Quantum 1/4 BPS dyons

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Classical properties of 1/4 Bogomol'nyi-Prasad-Sommerfield (BPS) dyons were previously well understood both in the field theory context and in the string theory context. Its quantum properties, however, have been more difficult to probe, although the elementary information of the supermultiplet structures is known from a perturbative construction. Recently, a low energy effective theory of monopoles was constructed and argued to contain these dyons as quantum bound states. In this paper, we find these dyonic bound states explicitly in the $N=4$ supersymmetric low energy effective theory. After identifying the correct angular momentum operators, we motivate an anti-self-dual ansatz for all BPS bound states. The wave functions are found explicitly, whose spin contents and degeneracies match exactly the expected results.

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I. INTRODUCTION

In the $N=4$ supersymmetric Yang-Mills theories, there can be 1/2 Bogomol'nyi-Prasad-Sommerfield (BPS) and 1/4 BPS configurations. The precise nature of these states depends on the asymptotic values of six Higgs fields in the theory. When the Higgs expectation values have only one independent component, only 1/2 BPS configurations can appear. Classically, 1/2 BPS configurations are made of monopoles or dyons, and their electric fields are proportional to their magnetic fields. When the Higgs expectation values have two or more independent components, 1/4 BPS configurations can also appear, which are all dyons whose electric charges are not proportional to their magnetic charges [1–3].

Any 1/2 BPS configuration of a given collection of monopoles is specified by its moduli parameters, and the low energy dynamics of these 1/2 BPS monopoles are determined by the metric on the manifold spanned by these moduli parameters [4]. On the other hand, as shown in Refs. [2,3], solutions to 1/4 BPS equations can be obtained in two steps. First one solves for a purely magnetic soliton, which may be regarded as 1/2 BPS in a technical sense. Then one solves for a certain linear combination of gauge zero modes in this purely magnetic background. Curiously enough, one can build the electric part of the 1/4 BPS dyons from such a gauge zero mode. Because the existence of gauge zero modes is guaranteed for any 1/2 BPS monopole [5], any 1/4 BPS configuration is again specified by the moduli parameters of the corresponding 1/2 BPS monopole configuration. Given a fixed set of moduli parameters, the solution of the second BPS equation determines the relative part of the electric charge of monopoles uniquely.

One may regard 1/4 BPS configurations as deformed 1/2 BPS configurations when the additional and independent Higgs expectation is turned on. When this second Higgs expectation is quite small compared to the first, the deviation of the 1/4 BPS configurations from the 1/2 BPS configuration is small. In such cases, the authors (with C. Lee) have shown in a recent paper [6] that one can describe the low energy dynamics of both 1/2 BPS and 1/4 BPS configurations with an effective nonrelativistic Lagrangian.

The kinetic part of the Lagrangian is given by the moduli space metric of the 1/2 BPS configurations. The potential is also present, and is given by the square of the norm of a triholomorphic Killing vector field related to an unbroken $U(1)$ gauge symmetry. The size of this attractive potential is proportional to the square of the additional Higgs expectation value. This effective Lagrangian can be interpreted as low energy dynamics of 1/2 BPS monopoles with attractive potential, in other words, and the 1/4 BPS configurations should be realized as BPS bound states of monopoles with additional electric quantum numbers.

In Ref. [6], the full $N=4$ supersymmetric low energy effective Lagrangian is written. This is a sigma model with potential that has extended complex supersymmetry with a central term and, as usual, the wave functions can be interpreted as differential forms on the moduli space. The BPS equation was found and translated to the language of differential forms.

The simplest nontrivial 1/4 BPS configurations appear as composites of two distinct fundamental magnetic monopoles in SU(3) gauge theory. Furthermore, the eight-dimensional moduli space of these two monopoles is known exactly. From Ref. [2], several facts are known about this case. First of all, classical 1/4 BPS configurations are made of two 1/2 BPS dyons at rest, whose mutual distance is determined by

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their relative electric charge. Also, the supermultiplet structures of all such dyons have been found by the perturbative method around the 1/2 BPS states of the zero relative electric charge, where the nonzero relative charge states are constructed by exciting certain massive excitations on 1/2 BPS configurations.

In this paper, we reconstruct these 1/4 BPS dyons as quantum bound states of two distinct SU(3) monopoles in the low energy dynamics described above. We construct all such SU(3) dyons. We also recover the phenomenon of instability found in Refs. [1,2]: The bound state wave function loses its normalizability exactly at the point where the instability should set in. Furthermore, we explicitly show that each (stable) dyon comes in the same supermultiplet as found in Ref. [2]

The plan of the paper is as follows. In Sec. II, we briefly discuss the moduli space of a pair of distinct monopoles in SU(3) theory. In Sec. III, we review briefly the supersymmetric Hamiltonian and BPS conditions on wave functions as shown in Ref. [6]. In Sec. IV we discuss the angular momentum and an ansatz for the BPS wave functions. In Sec. V we solve the BPS equations. In Sec. VI, we conclude with some comments.

II. A PAIR OF DISTINCT MONOPOLES IN THE SU(3) GAUGE THEORY

Consider $N=4$ SU(3) gauge theory spontaneously broken to $U(1)^2$. When the six Higgs expectations are all collinear, the theory contains two distinct types of fundamental monopoles, which we will label $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. The low energy interaction between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ monopoles can be described by the moduli space dynamics. There are four collective coordinates for each monopole, three for its position, and one for the U(1) phase. We call their positions and phases $\mathbf{x}_i$, χ_i, $i=1,2$, for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ monopoles, respectively. Let us parametrize the masses of these monopoles as μ_1 and μ_2. We are suppressing the gauge coupling constant in all subsequent formulas.

The exact nonrelativistic effective Lagrangian has been found to be a sum of the Lagrangians for the center of mass and the relative motion [7]. As there is no external force, the center of mass Lagrangian is a free one:

$$\mathcal{L}_{\rm cm}=\frac{(\mu_1+\mu_2)}{2}\dot{\mathbf{X}}^2+\frac{1}{2(\mu_1+\mu_2)}\dot{\chi}_T^2, \tag{1}$$

where the center of mass position is $\mathbf{X}=(\mu_1\mathbf{x}_1+\mu_2\mathbf{x}_2)/(\mu_1+\mu_2)$ and the center of mass phase is $\chi_T=\chi_1+\chi_2$. The relative motion between them is more complicated and described by the Taub-NUT (Newman-Unti-Tamburino) metric [8,9], and has the Lagrangian

$$\mathcal{L}_{\rm rel}=\frac{\mu}{2}\left[\left(1+\frac{1}{\mu r}\right)\dot{\mathbf{r}}^2+\frac{1}{\mu^2(1+1/\mu r)}[\dot{\chi}+\mathbf{w}(\mathbf{r})\cdot\dot{\mathbf{r}}]^2\right], \tag{2}$$

where the relative position is $\mathbf{r}=\mathbf{x}_2-\mathbf{x}_1$, the relative phase is $\chi=2(\mu_1\chi_2-\mu_2\chi_1)/(\mu_1+\mu_2)$, and $\mathbf{w}(\mathbf{r})$ is the Dirac potential such that $\nabla\times\mathbf{w}(\mathbf{r})=-\mathbf{r}/r^3$. The range of χ is $[0,4\pi]$. From now on, we will suppress the scale μ by setting $\mu=1$. The resulting monopole moduli space metric is then

$$g_{\rm rel}=\left(1+\frac{1}{r}\right)d\mathbf{r}^2+\left(\frac{1}{1+1/r}\right)[d\chi+\mathbf{w}(\mathbf{r})\cdot d\mathbf{r}]^2, \tag{3}$$

up to an overall scale. This Taub-NUT space, $\mathcal{M}_0$, has the topology of $R^4=R^+\times S^3$. The eight-dimensional total moduli space is then given by

$$\mathcal{M}=R^3\times\frac{R^1\times\mathcal{M}_0}{Z}, \tag{4}$$

where Z is the identification map

$$(\chi_T,\chi)=\left(\chi_T+2\pi,\chi+\frac{4\pi\mu_2}{\mu_1+\mu_2}\right). \tag{5}$$

For later convenience, we will make another choice of coordinates involving Euler angles on S^3,

$$g_{\rm rel}=\left(1+\frac{1}{r}\right)[dr^2+r^2\sigma_1^2+r^2\sigma_2^2]+\frac{1}{1+1/r}\sigma_3^2, \tag{6}$$

where the σ_a's are one-form frames on S^3 and satisfy the canonical relationship,

$$d\sigma_a=\frac{1}{2}\epsilon_{abc}\sigma_b\wedge\sigma_c. \tag{7}$$

More explicitly, we may write these one-forms in terms of SU(2) Euler angles as follows:

$$\sigma_1=-\sin\chi\, d\theta+\cos\chi\sin\theta\, d\phi,$$
$$\sigma_2=\cos\chi\, d\theta+\sin\chi\sin\theta\, d\phi,$$
$$\sigma_3=d\chi+\cos\theta\, d\phi. \tag{8}$$

The ranges of θ,τ,χ are respectively $\pi,2\pi,4\pi$. Let us define an orthonormal basis ω^μ by

$$\omega^0=\sqrt{1+1/r}\,dr,$$
$$\omega^1=\sqrt{r^2+r}\,\sigma_1,$$
$$\omega^2=\sqrt{r^2+r}\,\sigma_2,$$
$$\omega^3=\sqrt{\frac{r}{1+r}}\sigma_3. \tag{9}$$

Because the Taub-NUT manifold is a hyper-Kähler 4 manifold, its curvature is anti-self-dual with an appropriate choice of orientation.

When the Higgs vacua is slightly misaligned, the two monopoles are attracted to each other [10]. The effective low energy potential $\mathcal{U}$ of this static force has been found in Ref. [6] for all multimonopole configurations in all $N=4$ gauge theories. Specializing to the case of a pair of distinct monopoles in SU(3), the relative part of this potential is given by a squared norm of the Killing vector field ∂_χ up to an overall factor

$$\mathcal{U}_{\rm rel}=\frac{1}{2}a^2\left\langle\frac{\partial}{\partial\chi},\frac{\partial}{\partial\chi}\right\rangle, \tag{10}$$

where a is a measure of Higgs misalignment. The interacting part of the two monopole dynamics is dictated by an effective Lagrangian, whose bosonic part is

$$\mathcal{L}_{\rm rel}=\frac{1}{2}(g_{\rm rel})_{\mu\nu}\dot{z}^\mu\dot{z}^\mu-\mathcal{U}_{\rm rel}=\frac{1}{2}\left(1+\frac{1}{r}\right)\dot{\mathbf{r}}^2+\frac{1}{2}\left(\frac{1}{1+1/r}\right)[\dot{\chi}+\mathbf{w}(\mathbf{r})\cdot\dot{\mathbf{r}}]^2-\frac{1}{2}\left(\frac{a^2}{1+1/r}\right). \tag{11}$$

Note that the potential $\mathcal{U}_{\rm rel}$ increases from zero at origin to $a^2/2$ at infinity. This behavior allows new bound states of the dynamics which would not have been possible for $a=0$. Among these are certain dyonic states that preserve 1/4 of field theory supersymmetries. The purpose of this paper is to reconstruct these 1/4 BPS dyons as BPS quantum bound states in the low energy dynamics of monopoles.

III. SUPERSYMMETRY AND BPS BOUND

We begin by recapitulating generic properties of the $N=4$ supersymmetric quantum extension of the bosonic effective action [6,11,12]. Its form is rather similar to the usual supersymmetric sigma model action but supplemented by an attractive bosonic potential together with its fermionic counter part. These potentials are determined by a single Killing vector field G. The supersymmetric Lagrangian written with real fermions is

$$\mathcal{L}=\tfrac{1}{2}(g_{\mu\nu}\dot{z}^\mu\dot{z}^\nu+ig_{\mu\nu}\overline{\psi^\mu}\gamma^0D_t\psi^\nu+\tfrac{1}{6}R_{\mu\nu\rho\sigma}\overline{\psi^\mu}\psi^\rho\overline{\psi^\nu}\psi^\sigma-g^{\mu\nu}G_\mu G_\nu-D_\mu G_\nu\overline{\psi^\mu}\gamma_5\psi^\nu), \tag{12}$$

where ψ^μ is a two-component anticommuting Majorana spinor and $\gamma^0=\sigma_2$, $\gamma_5=\sigma_3$, and $\overline{\psi}=\psi^T\gamma^0$. In the case of relative dynamics of the two SU(3) monopoles, G is equal to $a\partial_\chi$. As required for the $N=4$ supersymmetry, the metric here is hyper-Kähler, endowed with three complex structures $\mathcal{I}^{(a)\mu}{}_\nu(a=1,2,3)$ that satisfy

$$\mathcal{I}^{(a)}\mathcal{I}^{(b)}=-\delta^{ab}+\epsilon^{abc}\mathcal{I}^{(c)}, \tag{13}$$

$$D_\mu\mathcal{I}^{(a)\nu}{}_\rho=0. \tag{14}$$

For the sake of $N=4$ supersymmetry, the Killing vector $G\equiv a\cdot K$ should be triholomorphic; namely its action preserves the three complex structures via

$$\mathcal{L}_G\mathcal{I}^{(a)}=0, \tag{15}$$

where $\mathcal{L}$ denotes the Lie derivative.

Upon quantization, the spinors $\psi^A=e^A_\mu\psi^\mu$ with vielbein e^A_μ commute with all the bosonic dynamical variables, especially with p's that are canonical momenta of the coordinates z's. The remaining canonical commutation relations are

$$[z^\mu,p_\nu]=i\delta^\mu_\nu,\qquad \{\psi^A_\alpha,\psi^B_\beta\}=\delta^{AB}\delta_{\alpha\beta}. \tag{16}$$

The Lagrangian (12) is invariant under the $N=4$ supersymmetry transformations,

$$\delta_{(0)}z^\mu=\overline{\epsilon}\psi^\mu, \tag{17}$$

$$\delta_{(0)}\psi^\mu=-i\dot{z}^\mu\gamma^0\epsilon-\Gamma^\mu_{\nu\lambda}\overline{\epsilon}\psi^\nu\psi^\lambda-\gamma_5G^\mu\epsilon, \tag{18}$$

$$\delta_{(a)}z^\mu=\mathcal{I}^{(a)\mu}{}_\nu\overline{\epsilon_{(a)}}\psi^\nu, \tag{19}$$

$$\delta_{(a)}(\mathcal{I}^{(a)\mu}{}_\nu\psi^\nu)=-i\dot{z}^\mu\gamma^0\epsilon_{(a)}-\Gamma^\mu_{\nu\lambda}\mathcal{I}^{(a)\nu}{}_\rho\mathcal{I}^{(a)\lambda}{}_\sigma\overline{\epsilon_{(a)}}\psi^\rho\psi^\sigma-\gamma_5G^\mu\epsilon_{(a)}, \tag{20}$$

where ϵ and $\epsilon_{(a)}$ are spinor parameters. In order to obtain supercharges, we define supercovariant momenta by

$$\pi_\mu\equiv p_\mu-\frac{i}{2}\omega_{AB\mu}\overline{\psi^A}\gamma^0\psi^B, \tag{21}$$

where $\omega^A{}_{B\mu}$ is the spin connection. The corresponding $N=4$ SUSY generators in real form are then

$$Q_\alpha=\psi^\mu_\alpha\pi_\mu+i(\gamma^0\gamma_5\psi^\mu)_\alpha G_\mu, \tag{22}$$

$$Q^{(a)}_\alpha=\mathcal{I}^{(a)\mu}{}_\nu\psi^\nu_\alpha\pi_\mu+i(\gamma^0\gamma_5\mathcal{I}^{(a)\mu}{}_\nu\psi^\nu)_\alpha G_\mu, \tag{23}$$

which satisfy the following SUSY algebra with a central extension of

$$\{Q_\alpha,Q_\beta\}=\{Q^{(a)}_\alpha,Q^{(a)}_\beta\}=2\delta_{\alpha\beta}\mathcal{H}+2i(\gamma^0\gamma_5)_{\alpha\beta}\mathcal{Z}, \tag{24}$$

$$\{Q_\alpha,Q^{(a)}_\beta\}=0, \tag{25}$$

$$\{Q^{(a)}_\alpha,Q^{(b)}_\beta\}=0\quad(a\neq b). \tag{26}$$

The Hamiltonian $\mathcal{H}$ and the central charge $\mathcal{Z}$ read

$$\mathcal{H}=\frac{1}{2}\left(\frac{1}{\sqrt{g}}\pi_\mu\sqrt{g}g^{\mu\nu}\pi_\nu+G_\mu G^\mu-\frac{1}{4}R_{\mu\nu\rho\sigma}\overline{\psi^\mu}\gamma^0\psi^\nu\overline{\psi^\rho}\gamma^0\psi^\sigma+D_\mu G_\nu\overline{\psi^\mu}\gamma_5\psi^\nu\right), \tag{27}$$

$$\mathcal{Z}=G^\mu\pi_\mu-\frac{i}{2}(D_\mu G_\nu)\overline{\psi^\mu}\gamma^0\psi^\nu. \tag{28}$$

It is easily checked that the central charge $\mathcal{Z}$ indeed commutes with all SUSY generators.

For spectrum analysis, SUSY generators in complex form are more useful. Introducing $\varphi^{\mu}\equiv(1/\sqrt{2})(\psi_1^{\mu}-i\psi_2^{\mu})$, and defining $Q\equiv(1/\sqrt{2})(Q_1-iQ_2)$, one finds

$$Q=\varphi^{\mu}\pi_{\mu}+i\varphi^{*\mu}G_{\mu}, \tag{29}$$

$$Q^{\dagger}=\varphi^{*\mu}\pi_{\mu}-i\varphi^{\mu}G_{\mu}, \tag{30}$$

which generates the following simple algebra:

$$\{Q,Q^{\dagger}\}=\{Q^{(a)},Q^{(a)\dagger}\}=2\mathcal{H}, \tag{31}$$

$$\{Q,Q\}=\{Q^{(a)},Q^{(a)}\}=-\{Q^{\dagger},Q^{\dagger}\}=-\{Q^{(a)\dagger},Q^{(a)\dagger}\}=2i\mathcal{Z}, \tag{32}$$

$$\{Q,Q^{(a)}\}=\{Q^{\dagger},Q^{(a)}\}=0, \tag{33}$$

$$\{Q^{(a)},Q^{(b)}\}=\{Q^{(a)\dagger},Q^{(b)}\}=0 \quad (a\neq b). \tag{34}$$

It is easy to read off the BPS condition for quantum states that preserves half of the supersymmetries. Depending on the sign of central charge, we find

$$(Q\mp iQ^{\dagger})|\Phi\rangle=0, \tag{35}$$

so that the given state may saturate the condition $\mathcal{H}=\pm\mathcal{Z}$. We can express this BPS condition in a more geometrical fashion by transcribing the wave function to differential forms on the moduli space [13]. Note that

$$[i\pi_{\mu},\varphi^{\nu}]=-\Gamma^{\nu}_{\mu\rho}\varphi^{\rho}, \tag{36}$$

$$[i\pi_{\mu},\varphi^{*}_{\nu}]=\Gamma^{\rho}_{\mu\nu}\varphi^{*}_{\rho}, \tag{37}$$

$$\{\varphi^{\mu},\varphi^{*}_{\nu}\}=\delta^{\mu}_{\nu}. \tag{38}$$

Furthermore, the wave function has the following general form,

$$|\Phi\rangle=\sum_{p}\frac{1}{p!}\Omega_{\mu_1\cdots\mu_p}(z^{\mu})\varphi^{\mu_1}\cdots\varphi^{\mu_p}|0\rangle, \tag{39}$$

$$\varphi^{*\mu}|0\rangle=0, \tag{40}$$

with an inner product defined by

$$\langle\Phi|\Phi'\rangle=\int dz\sqrt{g}\sum_{p}\frac{1}{p!}(\Omega^{\mu_1\cdots\mu_p})^{*}\Omega'_{\mu_1\cdots\mu_p}. \tag{41}$$

The coefficients $\Omega_{\mu_1\ldots\mu_p}$ are completely antisymmetric and may be regarded as those of a p form. In this language, where we interpret φ^{μ} and φ^{*}_{μ} as a natural cobasis dz^{μ} and a natural basis $\partial/\partial z^{\mu}$, one finds that the following replacement can be made:

$$i\varphi^{\mu}\pi_{\mu}\rightarrow d, \qquad i\varphi^{*\mu}\pi_{\mu}\rightarrow-\delta, \tag{42}$$

$$\varphi^{*\mu}G_{\mu}\rightarrow i_G, \quad i\mathcal{Z}\rightarrow\mathcal{L}_G\equiv di_G+i_Gd, \tag{43}$$

where i_G denotes the natural contraction of the vector field G with a differential form. The BPS equation now becomes

$$(d-i_G)\Omega=\mp i(\delta-G\wedge)\Omega, \tag{44}$$

where we use the same symbol G for both the Killing vector and the one form obtained by contraction with metric. By solving this first order system, we should recover all 1/2 BPS and 1/4 BPS states of the underlying Yang-Mills field theory.

IV. ANGULAR MOMENTUM AND EIGENSTATES ON S^3

A. Supermultiplet structure of 1/4 BPS dyons

1/4 BPS dyons have been constructed in several different guises. The first was as three-pronged strings ending on D3 branes [1], while the field theoretical construction was as exact classical solitons [2,3]. Neither of these was convenient for finding their supermultiplet structures; there are subtleties of the respective moduli space dynamics [14,15]. The third method, also present in Ref. [2], was a perturbative one. In this setup, one assumes very small electric coupling and works in vacua where the 1/4 BPS dyons would be stable. The construction proceeds by finding the lowest quantum excitation modes around a purely magnetic background. The lowest are massless moduli. The next lowest is massive and turns out to induce quantized electric charges to the system when excited, and produces many degenerate states with the same electromagnetic charges. In the simplest case of $\boldsymbol{\alpha}+\boldsymbol{\beta}$ magnetic charge in SU(3), the total degeneracy for dyons of relative charge $q\neq0$ is

$$2^6\times|2q|. \tag{45}$$

Note that q is quantized in half-integers. The highest spin of this supermultiplet is $|q|+1$. The relative charge, defined through the following expression of electric charge, that one may excite on the system is

$$\mathbf{q}=(n+q)\boldsymbol{\alpha}+(n-q)\boldsymbol{\beta}, \tag{46}$$

where $2n$ is an integer. The consistency with Dirac quantization condition, along with the spectrum of the original field theory, actually demands that $n\pm q$ are integers. Thus quantization of n is correlated with that of the relative charge q; a half-integral q comes with a half-integral n, and an integral q comes with an integral n.

When we reconstruct the dyons as bound states in the low energy dynamics, this correlation naturally emerges from the form of total moduli space

$$\mathcal{M}=R^3\times\frac{R^1\times\mathcal{M}_0}{Z}, \tag{47}$$

where the quotient action of Z is crucial. In this paper, we will not dwell on this point. It suffices to say that all dyonic states of relative charge q can be built, provided that the relative part of the wave function is found.

Once we consider the wave function to be a tensor product of two parts, one over the relative moduli space, and the other over the center-of-mass moduli space, the degeneracy is more naturally organized as,

$$2^4\times[(|2q|+1)+(|2q|)+(|2q|)+(|2q|-1)]. \tag{48}$$

The common factor 2^4 follows from the low energy dynamics trivially. Among the low energy degree of freedom, there are four bosonic and eight (real) fermionic coordinates that are associated with the center-of-mass motion and are thus free. These eight fermions act as four pairs of massless harmonic oscillators, whose excitations lead to $2^4=16$ degeneracy and the subsequent supermultiplet structure of $N=4$ vector multiplet.

The rest of the degeneracy factors must arise from the relative part of the dynamics, as we construct the 1/4 BPS dyons as bound states of monopoles. As the above decomposition suggests, the spin content of the supermultiplet found in Ref. [2] is such that the bound state wave functions over the relative moduli space are in four multiplets of angular momenta; one with $l=|q|$, two with $l=|q|-1/2$, the last with $l=|q|-1$. Thus, finding these BPS bound states explicitly presupposes detailed understanding of angular momentum in the low energy dynamics.

B. Isometries and symmetries

Let us recall the geometry of the Taub-NUT manifold. The metric is

$$\left(1+\frac{1}{r}\right)[dr^2+r^2\sigma_1^2+r^2\sigma_2^2]+\frac{1}{1+1/r}\sigma_3^2, \tag{49}$$

where the σ_a's are one-form frame on S^3 as in Sec. II. The Taub-NUT manifold has four Killing vectors, three of which generate SU(2) rotation of S^3. These SU(2) Killing vectors, which we denote by L_a, are [16]

$$L_1=-\sin\phi\,\partial_\theta-\cot\theta\cos\phi\,\partial_\phi+\frac{\cos\phi}{\sin\theta}\partial_\chi, \tag{50}$$

$$L_2=+\cos\phi\,\partial_\theta-\cot\theta\sin\phi\,\partial_\phi+\frac{\sin\phi}{\sin\theta}\partial_\chi, \tag{51}$$

$$L_3=\partial_\phi. \tag{52}$$

The σ_a's are easily seen to be invariant under the action of the vector fields

$$\mathcal{L}_{L_a}(\sigma_b)=0. \tag{53}$$

The operator $J_a=-i\mathcal{L}_{L_a}$ satisfies the usual SU(2) algebra.

The fourth Killing vector corresponding to internal U(1) gauge rotations of monopoles, to be denoted as K, is

$$K=\partial_\chi. \tag{54}$$

This is precisely the triholomorphic Killing vector field that enters the low energy dynamics. The vector fields do rotate σ_1 and σ_2 among themselves,

$$-i\mathcal{L}_K(\sigma_1\pm i\sigma_2)=\pm(\sigma_1\pm i\sigma_2), \tag{55}$$

but the Taub-NUT metric itself is invariant under such rotations. Since the range of χ is 4π, the eigenvalue of $-i\mathcal{L}_K$ is quantized at half-integers. Its origin as a gauge rotation generator also tells us that its eigenvalue should be identified with the relative charge q [9].

These isometries of the Taub-NUT manifold naturally generate symmetries of the low energy dynamics. In the absence of potential terms due to $G=aK$, this is especially clear since the Lagrangian is completely determined by the metric alone. Furthermore, the geometrical interpretation of the wave function $|\Omega\rangle$ as differential form Ω suggests that these symmetries act on Ω geometrically. In fact, it is easy to see that the Noether charge associated with each of these isometries acts on wave functions and/or differential forms as a Lie derivative. Thus, for the rotational SU(2), the above operator J_a may be regarded as the symmetry generators. And so is $-i\mathcal{L}_K$ for the relative U(1) gauge symmetry.

Symmetries generated by these vectors remain symmetries of the low energy dynamics when we turn on the potential term determined by $G=aK$. This is guaranteed because L_a's and K preserve G as in

$$J_a(G)=0=\mathcal{L}_K(G). \tag{56}$$

However, it turns out that the J_a's are not quite the physical angular momentum. We will come back to this crucial point shortly.

C. Angular momentum on S^3 and spin

As we observed from the known degeneracy of 1/4 BPS dyons, the bound state with relative charge q must have angular momentum that scales with q linearly. How do we realize such a multiplet? Note that the Taub-NUT space has the topology of $R^4=R^+\times S^3$. On the S^3, the SU(2), which itself is topologically S^3, acts freely as translations, so functions on S^3 naturally fall under various SU(2) representations. The angular momentum eigenfunctions on S^3 are well-known to those familiar with classic angular momentum theory, and are often denoted by D^j_{mk} [17]. Abstractly, it is defined as a finite rotation operator sandwiched between a pair of eigenstates of total angular momentum j,

$$D^j_{mk}(\theta,\phi,\chi)\equiv\sqrt{2j+1}\langle j;k|U(\theta,\phi,\chi)|j;m\rangle, \tag{57}$$

where we chose a normalization so that the norm is independent of j, m, or k. The definition makes it clear that m and k are bounded below and above by $\pm j$,

$$j\geqslant m,\quad k\geqslant -j, \tag{58}$$

and separated from j in integer steps. For any fixed k or m, the $2j+1$ functions indexed by m or by k would form a spin j multiplet of $SU(2)_L$ or $SU(2)_R$, respectively, where the isometry group of undeformed S^3 is

$$SO(4)=SU(2)_L\times SU(2)_R.$$

One might worry that the angular momentum eigenstates of our low energy dynamics may have little to do with these D functions. After all, a wave function should be regarded as a differential form whose transformation properties are generically more complicated than functions. However, as we pointed out earlier, the Taub-NUT manifold admits orthonormal basis w^μ, all four of which are invariant under J_a. This means that as long as we construct the wave function and/or differential form in the basis spanned by the orthonormal frame ω^μ, its transformation property under J_a originates entirely from its coefficient functions. Thus we expect the generic form of the wave function could be written as

$$\Omega=\sum D^j_{mk}(\theta,\phi,\chi)\Lambda_{j;k}(r;dr,\sigma_a), \tag{59}$$

where the differential forms $\Lambda_{j;k}$ have no explicit dependence on the three Euler angles except through the σ_a's. D^j_{mk} with $m=-j,-j+1,\ldots,j$ form a multiplet under J_a with $-i\partial_\chi$ charge k.

Given the definition of D functions, it is not difficult to show that the other lower indices, namely k, are eigenvalues of the operator $-i\partial_\chi$. [Because the actual geometry of the three sphere is deformed, the SU(2) under which D^j_{mk}, $k=-j,-j+1,\ldots,j$ would have formed a representation is no longer a symmetry.] Since j is bounded below by $|k|$, a large relative charge necessarily implies a large angular momentum; the degeneracy has to scale linearly with increasing $k\simeq q$. This is precisely the behavior we saw from the state counting of Sec. IV A.

However, there is something missing. Given a single eigenstate Ω, we expect to generate other physical states related to it by acting with a SUSY charge, such as Q. The underlying field theory tells us the new state $Q|\Omega\rangle$ must be fermionic and/or bosonic if $|\Omega\rangle$ is bosonic and/or fermionic. In particular their physical spin should differ by 1/2. On the other hand, Q itself is invariant under the action of $J_a=-i\mathcal{L}_{L_a}$, and cannot impart additional angular momentum quantum numbers, it seems.

The resolution of this dilemma is that we should modify the angular momentum operator by adding a ''spin'' piece. The hyper-Kähler structure of the Taub-NUT manifold supplies such additional conserved quantities, fortunately [18]. Let $\mathcal{I}^{(a)}$ be the three complex structures as before. Define a triplet of operators S_a acting on fermions by

$$S_a=\frac{i}{2}\mathcal{I}^{(a)}_{\mu\nu}\varphi^\mu\varphi^{*\nu}. \tag{60}$$

The pointwise action is, in geometrical terms,

$$S_a(dz^\mu)=\frac{i}{2}\mathcal{I}_\lambda{}^{(a)\mu}{}_\lambda dz^\lambda, \tag{61}$$

$$S_a\left(\frac{\partial}{\partial z^\mu}\right)=\frac{i}{2}\mathcal{I}_\mu{}^{(a)\lambda}{}_\mu\left(\frac{\partial}{\partial z^\lambda}\right), \tag{62}$$

which is nothing but the action of the three complex structures $\mathcal{I}^{(a)}$ up to a numerical factor. These S_a's span an SU(2) R symmetry of the $N=4$ superalgebra.

The S_a's themselves form a triple under the J_a's, and by using this fact we may write down a new set of angular momentum generators,

$$M_a\equiv J_a-S_a, \tag{63}$$

which also span an SU(2) algebra. Note that, unlike J_a, M_a commute with the R charges S_a.

We seem to have two possible choices of angular momentum; J_a, which rotate R charge, and M_a, which do not. Both commute with the Hamiltonian, but their commutators with supercharges are quite a different matter. Under J_a, the four complex supercharges fall into a singlet plus a triplet, i.e.,

$$[J_a,Q]=0,\quad [J_a,Q_{(b)}]=i\epsilon_{abc}Q_{(c)}. \tag{64}$$

On the other hand, since the four complex supercharges belong to doublets under S_a, they must form doublets under M_a as well. More specifically, the following linear combinations

$$Q_+=Q+iQ^{(3)},$$
$$Q_-=iQ^{(1)}+Q^{(2)}, \tag{65}$$

form one doublet under M_a with $[S_3,Q_\pm]=-\frac{1}{2}Q_\pm$, and the second combination

$$\tilde{Q}_+=iQ^{(1)}-Q^{(2)},$$
$$\tilde{Q}_-=Q-iQ^{(3)}, \tag{66}$$

form another doublet under M_a with $[S_3,\tilde{Q}_\pm]=+\frac{1}{2}\tilde{Q}_\pm$. Since supercharges should carry physical spin 1/2, we surmise that M_a rather than J_a should be interpreted as the physical angular momentum. We will denote the eigenvalues of M^2 by $l(l+1)$.

D. Anti-self-dual ansatz

Now we may proceed to write down the ansatz for dyonic BPS bound states. The BPS equation is easily seen to be invariant under the Hodge dual operation on the wave function. This property can be used to separate the self-dual part from the anti-self-dual part of the trial wave function, so a BPS wave function should be either self-dual or anti-self-dual.

Does the dynamics prefer one to the other? The Hamiltonian has three kinds of potential terms. In addition to the purely bosonic potential G^2, there are two more terms; one is a fermion bilinear contracted with dG, while the other is a fermion quadrilinear contracted with the Riemann curvature. The salient point is that dG and the Riemann curvature are both anti-self-dual tensors on the Taub-NUT manifold. Because of this, a self-dual ansatz will not be sensitive to some spin-spin type long range interaction, which could be crucial for the formation of bound states. In fact, the threshold bound state of SU(3) monopoles (when $G=0$) is known to be anti-self-dual, while no such self-dual bound state exists [9]. We expect that this behavior persists when $G=aK$ is turned on, which motivates us to look for 1/4 BPS dyonic bound states with an anti-self-dual ansatz.[1]

[1]Nonetheless, there is no reason to preclude self-dual bound states that do not saturate the BPS bound.

Let us start with anti-self-dual 2 forms. One interesting property of anti-self-dual 2 forms on hyper-Kähler 4 manifolds, is that they are of type (1,1), upon Hodge decompositions with respect to any one of three complex structures [19]. Recall that up to a numerical factor, the ''spin'' S_a act on differential forms as complex structures do. Since any form of type (n,n) is annihilated by the complex structure, we conclude that the two-form part of an anti-self-dual ansatz carries no spin.

On the other hand, the BPS equation connects even forms with even forms, so an ansatz containing anti-self-dual two-form may contain, in addition, zero form and four form. Neither carries ''spin'': zero form is obviously invariant under S_a, and a four form is always proportional to the volume form which is always of type (n,n). Thus, an anti-self-dual even form is always ''spinless''; l equals j. The angular dependence of anti-self-dual even forms can be written entirely in terms of D functions and the basis of σ_a's.

Of the four angular momentum multiplets, the cases of $l=q\geqslant 0$ and of $l=q-1\geqslant 0$ belong to this category. For $l=q$, the wave function should have the form

$$\Omega^q_{m;q}=D^q_{mq}\Lambda_{q;q}+D^q_{m(q-1)}\Lambda_{q;q-1}, \tag{67}$$

for any value of $m=q,q-1,\ldots,-q$, where differential forms Λ's can be written entirely with r and ω^μ only. This state has relative charge $q\geqslant 0$ if and only if the anti-self-dual form Λ's satisfy

$$-i\mathcal{L}_K\Lambda_{q;q}=0, \quad -i\mathcal{L}_K\Lambda_{q;q-1}=\Lambda_{q;q-1}. \tag{68}$$

Since the only charged combination one can build out of r and ω^μ is $\omega^1\pm i\omega^2$, which has ± 1 charge respectively, the Λ's may contribute ± 1 to the total charge q at most. Furthermore, only $\omega+i\omega^2$ may enter since $D^q_{m(q+1)}$, which should accompany $\omega-i\omega^2$, does not exist for positive q.

These considerations constrain possible form of Λ's quite severely. We find that only the following choice is consistent with the known quantum numbers of 1/4 BPS dyons,

$$\Lambda^{q;q}=f(r)+h(r)(\omega^0\wedge\omega^3+\omega^1\wedge\omega^2)+f(r)(\omega^0\wedge\omega^1\wedge\omega^2\wedge\omega^3), \tag{69}$$

$$\Lambda_{q;q-1}=[b(r)/r](\omega^0+i\omega^3)\wedge(\omega^1+i\omega^2), \tag{70}$$

for $l=q\geqslant 0$ BPS state.[2]

In case of $l=q-1$ with $q\geqslant 1$, the anti-self-dual ansatz is even more restrictive. The only such ansatz consistent with known quantum numbers is

$$\Omega^{q-1}_{m;q}=p(r)D^{q-1}_{m(q-1)}(\omega^0+i\omega^3)\wedge(\omega^1+i\omega^2), \tag{71}$$

for $m=q-1,q-2,\ldots,1-q$. No zero form may appear since a factor of $(\omega^1+i\omega^2)$ is necessary to make the electric charge to be q, while four form is ruled out subsequently by the fact that the BPS wave functions we are looking for are all anti-self-dual.

We will postpone discussion of the two remaining multiplets of $l=q-1/2\geqslant 0$ to the following section. The corresponding multiplets are in odd forms, which can be found by acting supercharges on $\Omega^q_{m;q}$ and $\Omega^{q-1}_{m;q}$. For these two multiplets, S_a contribution to the physical angular momentum does not vanish.

[2]We have defined the Hodge dual operation with respect to the volume form $-\omega^0\wedge\omega^1\wedge\omega^2\wedge\omega^3$.

V. DYONIC BPS BOUND STATES

Here, we will solve for the dyonic BPS bound states explicitly. Such dyonic bound state do not exist for all relative charge q. Rather, it is known that $|q|$ must be smaller than the critical charge $|q_{cr}|$ [1,2,15], where

$$q_{cr}=\lim_{r\to\infty} a\langle K,K\rangle. \tag{72}$$

With our current normalization, $\langle K,K\rangle$ asymptotes to 1 at infinity, so the critical charge q_{cr} is equal to the parameter a. Thus we expect to find the BPS bound state, that is, a normalizable and regular wave function that preserves half of low energy supersymmetry, only when $|q|\leqslant|a|$. Without loss of generality, we will take both a and q to be nonnegative.

Classical analysis of Ref. [1,2] leaves it unclear whether the bound state should exist (at threshold) when $a=q$. Classically two monopoles are infinitely separated, but quantum mechanically, there may be a bound state with powerlike decay. As the following analysis will show, however, no such threshold bound state exists, except for $a=q=0$ case.

A. The $l=q-1$ multiplet

The case of $l=q-1$ is the simplest. Starting with the ansatz,

$$\Omega^{q-1}_{m;q}=p(r)D^{q-1}_{m,q-1}(\omega^0+i\omega^3)\wedge(\omega^1+i\omega^2), \tag{73}$$

the BPS equation reduces to a single ordinary differential equation for $p(r)$,

$$\frac{d}{dr}rp(r)=-A(r)rp(r), \tag{74}$$

where the quantity A is defined to be the following combination:

$$A\equiv a-q\left(1+\frac{1}{r}\right). \tag{75}$$

A useful fact we employed is that

$$e^r(\omega^0+i\omega^3)\wedge(\omega^1+i\omega^2) \tag{76}$$

is a (nonnormalizable) harmonic two form. The equation is easily solved to give the wave function:

$$\Omega^{q-1}_{m;q} = D^{q-1}_{m(q-1)} r^{q-1} e^{-(a-q)r} (\omega^0 + i\omega^3) \wedge (\omega^1 + i\omega^2). \tag{77}$$

The wave function is exponentially small at large r and normalizable as long as $a > q$. This way, we have recovered the fact that a is the critical electric charge, beyond which no bound state may exist. Furthermore, the wave function is regular as long as $q \geqslant 1$, which is also consistent with the fact that $l = q-1$ BPS bound state exists only for $q \geqslant 1$.

B. The $l = q$ multiplet

The ansatz for $l = q \geqslant 0$ is a bit more involved;

$$\begin{aligned}\Omega_{q;q} = D^q_{mq}[f(r) + h(r)(\omega^0 \wedge \omega^3 + \omega^1 \wedge \omega^2)\\ + f(r)(\omega^0 \wedge \omega^3 \wedge \omega^1 \wedge \omega^2)] + D^q_{m(q-1)}\\ \times \{[b(r)/r](\omega^0 + i\omega^3) \wedge (\omega^1 + i\omega^2)\},\end{aligned} \tag{78}$$

under which BPS equations reduce to

$$\frac{d}{dr} f = -Ah + \frac{b}{r^2}, \quad \frac{d}{dr} h + \frac{2h}{1+r} = -Af - \frac{b}{r^2},$$

$$\frac{d}{dr} b + Ab = q(f-h). \tag{79}$$

In order to solve Eq. (79) for general $q > 0$, we proceed as follows. By substituting

$$f = u(r) e^{-\int A}, \quad h = v(r) e^{-\int A}, \quad b = w(r)\left(\frac{q}{A}\right) e^{-\int A}, \tag{80}$$

into Eq. (79), one obtains

$$\frac{d}{dr}(u-w) = 0, \quad \frac{d}{dr} u + \frac{1}{(1+r)^2} \frac{d}{dr}(1+r)^2 v = 0,$$

$$\frac{d}{dr}(w/A) = u - v. \tag{81}$$

We solve the first equation by $u = C_1 + w$ with an integration constant C_1. The remaining equations can be combined into a single second order equation,

$$\frac{d}{dr}\left[\frac{d}{dr}\left(\frac{(1+r)w}{A}\right) - 2A\left(\frac{(1+r)w}{A}\right) - 2C_1 r\right] = 0, \tag{82}$$

which is integrated with a second integration constant C_2 to

$$\frac{d}{dr}\left(\frac{(1+r)w}{A}\right) - 2A\left(\frac{(1+r)w}{A}\right) = +2C_1 r - 2C_2. \tag{83}$$

Integrating this equation gives us three-parameter family of solutions. But, fortunately, the simplest possible solution,

$$\frac{(1+r)w}{A} = -r, \tag{84}$$

turns out to be the only regular and normalizable solution. Using it to generate other radial functions, we find the $l = q$ bound states:

$$\begin{aligned}\Omega^q_{m;q} = D^q_{mq} \frac{r^q e^{-(a-q)r}}{1+r}\Bigg[a + \left(a + \frac{1}{1+r}\right)(\omega^0 \wedge \omega^3 + \omega^1 \wedge \omega^2)\\ + a\,\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3\Bigg]\\ - D^q_{m(q-1)} \frac{r^q e^{-(a-q)r}}{1+r} \sqrt{q/2}(\omega^0 + i\omega^3) \wedge (\omega^1 + i\omega^2).\end{aligned} \tag{85}$$

Again, we find that the wave function is normalizable as long as $a > q$. The solution is regular at origin for all non-negative q. We have recovered the $l = q$ multiplet of 1/4 BPS dyon of charge $q > 0$.

The case of $q = 0$ is a bit special, where the BPS state is a purely magnetic bound state of the two monopoles. For $q = 0$, the second D function does not exist, and we must solve a modified BPS equation. Nevertheless, the actual wave function is also obtained by taking $q = 0$ limit of the above result as follows:

$$\begin{aligned}\Omega^0_{0;0} = \frac{e^{-ar}}{1+r}\Bigg[a + \left(a + \frac{1}{1+r}\right)(\omega^0 \wedge \omega^3 + \omega^1 \wedge \omega^2)\\ + a\,\omega^0 \wedge \omega^1 \wedge \omega^2 \wedge \omega^3\Bigg].\end{aligned} \tag{86}$$

This BPS state actually preserves all supercharges of low energy dynamics. In fact, this is the lowest lying state of this low energy effective theory. In the limit of aligned vacua $(a = 0)$, this state also reverts to the threshold bound state of two monopoles found in Ref. [9], as it should.

C. The $l = q - 1/2$ multiplets

The remaining two multiplets of $l = q - 1/2$ can be found most easily by acting Q on $\Omega^q_{m;q}$ and $\Omega^{q-1}_{m;q}$ found above. From $\Omega^{q-1}_{m;q}$, we find $2q-1$ states

$$\frac{r^q e^{-(a-q)r}}{\sqrt{r + r^2}} (\omega^1 + i\omega^2) \wedge (1 + \omega^0 \wedge \omega^3) D^{q-1}_{m(q-1)}, \tag{87}$$

while $\Omega^q_{m;q}$ produces $2q+1$ states

$$\begin{aligned}\frac{r^q e^{-(a-q)r}}{\sqrt{r + r^2}}[(\omega^0 + i\omega^3) \wedge (1 + \omega^1 \wedge \omega^2)\\ \times \sqrt{2q} D^q_{mq} + i(\omega^1 + i\omega^2) \wedge (1 + \omega^0 \wedge \omega^3) D^q_{m(q-1)}].\end{aligned} \tag{88}$$

These account for all $4q$ states in the two $l = q - 1/2$ multiplets. (Alternatively we could have used $Q^{(a)}$ instead of Q. Since $\Omega^{q-1}_{m;q}$ and $\Omega^q_{m;q}$ are invariant under complex structures, and since $Q^{(a)}$ are essentially Q rotated by the complex structures, the resulting $4q$ states will be simply the above $4q$ states with complex structure $\mathcal{I}_a$ acting on them.)

However, because Q is a singlet under J_a, the above states form multiplets under J_a instead of the physical angular momentum M_a. To reconstruct M_a multiplets, we recall the relationship

$$J_a = M_a + S_a. \tag{89}$$

Since the two SU(2) generators, M_a and S_a, commute with each other, J_a multiplets are constructed from M multiplets and S multiplets by the rule of angular momentum addition. On the other hand, we actually need to reconstruct M_a eigenstates from J_a and S_a eigenstates, for which we need to reverse the procedure. Without delving into details of the computation, we present the two $l = q - \frac{1}{2}$ multiplets. Because S_3 commutes with M_a, we can label the two multiplets by its eigenvalue s_3. The first has $s_3 = 1/2$:

$$(\Omega_{(+)})^{(q-1/2)}_{m;q} = \frac{r^q e^{-(a-q)r}}{\sqrt{r+r^2}} \Bigg[(\omega^0 + i\omega^3)\wedge(1+\omega^1\wedge\omega^2) \sqrt{\frac{q+m+1/2}{2q+1}} D^q_{(m+1/2)q} + i(\omega^1 + i\omega^2)\wedge(1+\omega^0\wedge\omega^3) \times\left(\sqrt{\frac{q+m+1/2}{2q(2q+1)}} D^q_{(m+1/2)(q-1)} + \sqrt{\frac{q-m-1/2}{2q}} D^{q-1}_{(m+1/2)(q-1)} \right) \Bigg]. \tag{90}$$

The other multiplet has $s_3 = -1/2$;

$$(\Omega_{(-)})^{(q-1/2)}_{m;q} = \frac{r^q e^{-(a-q)r}}{\sqrt{r+r^2}} \Bigg[(\omega^0 + i\omega^3)\wedge(1+\omega^1\wedge\omega^2) \sqrt{\frac{q-m+1/2}{2q+1}} D^q_{(m-1/2)q} + i(\omega^1 + i\omega^2)\wedge(1+\omega^0\wedge\omega^3) \times\left(\sqrt{\frac{q-m+1/2}{2q(2q+1)}} D^q_{(m-1/2)(q-1)} - \sqrt{\frac{q+m-1/2}{2q}} D^{q-1}_{(m-1/2)(q-1)} \right) \Bigg]. \tag{91}$$

In both expressions, the index m takes values $q-1/2, q-3/2, \ldots, -q+1/2$.

In a direct construction of these two $l = q - \frac{1}{2}$ multiplets from the $l = q$ or $l = q-1$ multiplets, the role of the doublet supercharges in Eq. (65) and Eq. (66) can be easily identified. As a simple application of the angular momentum addition rule, the operations of the doublets on the even-form multiplets will produce $(q+\frac{1}{2})\oplus(q-\frac{1}{2})[=\frac{1}{2}\otimes q]$ or $(q-\frac{1}{2})\oplus(q-\frac{3}{2})[=\frac{1}{2}\otimes(q-1)]$, but one may check that both $l = q + \frac{1}{2}$ and $l = q - \frac{3}{2}$ multiplets vanish identically. Since the doublet $Q_{\pm}$ carries the spin eigenvalue $s_3 = -\frac{1}{2}$, the operation of this doublet on $l = q$ or $l = q-1$ multiplets produces the $l = q - \frac{1}{2}$ multiplet with $s_3 = -\frac{1}{2}$. Similarly, the application of $\tilde{Q}_{\pm}$ results in the $l = q - \frac{1}{2}$ multiplet with $s_3 = +\frac{1}{2}$.

D. Characteristics of the BPS states

In the construction of the BPS bound states, we have limited ourselves to the case of non-negative electric charge. For negatively charged bound states, we note the fact that the complex conjugation of a solution to the BPS equations in Eq. (44) gives another solution. Both the eigenvalues of the charge and M_3 reverse their signs under the complex conjugation. Thus the negatively charged solution $\Omega^l_{-m;-q}$ $(q \geq 0)$ is simply given by the complex conjugation of $\Omega^l_{m;q}$.

We now turn to the case where the Higgs misalignment parameter a is negative. When a is replaced by $-a$, only the fermionic term that couples to the Killing potential changes its sign in the supersymmetric Lagrangian (12). By the parity operation $\psi \to i\gamma^0\psi$ or, equivalently, $\varphi \to i\varphi$, one can bring this Lagrangian to the original form with $a>0$. The corresponding transformation amounts to replacement of dx^μ by $i dx^\mu$. The solutions for $-a$ are obtained if one replaces all ω^μ of the above BPS solutions with $i\omega^\mu$. These exhaust all the possibilities.

For the remainder of the section, we would like to comment briefly on some other aspects of the BPS states. With an excited charge q, the effective potential at large relative separation tends to $(a^2+q^2)/2$. Since the energy eigenvalue of the BPS multiplets is $|aq|$, one finds the that binding energy of the dyons is

$$E_{\text{binding}} = \frac{(|a|-|q|)^2}{2}, \tag{92}$$

which tends to zero as the charge approaches its critical value, as one expects from its classical counterpart. Another characteristic is the separation of the two monopole cores. In the classical limit, the separation between the two cores is given by $r_{\text{eq}} \equiv |q|/(|a|-|q|)$. We expect the vacuum expectation value

$$\langle r \rangle \equiv \frac{\langle \Omega | r | \Omega \rangle}{\langle \Omega | \Omega \rangle} \tag{93}$$

to approach r_{eq} in the classical limit. For a given supermultiplet with charge q, the expectation values are found to be dependent upon the angular momentum quantum number l. For instance, we find the expectation value

$$\langle r\rangle=\frac{|q|}{(|a|-|q|)}\left(1+\frac{1}{2|a|}\right)=r_{\rm eq}\left(1+\frac{1}{2|a|}\right) \tag{94}$$

for the $l=|q|-1$ multiplet, and

$$\langle r\rangle=\frac{|q|}{(|a|-|q|)}\left(1+\frac{1}{2|q|}\right)=r_{\rm eq}\left(1+\frac{1}{2|q|}\right) \tag{95}$$

for the $l=|q|-1/2$ multiplets. To restore the Planck constant $\hbar$, we simply observe that classical charge q has the same dimension as $\sqrt{\hbar}$. Since the difference between $\langle r\rangle$ and $r_{\rm eq}$ scales inversely with q or $a=q_{cr}$, it has to scale linearly with $\sqrt{\hbar}$:

$$\frac{\langle r\rangle-r_{\rm eq}}{r_{\rm eq}}\sim O(\sqrt{\hbar}). \tag{96}$$

Thus, $\langle r\rangle$ indeed approaches $r_{\rm eq}=|q|/(|a|-|q|)$ in the classical limit.

VI. CONCLUSION

In the low energy dynamics of 1/2 BPS monopoles, the misaligned Higgs vacua induces an attractive potential between monopoles of distinct types. This potential is crucial in the formation of new dyonic bound states of monopoles, some of which preserve 1/4 of the supersymmetries in a field theoretical sense or, equivalently, 1/2 of the supersymmetries of the low energy dynamics of monopoles. Starting from the full $N=4$ supersymmetric low energy effective Lagrangian, we expressed the BPS equation of the system in the language of the differential form, which was then further reduced to a set of coupled first-order ordinary differential equations. These equations were solved analytically, giving 1/4 BPS dyons as quantum bound states of two distinct SU(3) monopoles. Along the course of the construction, we have given a full account of the supermultiplet structures of the quantum 1/4 BPS dyons.

In this paper, we have focused on the BPS saturated states in pursuit of the 1/4 BPS dyons. However, it is expected that there exist spectra of other dyonic bound states that do not saturate the BPS bound. The problem of finding these non-BPS bound states is quite involved. Nevertheless, there is some additional information that might be of help. Supersymmetric sigma models on the Taub-NUT geometry are known to allow additional conserved quantities of the Runge-Lenz type [16,18]. It seems quite plausible that this new symmetry generalizes to the present low energy dynamics with potential. We have checked that the purely bosonic part indeed admits such conserved quantities. Such additional symmetries might be useful in finding the excited non-BPS bound states.

Another aspect of the dynamics we did not discuss here is the scattering of dyons. The supersymmetric quantum mechanics we used can be thought of as low energy dynamics for 1/4 BPS dyons in two ways. First, it produces these dyons as bound states. Second, it provides a framework where interaction among these dyons can be studied in a quantum mechanical setting. The dynamics is admittedly more involved than the usual moduli space dynamics, given the presence of the potential. It requires further study.

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