \tag{64}$$

The spinon field is defined in terms of the bosonic field as $\psi^{\pm}=\exp(\pm(i/\sqrt{2})\phi)$, where the sign stands for spinon polarization [27–30]. It is a free theory, which can also be expressed as a free fermionic theory, but we are going to do some non-trivial manipulations on it. First, we ''simplify'' the model by keeping just one spinon polarization, ''+'' say. The physical way to achieve this is to introduce a very strong magnetic field. Now we have a *semionic* CFT with central charge $c=3/5$ [31,32], which is certainly an interacting theory. However, its partition function is known, being the total partition function of the $SU(2)$ level-one Wess-Zumino-Witten model restricted to vanishing fugacity of ''−'' spinons, $z_-=0$, [29]. The thermodynamic quantities can be obtained with the help of the TBA.[4]

The second change consists of the addition of some tunable coupling, which perturbs the model away from criticality and allows one to probe the behavior of the entropy. If we impose that the interaction remains purely statistical, the only possibility is to give mass to the semions: We can replace the dispersion relation $\epsilon(p)=|p|$ with $\epsilon(p)=\sqrt{p^2+m^2}$. We assume that this perturbation fulfills the conditions for the application of the monotonicity theorem (4) and we shall see that it is easily implemented within the TBA approach and yields expressions which can be treated by algebraic methods. As a side remark, note that the massive relativistic dispersion relation differs from the nonrelativistic one assumed in the original Calogero-Sutherland model. Therefore, if we want to keep to the physics represented by this model, we must interpret the mass as a parameter unrelated to the real semion mass. For comparison, it can be shown that the 2D Dirac Lagrangian with a mass term appears as a low-energy effective Lagrangian for (nonrelativistic) conducting electrons in one-dimensional metals, but the Dirac mass is actually related to the electric potential ([30] Chap. 13). Regardless of the precise physical interpretation, we will consider the theory of relativistic massive semions as our first interacting field theory to investigate the properties of the entropy.

1. Application of the TBA to the semion gas

The full power of the TBA shows in the calculation of finite size corrections to thermodynamic quantities. One obtains for the critical semion gas [32,33]

$$\begin{aligned}c&=\frac{6\beta}{\pi^2}\int_0^\infty dk\log\left[\frac{2+\zeta^2+\zeta\sqrt{4+\zeta^2}}{2}\right]\\&=\frac{6}{\pi^2}L\left(\frac{\sqrt{5}-1}{2}\right)=\frac{3}{5},\end{aligned}\tag{65}$$

with $\zeta=e^{-\beta\epsilon(k)}$, $\epsilon(k)=|k|$. We give the semions a mass, replacing the dispersion relation $\epsilon(k)=|k|$ with $\epsilon(k)=\sqrt{k^2+m^2}$. Then

$$\begin{aligned}C(\beta m)&=-\frac{\beta}{\pi}\int_0^\infty dk\log\left[\frac{2+\zeta^2+\zeta\sqrt{4+\zeta^2}}{2}\right]\\&=-\frac{2\beta}{\pi}\int_0^\infty dk\,\mathrm{arcsinh}\left[\frac{\zeta}{2}\right]\\&=-\frac{2\beta}{\pi}\int_0^\infty dk\sum_{n=0}^{\infty}(-)^n\frac{(2n)!}{2^{2n}(1+2n)n!^2}\left[\frac{\zeta}{2}\right]^{(2n+1)}\\&=-\frac{2\beta}{\pi}m\sum_{n=0}^{\infty}(-)^n\frac{(2n)!}{2^{4n+1}(1+2n)n!^2}\\&\quad\times K_1[(2n+1)m\beta].\end{aligned}\tag{66}$$

One can easily obtain its perturbative expansion in powers of $m\beta$ by expanding first the modified Bessel function,

$$\begin{aligned}K_1[z]=&\frac{1}{z}+\ln\frac{z}{2}\,\frac{z}{2}\sum_{k=0}^{\infty}\frac{\left(\frac{z^2}{4}\right)^k}{k!(k+1)!}-\frac{z}{4}\\&\times\sum_{k=0}^{\infty}[\psi(k+1)+\psi(k+2)]\frac{\left(\frac{z^2}{4}\right)^k}{k!(k+1)!}\end{aligned}\tag{67}$$

$$=\frac{1}{z}+\frac{z}{2}\left(\ln\frac{z}{2}+\gamma-\frac{1}{2}\right)+\mathrm{O}(z)^2,\tag{68}$$

where $\psi(x)$ is the digamma function. However, we will content ourselves with extracting the bulk part,

$$\begin{aligned}C(x)|_{\mathrm{bulk}}&=-\frac{2x}{\pi}\sum_{n=0}^{\infty}(-)^n\frac{(2n)!}{2^{4n+1}(1+2n)n!^2}\\&\quad\times\left[\frac{z}{2}\left(\ln\frac{z}{2}+\gamma-\frac{1}{2}\right)\right]_{z=(2n+1)x}\\&=-\frac{x^2}{\pi}\left[S_1\ln\frac{x}{2}+S_2+S_1\left(\gamma-\frac{1}{2}\right)\right]\\&=-\frac{x^2}{\sqrt{5}\pi}\ln x+0.104744x^2,\end{aligned}\tag{69}$$

where

[4]Derivation of thermodynamic quantities from CFT usually demands a thermodynamic approach in the sense of [32], be the TBA or Schoutens' recursion method [31].

$$S_1=\sum_{n=0}^{\infty}(-)^n\frac{(2n)!}{2^{4n+1}n!^2}=\frac{1}{\sqrt{5}},$$

$$S_2=\sum_{n=0}^{\infty}(-)^n\frac{(2n)!}{2^{4n+1}n!^2}\ln(2n+1)$$
$$=-0.0536114.$$

Corresponding to $g=1/2$ statistics, in a segment of length L the semion momenta of completely filled single-particle levels are $k_n=(2\pi/L)|n/2+1/4|$ with $n\in\mathbb{Z}$. Hence, the *approximate* ground state energy is like in Eq. (41) but with wave numbers that are odd powers of $\pi/(2L)$ [31,32],

$$E_0=-\frac{1}{2}\sum_{n=-\infty}^{\infty}\sqrt{\left(\frac{(2n+1)\pi}{2L}\right)^2+m^2}=-\sum_{n=0}^{\infty}\sqrt{\left(\frac{(2n+1)\pi}{2L}\right)^2+m^2}. \tag{70}$$

Now, an expansion in powers of mL would yield a wrong Casimir energy, owing to the approximated nature of the ground state energy. We can however see that the bulk term

$$\frac{1}{L}\int_0^{\infty}dn\,\epsilon(n)=-2\int_0^{\infty}\frac{dp}{2\pi}\sqrt{p^2+m^2} \tag{71}$$

is twice that of fermions, as corresponds to the double average occupation number of semions. We can compare the approximate bulk non-analytic term in Eq. (71) with the exact result of the TBA (69). According to the combined form of IR and UV logarithmic terms (47) that we expect from Eq. (71), we have

$$C(x)|_{\rm bulk}=-\frac{x^2}{2\pi}\ln x+0.390536x^2. \tag{72}$$

We see that the first term of the sum S_1 reproduces the coefficient in this approximation, 1/2, but the total coefficient, $1/\sqrt{5}\simeq0.447214$, is slightly smaller.

The total non-analytic part is now an infinite series, obtained from

$$\frac{2x}{\pi}\sum_{n=0}^{\infty}(-)^n\frac{(2n)!}{2^{4n+1}(1+2n)n!^2}\left[\ln\frac{z}{2}\,\frac{z}{2}\sum_{k=0}^{\infty}\frac{\left(\frac{z^2}{4}\right)^k}{k!(k+1)!}\right]_{z=(2n+1)x}$$
$$=\frac{1}{\pi}\ln\frac{x}{2}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(-)^n\frac{(2n)!(2n+1)^{2k}}{2^{4n+1}n!^2}\frac{x^{2k+2}}{4^k k!(k+1)!}+\text{analytic}$$
$$=\frac{1}{\pi\sqrt{5}}\ln\frac{x}{2}\left(x^2+\frac{x^4}{25}-\frac{7x^6}{750}+\frac{353x^8}{225000}-\frac{2651x^{10}}{22500000}-\frac{619619x^{12}}{16875000000}+O(x^2)^7\right)+\text{analytic}. \tag{73}$$

Interestingly, the series coefficients seem to be rational numbers.

The entropic C functions are easily derived from the expression of C (66)

$$\mathcal{C}(x)=\frac{\pi}{10}-\frac{x^2}{2\sqrt{5}\pi}\log\left(\frac{x^2}{(\Lambda\beta)^2}\right)-\frac{x^2}{\pi}\times\sum_{n=0}^{\infty}(-)^n\frac{(2n)!}{2^{4n+1}n!^2}\{K_2[(2n+1)x]+K_0[(2n+1)x]\} \tag{74}$$

and

$$\tilde{\mathcal{C}}(x)=\frac{2x}{\pi}\sum_{n=0}^{\infty}(-)^n\frac{(2n)!}{2^{4n+1}n!^2}K_2[(2n+1)x]. \tag{75}$$

They are plotted in Fig. 2. We use again the value $\Lambda\beta=100$.

B. Deformed two-dimensional conformal field theories

A general class of theories amenable to derivation of the finite-size quantities of interest is that of deformed 2D CFT. Since CFT provides the *exact* dimensions of relevant fields, the results of conformal perturbation theory are more accurate in 2D than those of ordinary perturbation theory, which on the other hand is plagued with IR problems. In addition, many models admit integrable deformations, in the sense that the existence of an infinite number of conservation laws forces the *S*-matrix to factorize. Then the TBA provides a way to derive thermodynamic quantities. Many deformed CFT are known to be integrable and similar methods are applicable to all [34], although their complexity can be considerable for the most sophisticated models. Therefore, we shall choose one of the simplest cases. It should be intuitively clear how to generalize the computation of the entropic quantities to other integrable models.

The natural (and oldest) generalization of the Ising model consists of taking a site variable which can take three values instead of two, constituting the three-state Potts model. It is

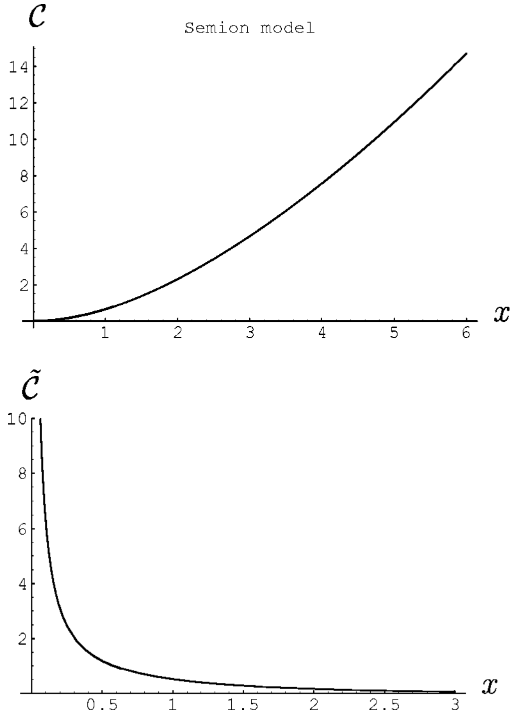

FIG. 2. $\mathcal{C}$ ($\Lambda\beta=100$) and $\tilde{\mathcal{C}}$ for semions.

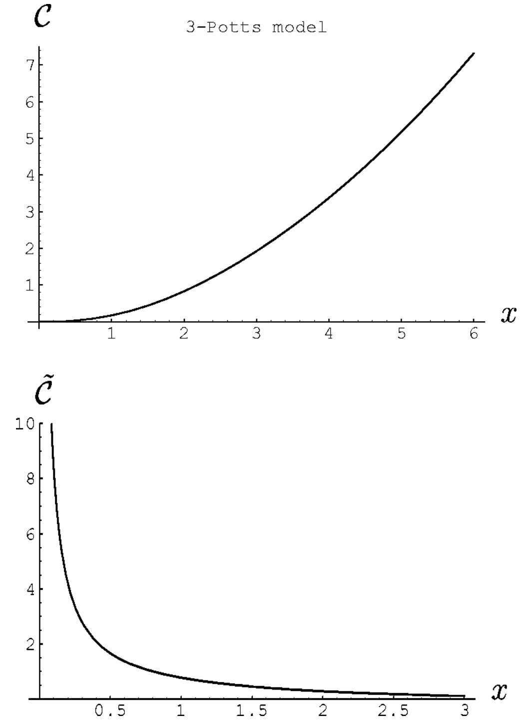

FIG. 3. $\mathcal{C}$ and $\tilde{\mathcal{C}}$ for the 3-state Potts model.

critical for $ß_c=\ln(\sqrt{3}+1)/3$ and its thermal critical exponent is $\nu=5/6$, implying that $y=6/5$. This model has been long known to be integrable and it has been long (but not as long) known to be describable in terms of particles with fractional statistics, which are generalizations of the Ising fermions and are called parafermions. This quasi-fermionic representation is in terms of two conjugate parafermions carrying $\mathbb{Z}_3$ charges $+1$ or -1 and spin 2/3, which are massless at the CP point but acquire a mass for $T>T_c$. Their interaction is purely statistical at the CP but it is more complicated off criticality. However, it is still integrable and its thermodynamic properties can be found with the TBA. It yields

$$C(\beta m)=-\beta m\int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\cosh(\theta)\ln(1+e^{-\epsilon(\theta)}), \quad (76)$$

which is apparently similar to the formula for free fermions but now $\epsilon(\theta)$ are unknown functions to be determined with the TBA equations. The concrete *S*-matrix elements of this model lead to the TBA equation

$$\epsilon(\theta)=\beta m\cosh\theta+\frac{2\sqrt{3}}{\pi}\int_{-\infty}^{\infty}d\theta'\frac{\cosh(\theta-\theta')}{1+2\cosh 2(\theta-\theta')} \times\ln(1+e^{-\epsilon(\theta-\theta')}). \quad (77)$$

It can be solved numerically by an iterative algorithm, yielding a set of numbers which can be displayed in a table [21]. Hence, according to the general formula (16),

$$\mathcal{C}(x)=\frac{\pi c}{6}+C(x)-\frac{5x}{6}\frac{dC(x)}{dx}, \quad (78)$$

where the bulk relative entropy $S_{\rm rel}(m)$ does not appear since it is now implicitly included. It is derived from the universal bulk free energy, which can be calculated exactly, yielding $-\sqrt{3}x^2/6$ [21]; hence, $S_{\rm rel}(m)=\sqrt{3}m^2/9$. We can calculate the monotonic functions $\mathcal{C}$ and $\tilde{\mathcal{C}}$ numerically as well. They are plotted in Fig. 3.

The TBA solution for $C(x)$ can be expressed as a power series in $x^{12/5}$ plus a bulk term [21],

$$\begin{aligned}C(x)=-\frac{\pi}{6}\Bigg[&\frac{4}{5}-\frac{\sqrt{3}x^2}{\pi}+0.339688x^{12/5}-0.00326095x^{24/5}\\&+0.000114199x^{36/5}-5.11209\times10^{-6}x^{48/5}\\&+4.01138\times10^{-7}x^{12}-1.38691\times10^{-8}x^{72/5}\\&+7.8336\times10^{-10}x^{84/5}-4.56\times10^{-11}x^{96/5}\\&+2.66\times10^{-12}x^{108/5}-1.68\times10^{-13}x^{24}+\mathrm{O}(x)^{132/5}\Bigg].\end{aligned} \quad (79)$$

Its radius of convergence can be estimated to $\delta x^{12/5}=14.3\pm0.4$, that is, $\delta x\simeq3.0$.[5] This expansion (79) can also be obtained by conformal perturbation theory [21]. The series coefficients are then expressed in terms of integrals of correlators of the perturbing conformal field, that is, the thermal field of dimension $d_\Phi=4/5$ in the three-state Potts model. The computation of these integrals is very laborious, except-

[5]A plot of this series sharply shows that it is very close to the TBA result for $x\lesssim3$ and quickly departs from it for larger *x*.

for the first ones, which in some cases admit explicit expressions as series [5]. A related computational method is the *truncated conformal-space approach*, in which one only takes a finite dimensional subspace of the Hilbert space of possible states, namely, a number of low-lying states, to numerically diagonalize the Hamiltonian. In the limit of increasing the number of states, this approach is equivalent to a numerical evaluation of the integrals giving the coefficients of the perturbation series [5]. We do not think it is worthwhile to dwell in detailed computational methods since the simple numerical integration of the TBA equation (77) suffices to show the monotonicity of $\mathcal{C}$ and $\tilde{\mathcal{C}}$.

V. DISCUSSION

For a 2D field theory, one can introduce two monotonic dimensionless functions, namely, $\mathcal{C}$ and $\tilde{\mathcal{C}}$, derived from the 2D relative entropy and the 1D quantum entropy, respectively. It has been shown that $\mathcal{C}$ is universal when the couplings are strongly relevant, that is, with dimension $1<y\leq 2$. They include the thermal perturbations of the unitary minimal models of conformal field theory (CFT), except for the Ising model, which we have also studied, notwithstanding. In contrast, $\tilde{\mathcal{C}}$ is always universal, since it only depends on the universal finite-size correction to the free energy. Given that general theorems may not be particularly useful if the quantities that they involve cannot be computed in practice, considerable time and effort has been devoted to compute $\mathcal{C}$ and $\tilde{\mathcal{C}}$ for a variety of models. In consequence, we have been able to show for them that those functions are monotonic.

The dimensionless entropies $\mathcal{C}$ and $\tilde{\mathcal{C}}$ play a similar role to Zamolodchikov's c function, constraining the structure of the RG flow, but they have a clear physical origin, unlike Zamolodchikov's c function. The existence of a monotonic function is usually argued on the grounds of the irreversible nature of the RG flow, which in the coarse-grained formulation implies a loss of information on microscopic degrees of freedom [10,35]. This idea inspired the adaptation of Boltzmann's H theorem to the RG flow in our previous work [1]. It has been shown here that this philosophy gives rise to the entropic functions $\mathcal{C}$ and $\tilde{\mathcal{C}}$, which are computable for a wide range of models. The non-perturbative computation of Zamolodchikov's c function is much harder and, in fact, it does not seem to have been carried out for any fully interacting model. For all these reasons, the entropic C functions proposed here arguably provide a new perspective in the long-standing problem of the irreversibility of the RG. Nevertheless, in comparison with Zamolodchikov's c theorem, it must be remarked that universality of $\mathcal{C}$, the entropic function more similar to Zamolodchikov's c function, has been proved only for deformations of the critical theory by fields with dimension $0\leq d_\Phi<1$ (strongly relevant), while Zamolodchikov's theorem covers the entire range of d_Φ.

Our entropic monotonicity theorem for the dimensionless relative entropy is

$$x\frac{d\mathcal{C}}{dx}=\frac{\beta^2}{y}\int d^2z\langle{:}\Theta(z){:}{:}\Theta(0){:}\rangle. \tag{80}$$

Even though it resembles Zamolodchikov's c theorem it is not quite the same: The correlator of Θ's in the second term appears integrated. Furthermore, a detailed calculation of Zamolodchikov's function $c(m)$ for the free boson or fermion shows that they differ from the respective values of $\mathcal{C}(m)=\mathcal{C}(x)|_{\beta=1}$. The essential discrepancy actually has a geometrical origin: A crucial step in the proof of Zamolodchikov's theorem relies on the assumption of rotation symmetry [21], which does not exist on the cylinder. Therefore, the theorem does not hold on it. However, the absence of rotation symmetry is traded for the appearance of a new parameter, the width β, which replaces the distance to the origin in Zamolodchikov's theorem and is used in the derivation of the entropic monotonicity theorems.

The reason for the introduction of a finite geometry is to have an IR scale to define a dimensionless relative entropy. We have used the cylinder because of its thermodynamic interpretation. Of course, other finite geometries are possible. For example, one can use a sphere. Its radius is then the IR scale. The advantage is that rotation symmetry is preserved on the sphere and Zamolodchikov's theorem holds. With this new geometry the monotonicity theorem for the dimensionless relative entropy would still involve an integral of the correlator of Θ's but a relation of $\mathcal{C}$ with Zamolodchikov's c function seems more feasible. At least in conformal perturbation theory one should be able to perform that integral in terms of the IR scale and a direct comparison with Zamolodchikov's theorem could be possible.

The existence of several monotonic functions prompts the question of which one is preferable. It is intuitively clear that a unique definition of RG monotonic function is not possible: The RG itself is not unique and one can choose a variety of RG parameters. Correspondingly, if $\mathcal{C}(x)$ is monotonic a monotonic change of the independent variable x will transform it into a different monotonic function. We might then consider what happens at the boundary, $x=0$ or $x\to\infty$. The point $x=0$ is the RG fixed point and it is sensible to define a function related to it, as are the dimensionless relative entropy $\mathcal{C}$ or Zamolodchikov's c function. The dimensionless absolute entropy $\tilde{\mathcal{C}}(x)$ is defined irrespective of the fixed point and actually diverges there. We could as well demand good behavior in the limit $x\to\infty$. This condition is satisfied by $\tilde{\mathcal{C}}$ but may not be satisfied by $\mathcal{C}$, owing to the bulk term. It is quite possible that a minor modification of the definition of $\mathcal{C}$ may remove the bulk term and make it well behaved in the limit $x\to\infty$ as well as at $x=0$.

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APPENDIX A: CALCULATION OF FINITE-SIZE CORRECTIONS WITH THE EULER-MACLAURIN FORMULA

For free models the energy series can be evaluated with the Euler-MacLaurin summation formula,[6]

$$\sum_{n=0}^{\infty} \epsilon(n) = \int_0^{\infty} dn\, \epsilon(n) + \frac{1}{2}\epsilon(0) - \frac{1}{12}\epsilon'(0) + \frac{1}{720}\epsilon'''(0) - \frac{1}{30240}\epsilon^{(v)}(0) + \cdots, \qquad \text{(A1)}$$

where $\epsilon(n)$ are single-particle energies. For the Gaussian model the first term can be proved to be proportional to L with the change of variable $p=2\pi n/L$ and leads to the integral in Eq. (34). One can see that all the odd derivatives of $\epsilon(n)$ vanish at $n=0$ because it is an even function. It is natural, because the finite size corrections are exponentially negligible when $L\to\infty$ and therefore nonanalytic: Every derivative pulls out a power of $2\pi/L$ and the subsequent series of powers of $1/L$ must have vanishing coefficients.

In the scaling zone $mL \ll 1$ the finite-size corrections are not negligible, despite the previous argument. To evaluate these corrections we can nevertheless make use of the Euler-MacLaurin expansion but using a non-zero value for the point at which the derivatives are computed, in the following form:

$$\sum_{n=1}^{\infty} \epsilon(n) = \int_1^{\infty} dn\, \epsilon(n) + \frac{1}{2}\epsilon(1) - \frac{1}{12}\epsilon'(1) + \frac{1}{720}\epsilon'''(1) - \frac{1}{30240}\epsilon^{(v)}(1) + \cdots. \qquad \text{(A2)}$$

Now the series can be transformed into an expansion in powers of mL. The reason why that trick works can be understood in several ways. One is that the derivatives $\epsilon^{(2k+1)}(n)$ as functions of m are ill behaved for small n. They converge to the null function for $n=0$ but nonuniformly. It is actually safer to choose the argument n of the derivatives larger than in Eq. (A2), $n=3$ or 4 say. Then the Euler-MacLaurin expansion converges very fast and the terms displayed above suffice to match the coefficients in Eq. (39) with about ten decimal places.

Alternatively, one may focus on the fact that for $m=0$ the function $\epsilon(n)\propto|n|$ is singular at $n=0$; its derivatives eventually diverging there. This is naturally an IR divergence, which does not exist for $m\neq 0$. However, one must be careful to evaluate $\epsilon(n)$ at $n\neq 0$ before taking $m=0$, or in other words, one must introduce an IR cutoff and evaluate the Euler-MacLaurin expansion at that point. Of course, the result shall be inpependent of the precise value of the cutoff, although its convergence properties are greatly affected by it. Within the realm of classical mathematics, it is interesting to recall that Legendre met a similar problem when he attempted to evaluate elliptic integrals numerically with the Euler-MacLaurin expansion. Since the integrand is an even function of the integration variable at the limits 0 and $\pi/2$, the odd derivatives vanish and the Euler-MacLaurin formula implies that the elliptic integral is equal to any of its rectangular approximations. The paradox was solved by Poisson, who showed that in this case the remainder term does not tend to zero as the number of terms increases and hence the series does not converge. If we further consider that the bulk free energy of the Gaussian or Ising models on a finite chain can be expressed as elliptic integrals, we may appreciate that Legendre actually encountered an IR divergence without being aware of the need of regularization.

For the Ising model (41) the odd derivatives of $\epsilon(n)$ do not vanish at $n=0$ but it also is necessary to choose $n=3$ or 4 for fast convergence.

[6]This is a common method to convert sums to integrals. However, since the function summed $\epsilon(n)$ diverges when $n\to\infty$ a preliminary regularization is required. A convenient form is to sum up to some arbitrary number $N\gg 1$, which for a chain can be the number of sites. This UV regularization renders meaningful the formal manipulations that follow. However, we do not need to be definite on the UV regularization for our focus is on universal quantities.

APPENDIX B: CALCULATION OF $\langle T_{ab}\rangle$ ON THE CYLINDER FOR FREE MODELS

For free field theories the expectation values of the components of the stress tensor can be expressed in terms of the Green function. Thus for a bosonic field

$$\langle\Theta\rangle = m^2\langle\varphi^2\rangle = m^2 \lim_{z\to 0} G_\beta(z,\bar{z}). \qquad \text{(B1)}$$

We use complex notation, $z=x_1+ix_2$. The Green function on a cylinder, $G_\beta(z,\bar{z})$ is nontrivial. Its Fourier transform includes a sum over discrete momenta in the compact direction,

$$G_\beta(z,\bar{z}) = \frac{1}{\beta}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{e^{i(\omega_n x_1 + k x_2)}}{\omega_n^2+k^2+m^2}, \qquad \text{(B2)}$$

where the allowed frequencies for bosons are $\omega_n=(2\pi/\beta)n$. It can be transformed into a more manageable form by the use of the proper-time representation [36]. In this representation the integral over k is elementary and one is left with the sum over n and the integral over proper time. After performing a convenient Poisson resummation one obtains

$$G_\beta(z,\bar{z}) = \sum_{n=-\infty}^{\infty}\int_0^{\infty}\frac{ds}{4\pi s}e^{-m^2 s-|z-n\beta|^2/4s}. \qquad \text{(B3)}$$

Since the Green function on the plane is just

$$G_\infty(z,\bar{z}) = \frac{1}{2\pi}K_0(m|z|),$$

which is the $n=0$ term in the sum (B3), this sum can be interpreted as the solution of the field equation for a point source by the method of images. It can be expressed in terms of the Jacobi theta function

$$\theta_3(\nu,\tau)=\sum_{n=-\infty}^{\infty} u^n q^{n^2/2}, \quad u=e^{2\pi i\nu}, \quad q=e^{2\pi i\tau},$$

as

$$G_\beta(z,\bar{z})=\int_0^\infty \frac{ds}{4\pi s} e^{-m^2 s-|z|^2/4s}\theta_3\left(-\frac{ix_1\beta}{4\pi s},\frac{i\beta^2}{4\pi s}\right). \tag{B4}$$

Then the Poisson resummation realizes the duality property of θ_3.

The formal $z\to 0$ limit of G_β is easily taken,

$$G_\beta(0)=\lim_{z\to 0} G_\beta(z,\bar{z})=\sum_{n=-\infty}^{\infty}\int_0^\infty \frac{ds}{4\pi s} e^{-m^2 s-(n\beta)^2/4s}. \tag{B5}$$

It contains the logarithmic divergence

$$G_\infty(0)=\frac{1}{2\pi}K_0(0)=\int_0^\infty \frac{ds}{4\pi s} e^{-m^2 s}.$$

Taking into account the integral representation of modified Bessel functions

$$K_\nu(z)=\frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{ds}{s} s^{-\nu} e^{-s-z^2/4s}, \tag{B6}$$

one obtains

$$G_\beta(0)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} K_0(nm\beta). \tag{B7}$$

The computation of $\langle T\rangle$ requires a little more work, for

$$\langle T\rangle=-4\langle(\partial_z\varphi)^2\rangle=4\lim_{z\to 0}\partial_z^2 G(z). \tag{B8}$$

From Eq. (B3),

$$\partial_z^2 G_\beta(z,\bar{z})=\sum_{n=-\infty}^{\infty}\int_0^\infty \frac{ds}{4\pi s}\left(\frac{\bar{z}+n\beta}{4s}\right)^2 e^{-m^2 s-|z-n\beta|^2/4s}. \tag{B9}$$

Hence,

$$\begin{aligned}\partial_z^2 G(0)&=\lim_{z\to 0}\partial_z^2 G(z)\\ &=\sum_{n=-\infty}^{\infty}\int_0^\infty \frac{ds}{4\pi s}\frac{(n\beta)^2}{16 s^2} e^{-m^2 s-(n\beta)^2/4s}\\ &=-\frac{m^2}{8\pi}\sum_{n=-\infty}^{\infty} K_2(nm\beta).\end{aligned} \tag{B10}$$

It contains a quadratic divergence in $K_2(0)$.

For Majorana fermions one could start from the known expressions of their stress tensor but it is simpler to consider them as bosons with antiperiodic boundary conditions and use again the Fourier transform (B2) with allowed frequencies $\omega_n=(2\pi/\beta)(n+\frac{1}{2})$. Before Poisson resummation, G_β can be expressed in terms of

$$\theta_2(\nu,\tau)=\sum_{n=-\infty}^{\infty} u^{n+1/2} q^{(1/2)(n+1/2)^2}.$$

The Poisson resummation transforms θ_2 into its dual θ_4,

$$\theta_4(\nu,\tau)=\sum_{n=-\infty}^{\infty}(-)^n u^n q^{n^2/2},$$

which is like the bosonic θ_3 but with an additional alternating sign.

We can write the final result in a condensed notation,

$$\langle\Theta\rangle=\pm\frac{m^2}{2\pi}\sum_{n=-\infty}^{\infty}(\pm)^n K_0(nm\beta), \tag{B11}$$

$$\langle T\rangle=\pm\frac{m^2}{2\pi}\sum_{n=-\infty}^{\infty}(\pm)^n K_2(nm\beta), \tag{B12}$$

where the upper signs stand for bosons and the lower signs for fermions.

However, to have well defined expressions we must further introduce a regularization that removes the divergences in $K_0(0)$ and $K_2(0)$. It is customary to begin defining normal-ordered composite fields, namely,

$$:\Theta:=m^2:\varphi^2:, \tag{B13}$$

$$:T:=-4:(\partial_z\varphi)^2:, \tag{B14}$$

in the sense of a point splitting regularization and a substraction of the divergent part, computed with the Wick prescription. It amounts to the substraction of $G_\infty(0)=\pm 1/(2\pi)K_0(0)$ or $4\partial_z^2 G_\infty(0)=\pm m^2/(2\pi)K_2(0)$. Point splitting on a lattice yields

$$K_0(0)=-\left(\ln\frac{ma}{2}+\gamma\right)+\mathrm{O}(a^2), \tag{B15}$$

$$K_2(0)=\frac{2}{m^2a^2}-\frac{1}{2}+\mathrm{O}(a^2). \tag{B16}$$

Considering $a\sim 1/\Lambda$ we have, for example, that according to Eq. (56)

$$\begin{aligned}e_0&=\pm\frac{m^2}{4\pi}[K_0(0)-K_2(0)]\\ &\sim\pm\frac{1}{4\pi}\left\{-2\Lambda^2+m^2\ln\frac{2\Lambda}{m}+m^2\left(\gamma-\frac{1}{2}\right)+\mathrm{O}(\Lambda^{-2})\right\}.\end{aligned} \tag{B17}$$

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