Superparticle models with tensorial central charges

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A generalization of the Ferber-Shirafuji formulation of superparticle mechanics is considered. The generalized model describes the dynamics of a superparticle in a superspace extended by tensorial central charge coordinates and commuting twistorlike spinor variables. The $D=4$ model contains a continuous real parameter $a\geqslant 0$ and at $a=0$ reduces to the $SU(2,2|1)$ supertwistor Ferber-Shirafuji model, while at $a=1$ one gets an $OSp(1|8)$ supertwistor model proposed by two of the authors which describes BPS states with all but one unbroken target space supersymmetries. When $0<a<1$ the model admits an $OSp(2|8)$ supertwistor description, and when $a>1$ the supertwistor group becomes $OSp(1,1|8)$. We quantize the model and find that its quantum spectrum consists of massless states of an arbitrary (half-)integer helicity. The independent discrete central charge coordinate describes the helicity spectrum. We also outline the generalization of the $a=1$ model to higher space-time dimensions and demonstrate that in $D=3$, 4, 6, and 10, where the quantum states are massless, the extra degrees of freedom (with respect to that of the standard superparticle) parametrize compact manifolds. These compact manifolds can be associated with higher-dimensional helicity states. In particular, in $D=10$ the additional ''helicity'' manifold is isomorphic to the sphere S^7.

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I. INTRODUCTION

In a recent paper [1] two of the present authors proposed a new superparticle model with tensorial central charges [2–8] and auxiliary fundamental spinor variables. An interesting peculiar feature of this model is that it describes a superparticle whose presence breaks only one target-space supersymmetry. In all previously known cases superparticles and (in general) superbranes break half or more of the supersymmetries of a target superspace vacuum.

As we shall show in this paper the model of Ref. [1] describes an infinite tower of massless particles of arbitrary (half-)integer helicities. The model can be regarded as an extension of a Ferber-Shirafuji formulation [9,10] of $D=4$ superparticle mechanics. In the framework of the $N=1$ Ferber-Shirafuji model one performs, at the classical level, the twistor transform [9,10] from the $N=1$, $D=4$ superspace description of massless superfields to their description in terms of supertwistors forming a fundamental representation of a superconformal group $SU(2,2|1)$.

In an analogous way the superparticle model of [1] admits the description in terms of $OSp(1|8)$ supertwistors [11,12]. The supergroups $SU(2,2|1)$ and $OSp(1|8)$ are not subgroups of each other, but they are different subgroups of the supergroup $OSp(2|8)$. Hence, one can assume that the Ferber-Shirafuji model and the model of Ref. [1] are different reductions of an $OSp(2|8)$ supertwistor model.

In this paper we construct such a generic $N=1$, $D=4$ superparticle action which depends on a numerical nonnegative real parameter a. When the value of a varies within the interval $0<a<1$ the model admits an $OSp(2|8)$ supertwistor description, while $a=0$ and $a=1$ are two critical points. At $a=0$ the model reduces to the Ferber-Shirafuji superparticle. And at $a=1$ one arrives at the $OSp(1|8)$ supertwistor model of Ref. [1].

For all values of a except for $a=1$ the superparticle breaks half of the target-space supersymmetries, while at $a=1$ only one supersymmetry is broken. When $a>1$ the supertwistor group becomes $OSp(1,1|8)$ which contains a noncompact group $SO(1,1)$ as a subgroup instead of $SO(2)$ in the case of $OSp(2|8)$.

The (super)twistor formulation of relativistic (super)particle dynamics is useful in many aspects. Let us recall that, since the relativistic (super)particle is a constrained dynamical system not all its dynamical variables are independent. By performing (super)twistor transform we deal directly with independent physical degrees of freedom of the (super)particle in a covariant way. This, for instance, simplifies the quantization procedure and the analysis of the spectrum of quantum states of the model.

We perform the quantization of the generic superparticle

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model for arbitrary values of the parameter a and find that in $D=4$ first–quantized states of the superparticle form an infinite tower of massless states of a (half-)integer helicity. We thus demonstrate that an extra dynamical (central charge) coordinate in the model under consideration has the physical meaning of a spin variable. This allows one to admit that the model considered might be related to the higher-spin field theory of Vasiliev [13] (see also relevant papers [14]).

We first quantize the superparticle in the supertwistor formulation where the quantization is almost straightforward since at $a\neq 0$ the supertwistor model is unconstrained and at $a=0$ there is only one first-class constraint. We then quantize the model in the $N=1$, $D=4$ superspace extended with the tensorial central charge coordinates, and show that the resulting spectrum of the quantum physical states coincides with that of the supertwistor formulation. Since in this formulation the model contains second class constraints our main tool in carrying out the quantization procedure will be the extension of the model in such a way that all the constraints of the initial model become first class constraints. This method, which can be traced back to the papers by Faddeev and Shatashvili [15,16], Batalin, Fradkin, and Fradkina [17,18], and Egorian and Manvelian [19] has already been applied to the quantization of ''standard'' massless superparticles by Moshe [20] and Eisenberg and Solomon [21–23]. The main advantage of this method is that it allows one to avoid problems with covariant splitting fermionic constraints into first and second class ones. The initial formulation of the model is recovered when we fix additional gauge symmetries (associated with new first-class constraints) by putting the conversion variables to zero. In its nature the conversion method is related to an old Stueckelberg formalism [24] which extends the theory of massive vector fields with an auxiliary scalar gauge degree of freedom.

The quantization of the model at $a=1$ has additional peculiarities. In this case superparticle dynamics is subject to only one second-class constraint, which is quite unusual. Dynamical systems with the odd number of fermionic second class constraints are rather rare. One of few known examples is a superparticle in $D=2$ superspace with a single chiral fermion direction [27]. So the quantization of such systems is an interesting exercise by itself which requires dealing with a single Clifford-like variable. In the case under consideration we shall use an auxiliary Clifford variable of Grassmann-odd parity to convert the single second class fermionic constraint into the one of a first class. Further we present two methods for quantizing the model with the single Clifford variable, both producing the same spectrum of first-quantized physical states.

The paper is organized as follows.

In Sec. II we consider the one-parameter family of actions describing the generalized $D=4$ superparticle models labeled by the real positive parameter $a\geqslant 0$ where the case $a=0$ corresponds to the Ferber-Shirafuji model, while at $a=1$ the action describes the superparticle model of Ref. [1]. We demonstrate that in the target space with four supersymmetries the $a\neq 1$ models possess two fermionic κ-symmetries and, hence, corresponding superparticle configurations preserve 1/2 of the supersymmetries, while if $a=1$ generalized superparticle has three κ symmetries and, hence, preserves $\frac{3}{4}$ of the supersymmetries. We also find that the $U(1)$ symmetry is inherent to the $a=0$ case only.

In Sec. III we describe the transform to a supertwistor form of the action. We show that $0<a<1$ models are described by a free $OSp(2|8)$ supertwistor action and thus link the Ferber-Shirafuji $SU(2,2|1)$ supertwistor model and the free $OSp(1|8)$ supertwistor model of [1]. The model with $a>1$ is transformed into a free $OSp(1,1|8)$ supertwistor action. We perform the quantization of the supertwistor models and find that the ''supertwistor'' wave function describes an infinite tower of short supersymmetric multiplets of massless fields of all possible helicities.

In Sec. IV we extend the initial phase space of the superparticle model with auxiliary variables and perform the conversion of the initial set of first and second class constraints into the first-class constraints generating new gauge symmetries. We then carry out the quantization of the extended model and find that the dependence of the wave functions on Grassmann-odd conversion variables is inessential and can be ignored. We show that the wave function of the first-quantized model of Sec. II can be identified with the supertwistor wave function of Sec. III if Cartan-Penrose twistor formulas relating superspace and supertwistor coordinates are imposed. Thus, we find that the infinite spectrum of the first-quantized states of the superparticle consists of massless fields of an arbitrary (half)integer helicity.

In Sec. V we consider a multidimensional generalization of the $a=1$ model and its quantization. It appears that after quantization the superwave function depends on only one Grassmann variable, and all other fermionic degrees of freedom can be eliminated by $N-1$ κ transformations, where N is the total number of supersymmetries. Thus, the corresponding superparticle configuration preserves the $(N-1)/N$ fraction of target-space supersymmetry. In the Appendix we analyze in detail the quantization of a supersymmetric system with one real Grassmann variable, which after quantization becomes a single Clifford variable.

II. $D=4$ MODEL WITH FUNDAMENTAL SPINOR AND TENSORIAL CENTRAL CHARGE COORDINATES

Let us consider the following $D=4$ superparticle action:

$$S=\int d\tau(\lambda_A\overline{\lambda}_{\dot B}\Pi_\tau^{A\dot B}+a\lambda_A\lambda_B\Pi_\tau^{AB}+\overline{a\lambda}_{\dot A}\overline{\lambda}_{\dot B}\Pi_\tau^{\dot A\dot B}), \tag{2.1}$$

where

$$\Pi^{A\dot B}\equiv d\tau\Pi_\tau^{A\dot B}=dx^{A\dot B}+i(d\Theta^A\overline{\Theta}^{\dot B}-\Theta^A d\overline{\Theta}^{\dot B}), \tag{2.2}$$

$$\Pi^{AB}\equiv d\tau\,\Pi_\tau^{AB}=dy^{AB}-i\Theta^{(A}d\Theta^{B)},$$

$$\overline{\Pi}^{\dot A\dot B}\equiv d\tau\,\overline{\Pi}_\tau^{\dot A\dot B}=d\overline{y}^{\dot A\dot B}-i\overline{\Theta}^{(\dot A}d\overline{\Theta}^{\dot B)}, \tag{2.3}$$

$A,B=1,2$, $\dot{A},\dot{B}=1,2$ are Weyl spinor indices, and the spin–tensors $x^{A\dot{B}}$ and y^{AB} are related to $D=4$ vector coordinates x^m and antisymmetric tensorial coordinates y^{mn} through the Pauli matrices

$$x^{A\dot{B}}=x^m\sigma_m^{A\dot{B}}, \quad y^{AB}=\frac{1}{2}y^{mn}(\sigma_{[m}\tilde{\sigma}_{n]})^{AB}=(\overline{y}^{\dot{A}\dot{B}})^*. \tag{2.4}$$

a is a numerical parameter which, without the loss of generality, can be taken to be real and positive definite $a=\overline{a}\in[0,\infty)$. Indeed, if a is complex its phase can always be absorbed by the bosonic spinor λ_A redefined in an appropriate way ($\lambda_A\to(\overline{a}/|a|)^{1/2}\lambda_A$). The action (2.1) describes a superparticle propagating in the extended superspace

$$M^{(4+6|4)}=\{Y^M\}\equiv\{(x^{A\dot{A}},y^{AB},\overline{y}^{\dot{A}\dot{B}};\Theta^A,\overline{\Theta}^{\dot{A}})\} \tag{2.5}$$

with tensorial central charge coordinates $y^{AB},\overline{y}^{\dot{A}\dot{B}}$.[1] The configuration space of the system

$$\begin{aligned}\mathcal{M}^{(4+6+4|4)}&=\{q^{\mathcal{M}}\}\equiv\{(Y^M;\lambda^A,\overline{\lambda}^{\dot{A}})\}\\&=\{(x^{A\dot{A}},y^{AB},\overline{y}^{\dot{A}\dot{B}};\lambda^A,\overline{\lambda}^{\dot{A}};\Theta^A,\overline{\Theta}^{\dot{A}})\}\end{aligned} \tag{2.6}$$

contains in addition four bosonic spinor coordinates $\lambda^A,\overline{\lambda}^{\dot{A}}$.

The presence of the parameter a in the action (2.1) reflects the property that each of its three terms is separately invariant under global supersymmetry transformations acting on $M^{(4+6|4)}$ as follows:

$$\begin{aligned}&\delta\Theta^A=\epsilon^A, \quad \delta\overline{\Theta}^{\dot{B}}=\overline{\epsilon}^{\dot{B}},\\&\delta x^{A\dot{B}}=i\epsilon^A\overline{\Theta}^{\dot{B}}-i\Theta^A\overline{\epsilon}^{\dot{B}},\\&\delta y^{AB}=i\epsilon^{(A}\Theta^{B)},\\&\delta\overline{y}^{\dot{A}\dot{B}}=i\overline{\epsilon}^{(\dot{A}}\overline{\Theta}^{\dot{B})},\\&\delta\lambda_A=0, \quad \delta\overline{\lambda}_{\dot{A}}=0.\end{aligned} \tag{2.7}$$

The generators of the transformations (2.7)

$$\delta Y^M=i(\epsilon^A Q_A+\overline{Q}_{\dot{A}}\overline{\epsilon}^{\dot{A}})Y^M$$

satisfy the supersymmetry algebra with central charges

$$\begin{aligned}&\{Q_A,Q_B\}=Z_{AB},\\&\{Q_A,\overline{Q}_{\dot{B}}\}=-2P_{A\dot{B}},\\&\{\overline{Q}_{\dot{A}},\overline{Q}_{\dot{B}}\}=\overline{Z}_{\dot{A}\dot{B}},\end{aligned} \tag{2.8}$$

and all other commutators of the generators vanish.

[1]For previous consideration of different models of superparticles and p-branes in superspaces with tensorial central charges see Refs. [22,29,8].

The superalgebra (2.8) has the following realization in the superspace (2.5):

$$\begin{aligned}&Q_A=i\partial_A-\partial_{A\dot{B}}\overline{\Theta}^{\dot{B}}-\frac{1}{2}\Theta^B\partial_{AB},\\&\overline{Q}_{\dot{A}}=-i\overline{\partial}_{\dot{A}}+\partial_{\dot{A}B}\Theta^B+\frac{1}{2}\overline{\partial}_{\dot{A}\dot{B}}\overline{\Theta}^{\dot{B}},\end{aligned} \tag{2.9}$$

$$P_{A\dot{B}}=-i\partial_{A\dot{B}}, \quad Z_{AB}=-i\partial_{AB}, \quad \overline{Z}_{\dot{A}\dot{B}}=-i\overline{\partial}_{\dot{A}\dot{B}}, \tag{2.10}$$

$$\partial_{A\dot{B}}\equiv\frac{\partial}{\partial x^{A\dot{B}}}, \quad \partial_{AB}\equiv\frac{\partial}{\partial y^{AB}}, \quad \overline{\partial}_{\dot{A}\dot{B}}\equiv\frac{\partial}{\partial\overline{y}^{\dot{A}\dot{B}}},$$

$$\partial_A\equiv\frac{\partial}{\partial\Theta^A}, \quad \overline{\partial}_{\dot{A}}\equiv\frac{\partial}{\partial\overline{\Theta}^{\dot{A}}}.$$

If $a=0$ the action (2.1) reduces to the Ferber-Shirafuji action [9,10], and if $a=1$ the action becomes the one considered in [1]. We shall see that $a=0$ and $a=1$ are ''critical'' values of the parameter where symmetries of the action (2.1) as well as the physical content of the model are modified.

A. Critical points $a=0$ and $a=1$

In order to analyze symmetry properties of the action (2.1) at different values of a we consider the general variation of (2.1) which (modulo boundary terms) has the form (2,1) for $a=1$:

$$\begin{aligned}\delta S=&\int[\delta\lambda_A(\overline{\lambda}_{\dot{B}}\Pi^{A\dot{B}}+2a\lambda_B\Pi^{AB})+\delta\overline{\lambda}_{\dot{A}}(\Pi^{B\dot{A}}\lambda_B\\&+2a\overline{\lambda}_{\dot{B}}\Pi^{\dot{A}\dot{B}})]-\int[d(\lambda_A\overline{\lambda}_{\dot{B}})i_\delta\Pi^{A\dot{B}}+ad(\lambda_A\lambda_B)i_\delta\Pi^{AB}\\&+ad(\overline{\lambda}_{\dot{A}}\overline{\lambda}_{\dot{B}})i_\delta\Pi^{\dot{A}\dot{B}}]+\int[2i(d\overline{\Theta}^{\dot{B}}\overline{\lambda}_{\dot{B}}\\&+ad\Theta^B\lambda_B)\delta\Theta^A\lambda_A+2i(d\Theta^B\lambda_B+ad\overline{\Theta}^{\dot{B}}\overline{\lambda}_{\dot{B}})\delta\overline{\Theta}^{\dot{B}}\overline{\lambda}_{\dot{B}}],\end{aligned} \tag{2.11}$$

where the basis in the space of variations of x and y is chosen in the form

$$i_\delta\Pi^{A\dot{B}}\equiv\delta x^{A\dot{B}}+i(\delta\Theta^A\overline{\Theta}^{\dot{B}}-\Theta^A\delta\overline{\Theta}^{\dot{B}}), \tag{2.12}$$

$$i_\delta\Pi^{AB}\equiv\delta y^{AB}-i\Theta^{(A}\delta\Theta^{B)},$$

$$i_\delta\overline{\Pi}^{\dot{A}\dot{B}}\equiv\delta\overline{y}^{\dot{A}\dot{B}}-i\overline{\Theta}^{(\dot{A}}\delta\overline{\Theta}^{\dot{B})}. \tag{2.13}$$

1. U(1) gauge symmetry of a=0 model

Let us consider the variation of the action when the variations of all fields except for λ_A, $\overline{\lambda}_{\dot{A}}$ are zero

$$\delta S=\int [\delta\lambda_A(\overline{\lambda}_{\dot B}\Pi^{A\dot B}+2a\lambda_B\Pi^{AB}) + \delta\overline{\lambda}_{\dot A}(\Pi^{B\dot A}\lambda_B+2a\overline{\lambda}_{\dot B}\Pi^{\dot A\dot B})]. \tag{2.14}$$

If the variation of λ corresponds to local infinitesimal $U(1)$ rotations

$$\lambda'_A(\tau)=\lambda_A e^{i\alpha(\tau)},\quad \overline{\lambda}'_{\dot A}=\overline{\lambda}_{\dot A}e^{-i\alpha(\tau)} \tag{2.15}$$

the Eq. (2.14) takes the form

$$\delta S=\int [i\alpha(\tau)\lambda_A(\overline{\lambda}_{\dot B}\Pi^{A\dot B}+2a\lambda_B\Pi^{AB}) - i\alpha(\tau)\overline{\lambda}_{\dot A}(\Pi^{B\dot A}\lambda_B+2a\overline{\lambda}_{\dot B}\Pi^{\dot A\dot B})].$$

Such a variation vanishes at $a=0$. Hence at this value of a the $U(1)$ transformations (2.15) describe local symmetry of the model which is inherent to the Ferber-Shirafuji formulation [9,10] of the massless superparticle.

Note that for all values of the parameter a the spinors λ are constants on the mass shell. Indeed, the equations of motion

$$\frac{\delta S}{\delta x^{A\dot B}}=0,\quad \frac{\delta S}{\delta y^{AB}}=0,\quad \frac{\delta S}{\delta \overline{y}^{\dot A\dot B}}=0$$

which follow from Eq. (2.11) have the form

$$d(\lambda_A\overline{\lambda}_{\dot B})=0,\quad ad(\lambda_A\lambda_B)=0,\quad ad(\overline{\lambda}_{\dot A}\overline{\lambda}_{\dot B})=0. \tag{2.16}$$

In the framework of any twistor or twistorlike approach [30] one assumes that the bosonic spinors parametrize a projective space. This requirement does not allow λ to have all its components equal to zero simultaneously.

Then in the generic case $a\neq 0$ Eqs. (2.16) imply

$$d(\lambda_A)=0,\quad d(\overline{\lambda}_{\dot A})=0, \tag{2.17}$$

i.e., the bosonic spinor is constant on the mass shell

$$\lambda_A(\tau)=\lambda^0_A=\text{const},\quad \overline{\lambda}_{\dot A}=\overline{\lambda}^0_{\dot A}=\text{const}. \tag{2.18}$$

When $a=0$ only one equation is left in Eq. (2.16)

$$a=0:\quad d(\lambda_A\overline{\lambda}_{\dot B})=0. \tag{2.19}$$

The general solution of Eq. (2.19) is

$$\lambda_A(\tau)=\lambda^0_A e^{i\tilde\alpha(\tau)},\quad \overline{\lambda}_{\dot A}=\overline{\lambda}^0_{\dot A}e^{-i\tilde\alpha(\tau)}. \tag{2.20}$$

The arbitrary function $\tilde\alpha(\tau)$ [whose presence in Eq. (2.20) reflects the $U(1)$ gauge symmetry of the $a=1$ model] can be gauged away by the local $U(1)$ transformation (2.15), and we are again left with constant λ on the mass shell.

2. Fermionic variations and κ symmetry

Let us consider now the formula (2.11) with the variations of fermionic coordinates accompanied by the following variations of x and y

$$\delta x^{A\dot B}=-i(\delta\Theta^A\overline{\Theta}^{\dot B}-\Theta^A\delta\overline{\Theta}^{\dot B})\Rightarrow i_\delta\Pi^{A\dot B}=0,$$
$$\delta y^{AB}=i\Theta^{(A}\delta\Theta^{B)}\Rightarrow i_\delta\Pi^{AB}=0,$$
$$\delta\overline{y}^{\dot A\dot B}=i\overline{\Theta}^{(\dot A}d\overline{\Theta}^{\dot B)}\Rightarrow i_\delta\overline{\Pi}^{\dot A\dot B}=0. \tag{2.21}$$

The bosonic spinor λ_α remains unchanged. In such a case Eq. (2.11) takes the form

$$\delta S=\int [2i\,d\Theta^A\lambda_A(\delta\overline{\Theta}^{\dot B}\overline{\lambda}_{\dot B}+a\,\delta\Theta^B\lambda_B) + 2i\,d\overline{\Theta}^{\dot B}\overline{\lambda}_{\dot B}(\delta\Theta^B\lambda_B+a\,\delta\overline{\Theta}^{\dot B}\overline{\lambda}_{\dot B})]. \tag{2.22}$$

We see that for $a\neq 1$ only two out of four variations of the fermionic coordinates $\delta\Theta^A,\delta\overline{\Theta}^{\dot A}$ are effectively involved into the variation (2.22) of the action (2.1). This reflects the presence of local fermionic κ symmetry [25] with *two* independent parameters $\kappa=(\kappa_1+i\kappa_2)$, $\overline{\kappa}=(\kappa_1-i\kappa_2)$. The κ transformations of the coordinates are given by Eq. (2.21) and

$$\delta\Theta^A=\kappa\lambda^A=(\kappa_1+i\kappa_2)\lambda^A,$$
$$\delta\overline{\Theta}^{\dot B}=\overline{\kappa\lambda}^{\dot B}=(\kappa_1-i\kappa_2)\overline{\lambda}^{\dot B}. \tag{2.23}$$

At the critical point $a=1$ the number of independent κ–symmetries increases from 2 to 3, since in this case only one linear combination $(\delta\Theta^B\lambda_B+\delta\overline{\Theta}^{\dot B}\overline{\lambda}_{\dot B})$ of four real fermionic variations enters into the variation of the action

$$a=1: \tag{2.24}$$
$$\delta S=\int 2i(d\Theta^A\lambda_A+d\overline{\Theta}^{\dot A}\overline{\lambda}_{\dot A})(\delta\Theta^B\lambda_B+d\overline{\Theta}^{\dot B}\overline{\lambda}_{\dot B}).$$

Thus remaining three fermionic variations correspond to the local fermionic symmetries of the $a=1$ model [1].

In order to present an explicit form of these *three κ symmetries* we should introduce an additional bosonic spinor u_A such that

$$\lambda^A u_A=\overline{\lambda}^{\dot A}\overline{u}_{\dot A}=1. \tag{2.25}$$

Then one can perform the decomposition of the unit matrix in the spinor space[2]

$$\delta_B{}^A=\lambda^A u_B-u^A\lambda_B,\quad \delta^{\dot A}_{\dot B}=\overline{\lambda}^{\dot A}\overline{u}_{\dot B}-\overline{u}^{\dot A}\overline{\lambda}_{\dot B} \tag{2.26}$$

[2]The pair of Weyl spinors λ^A,u_A is analogous to the Newman-Penrose dyad [31] widely used in general relativity.

and use it to decompose the fermionic variation of Θ. As a result we find that the κ−symmetry transformations of the $a=1$ model are given by Eqs. (2.21) and

$$\delta\Theta^A=(\kappa_1+i\kappa_2)\lambda^A+i\kappa_3 u^A,$$

$$\delta\overline{\Theta}^{\dot B}=(\kappa_1-i\kappa_2)\overline{\lambda}^{\dot B}-i\kappa_3\overline{u}^{\dot A}. \tag{2.27}$$

B. Hamiltonian analysis

We now turn to the Hamiltonian analysis of the model with the purpose of getting all the constraints on the dynamics of the system, classifying them in the manner of Dirac and thus identifying all local symmetries of the model. For the case $a=1$ the analysis has been performed in Ref. [1]. The generic model ($a\neq 0$) has the same total number of constraints as the $a=1$ model, the only difference being that when the parameter a takes the value $a=1$ one of the fermionic second-class constraints becomes the first-class constraint generating the third κ symmetry. So what we should do is just to adapt the results of the Hamiltonian analysis of Ref. [1] to the generic case.

The constraints corresponding to the case $a=0$ of the Ferber-Shirafuji superparticle are obtained from the generic set of constraints by putting the canonical momenta for the central charge coordinates identically equal to zero. The canonical momenta of the generic system are

$$\mathcal{P}_{\mathcal{M}}=\frac{\partial L}{\partial\dot q^{\mathcal{M}}}=(P_{A\dot A},Z_{AB},\overline{Z}_{\dot A\dot B};P^A,\overline{P^{\dot A}};\pi_A,\overline{\pi}_{\dot A}), \tag{2.28}$$

$$[\mathcal{P}_{\mathcal{M}},q^{\mathcal{N}}\}_P=-(-1)^{\mathcal{M}\mathcal{N}}[q^{\mathcal{N}},\mathcal{P}_{\mathcal{M}}\}_P=\delta^{\mathcal{N}}_{\mathcal{M}}: \tag{2.29}$$

$$[P_{A\dot A},x^{B\dot B}]_P=\delta^B_A\delta^{\dot B}_{\dot A},$$

$$[Z_{AB},y^{CD}]_P=2\,\delta^C_{[A}\delta^D_{B]},$$

$$[\overline{Z}_{\dot A\dot B},\overline{y^{C\dot D}}]_P=2\,\delta^{\dot C}_{[\dot A}\delta^{\dot D}_{\dot B]},$$

$$[P^A,\lambda_B]_P=\delta^A_B,\quad [\overline{P^{\dot A}},\overline{\lambda}_{\dot B}]_P=\delta^{\dot A}_{\dot B},$$

$$\{\pi_A,\Theta^B\}_P=\delta^B_A,\quad \{\overline{\pi}_{\dot A},\overline{\Theta}^{\dot B}\}_P=\delta^{\dot B}_{\dot A}.$$

They satisfy the following set of constraints:

$$\Phi_{A\dot B}\equiv P_{A\dot B}-\lambda_A\overline{\lambda}_{\dot B}=0, \tag{2.30}$$

$$\Phi_{AB}\equiv Z_{AB}-a\lambda_A\lambda_B=0, \tag{2.31}$$

$$\overline{\Phi}_{\dot A\dot B}\equiv\overline{Z}_{\dot A\dot B}-a\overline{\lambda}_{\dot A}\overline{\lambda}_{\dot B}=0, \tag{2.32}$$

$$P_A=0,\quad \overline{P}_{\dot A}=0, \tag{2.33}$$

$$D_A\equiv-\pi_A+iP_{A\dot B}\overline{\Theta}^{\dot B}+iZ_{AB}\Theta^B=0, \tag{2.34}$$

$$\overline{D}_{\dot A}\equiv\overline{\pi}_{\dot A}-i\Theta^BP_{B\dot A}-i\overline{Z}_{\dot A\dot B}\overline{\Theta}^{\dot B}=0. \tag{2.35}$$

To separate the constraints (2.30)–(2.35) into the first and second class let us project them on the bosonic spinors λ and u [Eqs. (2.25), (2.26)]. We get

$$B_1=\lambda^A\overline{\lambda}^{\dot B}P_{A\dot B}=0, \tag{2.36}$$

$$B_2=\lambda^A\overline{u}^{\dot B}P_{A\dot B}-\lambda^Au^BZ_{AB}=0, \tag{2.37}$$

$$B_3\equiv(B_2)^*=u^A\overline{\lambda}^{\dot B}P_{A\dot B}-\overline{\lambda}^{\dot A}\overline{u}^{\dot B}\overline{Z}_{\dot A\dot B}=0, \tag{2.38}$$

$$B_4=2u^A\hat{\overline{u}}^{\dot B}P_{A\dot B}-\frac{1}{a}u^Au^BZ_{AB}-\frac{1}{a}\overline{u}^{\dot A}\overline{u}^{\dot B}\overline{Z}_{\dot A\dot B}=0, \tag{2.39}$$

$$B_5=\lambda^A\overline{\lambda}^{\dot B}Z_{AB}=0, \tag{2.40}$$

$$B_6\equiv(B_5)^*=\overline{\lambda}^{\dot A}\overline{\lambda}^{\dot B}\overline{Z}_{\dot A\dot B}=0, \tag{2.41}$$

$$B_7\equiv\lambda^A\overline{u}^{\dot B}P_{A\dot B}+\lambda^Au^BZ_{AB}=0,$$

$$B_8\equiv u^A\overline{\lambda}^{\dot B}P_{A\dot B}+\overline{\lambda}^{\dot A}\overline{u}^{\dot B}\overline{Z}_{\dot A\dot B}=0, \tag{2.42}$$

$$B_9\equiv u^Au^BZ_{AB}-a=0,\quad B_{10}\equiv\overline{u}^{\dot A}\overline{u}^{\dot B}\overline{Z}_{\dot A\dot B}-a=0, \tag{2.43}$$

$$B_{11}\equiv i(\lambda^AP_A-\overline{\lambda}^{\dot A}\overline{P}_{\dot A})=0, \tag{2.44}$$

$$B_{12}\equiv\lambda^AP_A+\overline{\lambda}^{\dot A}\overline{P}_{\dot A}=0, \tag{2.45}$$

$$B_{13}\equiv u^AP_A=0,\quad B_{14}\equiv\overline{u}^{\dot A}\overline{P}_{\dot A}=0.$$

$$F_1=\lambda^AD_A=0, \tag{2.46}$$

$$F_2\equiv(F_1)^*=\overline{\lambda}^{\dot A}\overline{D}_{\dot A}=0, \tag{2.47}$$

$$F_3=u^AD_A+\overline{u}^{\dot A}\overline{D}_{\dot A}=0, \tag{2.48}$$

$$F_4\equiv u^AD_A-\overline{u}^{\dot A}\overline{D}_{\dot A}=0. \tag{2.49}$$

For arbitrary $a\neq 0,1$ it can be checked that the bosonic constraints (2.36)–(2.41) and the fermionic constraints (2.46) and (2.47) belong to the first class, i.e., their Poisson brackets with all constraints vanish on the constraint surface, and the constraints (2.42)–(2.45), (2.48), and (2.49), are second class. When computing the Poisson brackets of the constraints one should take into account that, because of the normalization condition (2.25), the spinor u^A should be regarded as a variable depending on λ_A. The simplest way of taking this into account is to assume the following Poisson (actually Dirac) brackets of u^A with the λ momentum P_A:[3]

[3]These brackets appear as Dirac brackets with respect to the pair of the second class constraints (2.25), $u^AP^{(u)}_A=0$ and their complex conjugate pair, when the bosonic spinor u is considered as an independent variable whose momentum is constrained to be zero $P^{(u)}_A=0$. Then it is not hard to verify that the new phase space variables $u^A,P^{(u)}_A$ do not introduce new redundant degrees of freedom into the system under consideration.

$$[P_A, u^B]_P = -u^B u_A .$$

Thus in the case $a \neq 0,1$ among 14 bosonic and 4 fermionic constraints 6 bosonic and 2 fermionic constraints are of the first class and 8 bosonic and 2 fermionic constraints are of the second class.

The first class constraints generate local symmetries of the dynamical system. For instance, the constraints (2.36), and $(B_5 + B_6)$ of Eqs. (2.40),(2.41) generate worldline reparametrizations of the coordinates x and y. The fermionic constraints (2.46) and (2.47) generate the κ-symmetry transformations (2.21) and (2.23).

Each first class constraint reduces the number of independent phase space variables by 2, while each second class constraint eliminates only one degree of freedom. Hence, in the case $a \neq 0,1$ the phase space of $2\times(4+6+4)=28$ bosonic and $2\times 4 = 8$ fermionic canonical variables of the system is reduced to

$$a \neq 0,1: \quad n_{ph} = \mathbf{8}_b + \mathbf{2}_f , \tag{2.50}$$

i.e., we get eight bosonic and two fermionic physical degrees of freedom.

In order to see how at $a=1$ the fermionic second-class constraint (2.48) transforms into the first-class constraint generating the third κ symmetry (2.27) let us consider the Poisson bracket of the constraint (2.48) with itself

$$\{F_3, F_3\}_P = 2(a-1). \tag{2.51}$$

When $a \neq 1$ the right-hand side (RHS) of Eq. (2.51) is nonzero and hence this constraint is second class, but at $a=1$ the RHS of Eq. (2.51) vanishes. Since F_3 weakly commutes with all other constraints, at this critical value of a we obtain one more fermionic first class constraint, and we achieve the reduction of the number of independent fermionic physical degrees of freedom from two to one

$$a = 1: \quad n_{ph} = 8_b + 1_f . \tag{2.52}$$

Finally, when $a=0$ the tensorial coordinates y disappear from the action (2.1), and in Eqs. (2.36)–(2.49) we must put to zero their canonical momenta Z. The remaining set of the constraints takes the following form:

$$B_1 = \lambda^A \overline{\lambda}^{\dot B} P_{A\dot B} = 0, \tag{2.53}$$

$$B_2 = \lambda^A \overline{u}^{\dot B} P_{A\dot B} = 0, \tag{2.54}$$

$$B_3 \equiv (B_2)^* = u^A \overline{\lambda}^{\dot B} P_{A\dot B} = 0, \tag{2.55}$$

$$B_4 = u^A \overline{u}^{\dot B} P_{A\dot B} - 2 = 0, \tag{2.56}$$

$$B_5 \equiv i(\lambda^A P_A - \overline{\lambda}^{\dot A} \overline{P}_{\dot A}) = 0, \tag{2.57}$$

$$B_6 \equiv \lambda^A P_A + \overline{\lambda}^{\dot A} \overline{P}_{\dot A} = 0, \tag{2.58}$$

$$B_7 \equiv u^A P_A = 0, \quad B_8 \equiv \overline{u}^{\dot A} \overline{P}_{\dot A} = 0,$$

$$F_1 = \lambda^A D_A = 0, \tag{2.59}$$

$$F_2 \equiv (F_1)^* = \overline{\lambda}^{\dot A} \overline{D}_{\dot A} = 0, \tag{2.60}$$

$$F_3 = u^A D_A + \overline{u}^{\dot A} \overline{D}_{\dot A} = 0, \tag{2.61}$$

$$F_4 \equiv u^A D_A - \overline{u}^{\dot A} \overline{D}_{\dot A} = 0. \tag{2.62}$$

These are the constraints of the Ferber–Shirafuji formulation of the superparticle which has been analyzed in detail in a number of papers [10,20–23]. Now two bosonic constraints (2.53) and (2.57) are first class and the other six are second class, while, as in the generic case $a \neq 1$, two of the fermionic constraints are first class (2.59), (2.60) and two are second class (2.61), (2.62).

Therefore, the number of independent phase-space physical degrees of freedom of the standard $N=1$, $D=4$ superparticle consists of six bosonic and two fermionic variables

$$a = 0: \quad n_{ph} = 6_b + 2_f . \tag{2.63}$$

In the next section we shall show that the independent phase-space physical degrees of freedom (2.50), (2.52), and (2.63) of the generic superparticle model can be covariantly described by $OSp(2|8)$ [or $OSp(1,1|8)$], $OSp(1|8)$, and $SU(2,2|1)$ supertwistors, respectively.

III. SUPERTWISTOR TRANSFORM: $OSp(2|8)$, $OSp(1|8)$, AND $SU(2,2|1)$ SUPERTWISTORS

Let us integrate the action (2.1) by parts and neglect the boundary term. The result is

$$S = -\int (\mu^A d\lambda_A + \overline{\mu}^{\dot A} d\overline{\lambda}_{\dot A}) - i\int (\chi\, d\overline{\chi} + \overline{\chi}\, d\chi + a\chi\, d\chi + a\overline{\chi}\, d\overline{\chi}) \tag{3.1}$$

or

$$S = -\int (\mu^A d\lambda_A + \overline{\mu}^{\dot A} d\overline{\lambda}_{\dot A}) - 2i\int [(1+a)\chi_1 d\chi_1 + (1-a)\chi_2 d\chi_2]$$
$$= -\int (\mu^A d\lambda_A + \overline{\mu}^{\dot A} d\overline{\lambda}_{\dot A}) - 2i\int \overline{\chi}(a) d\chi(a), \tag{3.2}$$

where

$$\mu^A = x^{A\dot B}\overline{\lambda}_{\dot B} + 2a y^{AB}\lambda_B + i\Theta^A[(\overline{\Theta\lambda}) + a(\Theta\lambda)], \tag{3.3}$$

$$\overline{\mu}^{\dot A} = \lambda_B x^{B\dot A} + 2a\overline{y}^{\dot A\dot B}\overline{\lambda}_{\dot B} + i\overline{\Theta}^{\dot A}[(\Theta\lambda) + a(\overline{\Theta\lambda})],$$

$$\chi = (\Theta\lambda) \equiv \Theta^A\lambda_A, \quad \overline{\chi} = (\overline{\Theta\lambda}) \equiv \overline{\Theta}^{\dot A}\overline{\lambda}_{\dot A}, \tag{3.4}$$

$$\chi_1 = \frac{1}{2}(\chi + \overline{\chi}), \quad \chi_2 = \frac{i}{2}(\overline{\chi} - \chi), \tag{3.5}$$

$$\chi(a) = \sqrt{(1+a)}\chi_1 + i\sqrt{(1-a)}\chi_2 ,$$

$$\overline{\chi}(a)=\sqrt{(1+a)}\chi_1-i\sqrt{(1-a)}\chi_2. \tag{3.6}$$

[Note that $\chi(a)$ and $\overline{\chi}(a)$ are complex conjugate to each other only for $a<1$, while for $a\geqslant 1$ they are real spinors.]

Thus, in the generic case $a\neq 0,1$ one can reformulate the dynamical system in terms of 8 bosonic variables $\lambda_A,\mu^A;\overline{\lambda}_{\dot{B}},\overline{\mu}^{\dot{B}}$ and two real fermionic variables χ_1, χ_2. These variables can be regarded as components of a real (8,2) component supertwistor (cf. with [1])

$$Y_{\mathcal{A}}=(y_1,\ldots,y_8;\chi_1,\chi_2)=(\lambda^\alpha,\mu^\alpha,\chi_1,\chi_2), \tag{3.7}$$

where λ^α and μ^α are real Majorana spinors formed of the Weyl spinors

$$\lambda^\alpha\Leftrightarrow\begin{pmatrix}\lambda_A\\ \overline{\lambda}^{\dot{A}}\end{pmatrix},\quad \mu^\alpha\Leftrightarrow\begin{pmatrix}\mu_A\\ \overline{\mu}^{\dot{A}}\end{pmatrix}.$$

One can write the action (3.2) in the form

$$S=-\frac{1}{2}\int d\tau\, Y_{\mathcal{A}}G^{\mathcal{AB}}\dot{Y}_{\mathcal{B}}, \tag{3.8}$$

where

$$G^{\mathcal{AB}}=\begin{pmatrix}\omega^{(8)} & 0\\ 0 & i\omega^{(2)}\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}0_2 & I_2 & 0_2 & 0_2\\ -I_2 & 0_2 & 0_2 & 0_2\\ 0_2 & 0_2 & 0_2 & I_2\\ 0_2 & 0_2 & -I_2 & 0_2\end{pmatrix} & 0\\ 0 & i\begin{pmatrix}2(1+a) & 0\\ 0 & 2(1-a)\end{pmatrix}\end{pmatrix}. \tag{3.9}$$

$\omega^{(8)}$ is the $Sp(8)$ invariant simplectic metric.

When $a\neq 1$ we can rescale the fermionic variables χ_1 and χ_2, i.e. multiply them, respectively, by $\sqrt{1+a}$ and $\sqrt{|1-a|}$. This results in the following form of the metric $\omega^{(2)}$ in Eq. (3.9):

$$\omega^{(2)}\to\omega^{(2)}=2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\quad \text{for}\quad a<1 \tag{3.10}$$

or

$$\omega^{(2)}\to\omega^{(2)}=2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\quad \text{for}\quad a>1. \tag{3.11}$$

We see that the symmetry group of the fermionic sector of the metric (3.9) is $SO(2)=U(1)$ when $a<1$ and $SO(1,1)$ when $a>1$.

Hence, when $a<1$ the complete symmetry group of the metric (3.9) is the supergroup $OSp(2|8)$, while in the case $a>1$ the symmetry group becomes $OSp(1,1|8)$. The supertwistors (3.7) transform under the fundamental representations of these supergroups.

When $a=1$ the metric becomes degenerate

$$\omega^{(2)}\to\omega^{(2)}_{a=1}=2\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}. \tag{3.12}$$

This reflects the absence of the second fermionic variable χ_2 from the action (3.2). Thus at $a=1$ the supergroups $OSp(2|8)$ and $OSp(1,1|8)$ reduce to $OSp(1|8)$ with the corresponding supertwistor representation having one real fermionic component (cf. with [1])

$$Y_{\mathcal{A}}=(y_1,\ldots,y_8;\chi_1). \tag{3.13}$$

Consider now the case $a=0$. At $a=0$ the action has the same form as for $a<1$ and hence is formally $OSp(2|8)$ invariant. But, as we have seen in the previous section, at this critical point the model acquires additional local $U(1)$ symmetry, which must have its counterpart in the supertwistor description, i.e., there should appear a first-class constraint on the supertwistor variables which generates this symmetry. In order to identify this constraint we use the defining relations (3.3) and consider the following bilinear combination of supertwistor components for an arbitrary value of a

$$\mu^A(a)\lambda_A-\overline{\mu}^{\dot{A}}(a)\overline{\lambda}_{\dot{A}}+2i\overline{\chi}\chi=2a\lambda y\lambda-2\overline{a\lambda y\lambda}. \tag{3.14}$$

At $a=0$ Eq. (3.14) does not involve central charge coordinates y and thus we obtain the pure supertwistor constraint

$$\mu^A(0)\lambda_A-\overline{\mu}^{\dot{A}}(0)\overline{\lambda}_{\dot{A}}+2i\overline{\chi}\chi=0. \tag{3.15}$$

Hence, at $a=0$ for the action (3.1) to be equivalent to Eq. (2.1) it must be supplemented with the (first-class) constraint (3.15) introduced through a Lagrange multiplier term

$$\begin{aligned}a=0:\quad S=&-\int(\mu^A d\lambda_A+\overline{\mu}^{\dot{A}}d\overline{\lambda}_{\dot{A}})\\ &-i\int(\chi\, d\overline{\chi}+\overline{\chi}\, d\chi)\\ &+i\int d\tau\,\Lambda(\mu\lambda-\overline{\mu\lambda}+2i\overline{\chi}\chi).\end{aligned} \tag{3.16}$$

The constraint (3.15) generates the $U(1)$ gauge symmetry appearing only in the $a=0$ case. This constraint introduces the complex structure and thus breaks the $OSp(2|8)$ symmetry of the $a<1$ model down to $SU(2,2|1)$. As a result one gets the Ferber-Shirafuji formulation [9,10] of a conventional massless superparticle [37–40] in terms of $SU(2,2|1)$ supertwistors

$$\mathcal{Z}_{\mathcal{A}}=(\lambda_A,\overline{\mu^{\dot A}},\chi),\quad \overline{\mathcal{Z}}^{\mathcal{A}}=(\mu^A,\overline{\lambda}_{\dot A},\overline{\chi}),$$
$$a=0:\quad S=-\int(\overline{\mathcal{Z}}^{\mathcal{A}}d\mathcal{Z}_{\mathcal{A}})+i\int d\tau\,\Lambda(\overline{\mathcal{Z}}^{\mathcal{A}}\mathcal{Z}_{\mathcal{A}}-s), \tag{3.17}$$

where the constant s has been introduced in order to have the possibility of describing massless superparticles with nonzero (super)helicity [30] (see Refs. [22,32], and references therein for details).

A. Quantization of the supertwistor model

1. Canonical supertwistor quantization

The quantization of the dynamical system (3.2) with $a\neq 0,1$ is quite straightforward. The action is of the first order form, therefore $\mu,\overline{\mu}$ should be identified with the canonical momenta conjugate to $\lambda_A,\overline{\lambda}_{\dot A}$, and $2i\overline{\chi}(a)$ is the momentum conjugate to $\chi(a)$ [remember that $\chi(a)$ and $\overline{\chi}(a)$ are defined by Eq. (3.6)]. The canonical Poisson brackets are

$$[\mu_A,\lambda^B]_P=\delta_A{}^B,$$
$$[\overline{\mu}_{\dot A},\overline{\lambda}^{\dot B}]_P=\delta_{\dot A}^{\dot B},$$
$$\{\overline{\chi}(a),\chi(a)\}_P=-\frac{i}{2}. \tag{3.18}$$

At the quantum level the dynamical variables become operators, and the Poisson brackets are replaced by (anti)commutators ($[\dots,\dots]_P\to i[\dots,\dots]$, $\{,\}_P\to -i\{,\}$). For instance, in the ''coordinate'' representation the momenta are the derivatives of corresponding coordinates

$$\mu^A=i\frac{\partial}{\partial\lambda_A},\quad \overline{\mu}^{\dot A}=i\frac{\partial}{\partial\overline{\lambda}_{\dot A}},\quad \overline{\chi}(a)=-\frac{1}{2}\frac{\partial}{\partial\chi(a)}. \tag{3.19}$$

The canonical Hamiltonian of the system vanishes identically. The wave function of the system in the supertwistor ''coordinate'' representation is

$$a\neq 1:\Phi[\lambda_A,\lambda_{\dot A},\chi(a)]=\phi(\lambda_A,\lambda_{\dot A})+i\chi(a)\psi(\lambda_A,\lambda_{\dot A}) \tag{3.20}$$

and the spectrum of quantum states is described by one bosonic and one fermionic function depending on Weyl spinor variables.

At $a=1$ $\chi(1)$ becomes a real Clifford variable and the field (3.20) becomes a Clifford algebra valued function. We shall discuss this case in detail in Secs. III A4, V B, and the Appendix. To understand what kind of physical states are described by the function (3.20) in the case $a\neq 0,1$ we shall first consider the well known case $a=0$.

2. a=0

At $a=0$ the dynamics of the system is subject to the first-class constraint (3.15) which at the quantum level is imposed on the wave function (3.20)[4]

$$\left(D^{(0)}+\chi\frac{\partial}{\partial\chi}-s\right)\Phi(\lambda_A,\overline{\lambda}_{\dot A};\chi)=0, \tag{3.21}$$

where

$$D^{(0)}=\lambda_A\frac{\partial}{\partial\lambda_A}-\overline{\lambda}_{\dot A}\frac{\partial}{\partial\overline{\lambda}_{\dot A}}$$

is the supertwistor representation of the bosonic part of the $U(1)$ generator, $\chi=\chi(0)$ [see Eqs. (3.4)–(3.6)] and s is an integer constant which appears due to the ambiguity in ordering the operators in Eq. (3.21) (see Refs. [30,22,32], and references therein for details, here we only note that the quantization of s follows from the requirement for the wave function to be single valued). The (half-)integer values of $s/2$ describe helicities of massless quantum states.

Let us consider first the case $s=0$. Equation (3.21) requires the bosonic and fermionic components of the superfield (3.20) to be homogeneous functions of $\lambda,\overline{\lambda}$ of the degree 0 and -1, respectively,

$$D^{(0)}\phi(\lambda_A,\lambda_{\dot A})=0,\quad D^{(0)}\psi(\lambda_A,\lambda_{\dot A})=-\psi(\lambda_A,\lambda_{\dot A}). \tag{3.22}$$

The solution is[5]

$$\phi=\phi_0(p_m),\quad \psi=\overline{\lambda}^{\dot A}\overline{\psi}_{\dot A}(p_m), \tag{3.23}$$

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[4]In this section we basically follow the quantization procedure of references [21]. The operator $D=D^{(0)}+\chi\partial/\partial\chi$ is the superhelicity operator.

[5]A rigorous approach [32,33] consists of the consideration of the decomposition of the wave function $\phi(\lambda_A,\overline{\lambda}_{\dot A})$ in the basis of the functions on $\mathbf{C}^2-\{0\}$ formed by homogeneous infinite-differentiable functions $\phi_{\nu_1,\nu_2}(z\lambda_A,\overline{z\lambda}_{\dot A})=z^{\nu_1}\overline{z}^{\nu_2}\phi_{\nu_1,\nu_2}(\lambda_A,\overline{\lambda}_{\dot A})$ of a homogeneity index $\chi=(\nu_1,\nu_2)$ [34]. The homogeneous functions are defined by the Mellin transformation

$$\phi_{\nu_1,\nu_2}(\lambda_A,\overline{\lambda}_{\dot A})=\frac{i}{2}\int dz\,d\overline{z}\,z^{\nu_1+1}\overline{z}^{\nu_2+1}\phi(z\lambda_A,\overline{z\lambda}_{\dot A}).$$

The decomposition

$$\phi(\lambda_A,\overline{\lambda}_{\dot A})=\Sigma_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}d\rho\;\phi_{(n+i\rho)/2,(-n+i\rho)/2}(\lambda_A,\overline{\lambda}_{\dot A})$$

can be substituted into Eq. (3.22) instead of the power series in λ_A, $\lambda_{\dot A}$ to obtain the general solution. We refer the reader to [32,33] for further details and to [34] for an excellent presentation of related mathematics and, for simplicity, use a physical ''shortcut'' of the rigorous approach.

where, by definition, p_m is a lightlike vector composed of λ and $\bar{\lambda}$ [see also Eq. (2.30)]

$$p_{A\dot{A}}=p_m\sigma^m_{A\dot{A}}=\lambda_A\bar{\lambda}_{\dot{A}}. \tag{3.24}$$

We see that the spectrum of the Ferber-Shirafuji model at $s=0$ consists of a massless $N=1$, $D=4$ (anti)chiral supermultiplet containing a complex scalar field of zero helicity and a Weyl fermion field of helicity $-\frac{1}{2}$. This supermultiplet can be described either by the set of bosonic and fermionic wave functions depending on the bosonic Weyl spinor variables $\lambda_A,\lambda_{\dot{A}}$ in accordance with the formula (3.23),(3.24), or as a set of the unrestricted bosonic scalar function $\phi_0(p_m)$ and the fermionic spinor function $\bar{\psi}_{\dot{A}}(p_m)$ depending on the lightlike vector $p_mp^m=0$ which we identify with the momentum of the massless superparticle. In such a way we establish the relation of the supertwistor formulation with the space-time description of the massless superparticle, and this dual description can be extended to the case of more general model with nonvanishing central charge coordinates.

Finally, let us consider the case of the nonvanishing operator ordering constant s in Eq. (3.21) which we shall call the superhelicity parameter, characterizing the helicity properties of the superfield solutions. The component form of the constraint (3.21) now reads

$$D^{(0)}\phi(\lambda_A,\lambda_{\dot{A}})=s\,\phi(\lambda_A,\lambda_{\dot{A}}),$$
$$D^{(0)}\psi(\lambda_A,\lambda_{\dot{A}})=(s-1)\,\psi(\lambda_A,\lambda_{\dot{A}}). \tag{3.25}$$

For integer $s>0$ the solution of Eq. (3.25) is

$$\phi=\lambda^{A_1}\ldots\lambda^{A_s}\phi_{A_1\ldots A_s}(p_m),$$
$$\psi=\lambda^{A_1}\ldots\lambda^{A_{s-1}}\psi_{A_1\ldots A_{s-1}}(p_m). \tag{3.26}$$

We thus obtain supermultiplets whose components have the helicities $s/2$ and $s/2-1/2$, respectively. The choice of the statistics of the superfields (3.26) should be made in accordance with the general spin-statistics theorem, such that for the even values of s (integer superhelicities) the superfields (3.26) are bosonic and for odd s (half-integer superhelicities) they are fermionic.

Notice that the Grassmann parity of the superfield Φ (3.21) (and its components ϕ and ψ) is related to the parity of $\Phi(\lambda,\bar{\lambda},\chi)$ under the change $\lambda\to-\lambda$ (λ parity) which implies $\chi\to-\chi$. If $\Phi(-\lambda,-\bar{\lambda},-\chi)=\Phi(\lambda,\bar{\lambda},\chi)$ then from Eq. (3.26) follows that s is even (integer superhelicities) and such a superfield is Grassmann even (ϕ is bosonic and ψ is fermionic). Analogously if the superfield $\Phi(\lambda,\bar{\lambda},\chi)$ changes the sign under the λ parity, then s is odd (half-integer superhelicities) and the superfield is Grassmann odd (ϕ is fermionic, and ψ is bosonic).

For integer $s<0$ the solution of Eq. (3.25) is

$$\phi=\lambda^{\dot{A}_1}\ldots\lambda^{\dot{A}_{-s}}\bar{\phi}_{\dot{A}_1\ldots\dot{A}_{-s}},$$
$$\psi=\lambda^{\dot{A}_1}\ldots\lambda^{\dot{A}_{-s+1}}\bar{\psi}_{\dot{A}_1\ldots\dot{A}_{-s+1}} \tag{3.27}$$

and thus the spectrum of the quantum states of the model is represented by a supermultiplet of helicity $[-s/2,(-s+1)/2]$.

3. a≠0,1

Let us return to the generic $a\neq0,1$ models. Their spectrum is defined by arbitrary scalar bosonic and fermionic functions of the Weyl bosonic spinors $\phi(\lambda_A,\lambda_{\dot{A}})$ and $\psi(\lambda_A,\lambda_{\dot{A}})$ which, in contrast to the $a=0$ case, are not subject to any constraints. The bosonic spinor components can be regarded to be defined through the components of the lightlike vector $p_mp^m=0$ (3.24) up to the phase transformations

$$\lambda_A\to e^{i\alpha(\tau)}\lambda_A,\quad \lambda_{\dot{A}}\to e^{-i\alpha(\tau)}\lambda_{\dot{A}}. \tag{3.28}$$

Thus for $a\neq0$ we can consider the bosonic and fermionic wave functions to depend on the lightlike vector and a $U(1)$ angle variable $\alpha\sim\alpha+2\pi k$

$$\phi(\lambda_A,\lambda_{\dot{A}})=\phi(p_m,\alpha),\quad \psi(\lambda_A,\lambda_{\dot{A}})=\psi(p_m,\alpha). \tag{3.29}$$

Hence, in contrast to the Ferber-Shirafuji model, the wave function of the generic dynamical system (3.1) with $a\neq0$ depends on *one additional variable* which *parametrizes a compact manifold* $U(1)=S^1$. This means that the functions ϕ and ψ, as the single valued functions, can be expanded in the Fourier series

$$\phi(p_m,\alpha)=\Sigma_{k\in\mathbf{z}}e^{ik\alpha}\phi_k(p_m),$$
$$\psi(p_m,\alpha)=\Sigma_{k\in\mathbf{z}}e^{ik\alpha}\psi_k(p_m). \tag{3.30}$$

The meaning of this series expansion becomes clear if we use the Lorentz-covariant representation of ϕ and ψ as single valued functions of λ_A (3.20) where $\lambda_A\bar{\lambda}_{\dot{A}}$ are replaced by p_m. Then the series (3.30) acquires the form

$$\phi(\lambda,\bar{\lambda})=\phi^0(p_m)+\Sigma_{k\in\mathbf{z}_+}[\lambda^{A_1}\ldots\lambda^{A_k}\phi_{A_1\ldots A_k}(p_m)+\bar{\lambda}^{\dot{A}_1}\ldots\bar{\lambda}^{\dot{A}_k}\bar{\phi}_{\dot{A}_1\ldots\dot{A}_k}(p_m)], \tag{3.31}$$

$$\psi(\lambda,\bar{\lambda})=\psi^0(p_m)+\Sigma_{k\in\mathbf{z}_+}[\lambda^{A_1}\ldots\lambda^{A_k}\psi_{A_1\ldots A_k}(p_m)+\bar{\lambda}^{\dot{A}_1}\ldots\bar{\lambda}^{\dot{A}_k}\bar{\psi}_{\dot{A}_1\ldots\dot{A}_k}(p_m)].$$

We therefore conclude that the most general solution of the model with $a\neq0$ describes an infinite doubly degenerate spectrum of massless fields of an arbitrary helicity, with the additional compact S^1 coordinate in the momentum space conjugate to the discrete helicity variable.

If we assume the validity of spin-statistics theorem the bosonic fields should have positive λ parity, and fermionic fields should have odd λ parity. Thus, the λ-even part $\Phi_+(\lambda,\bar{\lambda},\chi)\equiv\Phi_+(-\lambda,-\bar{\lambda},-\chi)=\phi_+(\lambda,\bar{\lambda})+i\chi\psi_-(\lambda,\bar{\lambda})$ of the general superfield solution $\Phi(\lambda,\bar{\lambda},\chi)=\phi(\lambda,\bar{\lambda})+i\chi\psi(\lambda,\bar{\lambda})$ [see Eqs. (3.20) and (3.31)] should be regarded as bosonic (i.e., Grassmann–even). Consequently this im-

plies that the wave function $\phi_+(\lambda,\bar{\lambda})$ has positive λ parity (even powers of λ) i.e., it is bosonic, and the fermionic wave function $\psi_-(\lambda,\bar{\lambda})$ has negative λ parity (odd powers of λ). Another sector of the full quantum state spectrum is described by the fermionic λ-odd superfield $\Phi_-(\lambda,\bar{\lambda},\chi)\equiv -\Phi_-(-\lambda,-\bar{\lambda},-\chi)=\phi_-(\lambda,\bar{\lambda})+\chi\psi_+(\lambda,\bar{\lambda})$ which is composed of the fermionic λ-odd field $\phi_-(\lambda,\bar{\lambda})$ and the bosonic λ-even field $\psi_+(\lambda,\bar{\lambda})$.

We, therefore, see that in order to obtain physically meaningful solutions described by the superfields $\Phi_+(\lambda,\bar{\lambda},\chi)$ and $\Phi_-(\lambda,\bar{\lambda},\chi)$ with definite Grassmann parity one should divide the general solution (3.20) into two parts with even and odd λ parity. Note that the superfield solutions with definite even/odd λ parity have the even/odd superhelicities, but each of them contains a complete nondegenerate spectrum of states with both even and odd helicities.

It is instructive to compare the consequences of the presence of the ''internal'' compact coordinate in our case and in Kaluza-Klein theories. In the Kaluza-Klein theories the compact variables arise in an extension of space time with extra directions and lead to the quantization of corresponding ''internal'' momenta in the extended momentum space. The ''internal'' quantized momenta describe masses and gauge charges of Kaluza-Klein fields in the dimensionally reduced theory.[6] In our case the compactification is achieved by expressing the generalized momenta in terms of bosonic spinor (twistor) componenta. Thus, we have the opposite situation: the compact ''internal'' manifold is in the extended *momentum* (twistor) space and a *quantized (discrete)* central charge coordinate is in the extended *coordinate* space (space-time + central charge coordinates). The Fourier transform of the compact ''internal'' momentum results in the discrete values of the conjugate coordinate, which are described by an integer s. From the physical point of view the (half)integer number $s/2$ describes the possible helicities of the massless quantum states.

The quantum states of our model form a reducible (infinite-dimensional) representation of target space supersymmetry. Indeed, as the bosonic spinor is inert under global supersymmetry, the fields (3.31) can be collected into the superfield series expansion with each term having definite superhelicity

$$\begin{aligned}\Phi[\lambda^A,\bar{\lambda}^{\dot{A}},\chi(a)]=&\Phi^0[p_m,\chi(a)]\\&+\Sigma_{k\in\mathbf{Z}_+}\lambda^{A_1}\cdots\lambda^{A_k}\Phi_{A_1\cdots A_k}[p_m,\chi(a)]\\&+\Sigma_{k\in\mathbf{Z}_+}\bar{\lambda}^{\dot{A}_1}\cdots\bar{\lambda}^{\dot{A}_k}\bar{\Phi}_{\dot{A}_1\cdots\dot{A}_k}[p_m,\chi(a)].\end{aligned}\tag{3.32}$$

It is easy to see that each term is separately invariant under supersymmetry.[7]

In the case $a=0$ the additional $U(1)$ constraint (3.21) appears. It singles out one irreducible superfield with a definite superhelicity out of the infinite series (3.32).

4. $a=1$

Consider now a peculiarity of the model at $a=1$. In this case the action (3.1), (3.2) contains only one real fermionic variable χ_1. The corresponding term in the action is

$$S_\chi=-4i\int\chi_1 d\chi_1\quad \chi_1=\frac{1}{2}(\Theta\lambda+\bar{\Theta}\bar{\lambda}).\tag{3.33}$$

From Eq. (3.33) we conclude that the odd momentum of χ is proportional to χ itself

$$S_\chi=\pi_{\chi_1}-4i\chi_1=0.\tag{3.34}$$

Equation (3.34) is the second-class constraint being typical of any free fermion theory

$$\{S_\chi,S_\chi\}_P=-8i.\tag{3.35}$$

It can be regarded to be satisfied in the strong sense [28] after we pass from the Poisson brackets to the Dirac brackets

$$[f,g]_D=[f,g]_P-\frac{i}{8}[f,S_\chi]_P[S_\chi,g]_P\tag{3.36}$$

which imply

$$\{\chi_1,\chi_1\}_D=2i.\tag{3.37}$$

Hence, upon quantization ξ becomes a Clifford variable of odd Grassmann parity

$$(\hat{\chi}_1)^2=1.\tag{3.38}$$

The Clifford algebra generated by this variable consists of two elements, the unit element and $\hat{\chi}_1$. Hence, all functions of $\hat{\chi}_1$ can be written as a ''Clifford algebra valued superfield'' having two components [27]

$$\Phi(\hat{\xi})=\phi+i\hat{\chi}_1\psi,\tag{3.39}$$

where ϕ and ψ do not depend on $\hat{\chi}_1$. We conclude that *at* $a=1$ *the wave functions (3.20), (3.32) become Clifford ''superfields'' whose components again (as in the case* $a\neq 0,1$*) describe an infinite tower of fields of all possible helicities.* We can decompose the superfields (3.39) into the even and odd parts with respect to λ parity and thus have the wave functions with definite Grassmann parity (bosonic and fermionic superfields).

[6]It is worth mentioning that ''usual'' Lorentz-scalar central charges can be interpreted as Kaluza-Klein momenta [35]

[7]Remember that the supersymmetry transformations of the superfield (3.32) are generated by the transformations of χ as functions of Θ [Eqs. (3.4)].

We see that at $a=1$ the model has the same spectrum of quantum physical states as in the generic case. The difference with the generic case ($a\neq 1$) is only in the transformation properties of the field components with respect to target space supersymmetry—the $a=1$ supersymmetry multiplets are shortened (see Ref. [1], Sec. 2). Note also that in the models with $a\geqslant 1$ one can impose additional reality condition on the quantum wave functions. In the Appendix we shall present another way of quantizing a single classical fermionic variable χ_1 [see Eq. (3.33)] which allows to treat it as a usual Grassmann variable and quantum superwave functions as standard superfields.

IV. QUANTIZATION BY USING THE CONVERSION METHOD

In order to justify the results of the supertwistor quantization of the model presented in Sec. III and to clarify the space-time structure of the quantum wave functions, in this section we shall perform the quantization directly in the coordinate representation. Because of the appearance of a particular mixture of fermionic first and second class constraints there appears a problem of quantizing the system covariantly. However, there exists a powerful method to handle this problem [19,17,18], which is based on the conversion of the second class constraints into the first class ones.

The quantization of the Ferber-Shirafuji model by the conversion method was considered in Refs. [20–23]. In Ref. [22] a $D=10$ supersymmetric particle with extra tensorial coordinates has been also discussed. In the present paper, however, the relation between spinor variables and the tensorial central charges, as well as their physical interpretation, goes far beyond the results presented in Ref. [22].

A. Conversion degrees of freedom

To convert the second class constraints into the first class ones we introduce additional (conversion) phase space degrees of freedom, whose number is equal to the number of the second class constraints.

Thus, for the $a\neq 0,1$ models we need $\mathbf{8}_b+\mathbf{2}_f$ conversion degrees of freedom. For this purpose we introduce bosonic spinors ρ_A, $\overline{\rho}_{\dot A}$ plus its canonical momenta $P^A_{(\rho)}$, $\overline{P}^{\dot A}_{(\rho)}$

$$[P^A_{(\rho)},\rho_B]_P=\delta_B{}^A,\quad [\overline{P}^{\dot A}_{\rho},\overline{\rho}_{\dot B}]_P=\delta^{\dot A}_{\dot B}, \tag{4.1}$$

and two real fermionic variables f_1 and f_2 whose Poisson brackets form a Clifford algebra

$$\{f_1,f_1\}_P=-i(1-a),\quad \{f_1,f_2\}_P=0,$$
$$\{f_2,f_2\}_P=-i(1+a). \tag{4.2}$$

Instead of f_1 and f_2 we shall also use two conjugate fermionic variables

$$S=\sqrt{(1+a)}f_1-i\sqrt{(1-a)}f_2,$$
$$\overline{S}=\sqrt{(1+a)}f_1+i\sqrt{(1-a)}f_2, \tag{4.3}$$

$$\{S,S\}_P=0,\quad \{\overline{S},\overline{S}\}_P=0,\quad \{S,\overline{S}\}_P=-2i(1-a^2). \tag{4.4}$$

Note, that $\overline{S}$ is complex conjugate of S only for the case $0<a<1$, while for $a>1$, where $\sqrt{(1-a)}=i\sqrt{|1-a|}$ is imaginary, both S and $\overline{S}$ are independent real variables. For $a\neq 1$ $\overline{S}$ can be regarded as the momentum conjugate to S.

B. Conversion of the constraints

We use the additional degrees of freedom (4.1) and (4.2) in order to convert the mixture of first and second class constraints (2.30)–(2.35) into the first class ones. As it was shown in Ref. [26], in twistorlike formulations of particle mechanics it is convenient to perform conversion of the whole set of primary constraints, without dividing them into the sets of first and second class constraints.

For any a the first class constraints obtained as the result of conversion are[8]

$$\Phi_{A\dot B}\equiv P_{A\dot B}-(\lambda_A+\rho_A)(\overline{\lambda}_{\dot B}+\overline{\rho}_{\dot B})=0, \tag{4.5}$$

$$\Phi_{AB}\equiv Z_{AB}-a(\lambda_A+\rho_A)(\lambda_B+\rho_B)=0, \tag{4.6}$$

$$\overline{\Phi}_{\dot A\dot B}\equiv \overline{Z}_{\dot A\dot B}-a(\overline{\lambda}_{\dot A}+\overline{\rho}_{\dot A})(\overline{\lambda}_{\dot B}+\overline{\rho}_{\dot B})=0, \tag{4.7}$$

$$\Phi_A\equiv P^A_{(\lambda)}+P^A_{(\rho)}=0,\quad \overline{\Phi}_{\dot A}\equiv \overline{P}^{\dot A}_{(\lambda)}+\overline{P}^{\dot A}_{(\rho)}=0, \tag{4.8}$$

$$P^A_{(\lambda)}-P^A_{(\rho)}=0,\quad \overline{P}^{\dot A}_{(\lambda)}-\overline{P}^{\dot A}_{(\rho)}=0, \tag{4.9}$$

$$D_A\equiv-\pi_A+iP_{A\dot B}\overline{\Theta}^{\dot B}+iZ_{AB}\Theta^B+(f_1+if_2)(\lambda_A+\rho_A)=0, \tag{4.10}$$

$$\overline{D}_{\dot A}\equiv\overline{\pi}_{\dot A}-i\Theta^BP_{B\dot A}-i\overline{Z}_{\dot A\dot B}\overline{\Theta}^{\dot B}+(f_1-if_2)(\overline{\lambda}_{\dot A}+\overline{\rho}_{\dot A})=0. \tag{4.11}$$

The algebra of the first class constraints (4.5)–(4.11) is quite simple. The only nonvanishing brackets (in the strong sense) appear in the fermionic sector and have the form

$$\mathcal{D}_{\alpha\beta}\equiv\begin{pmatrix}\{D_A,D_B\}_P & \{D_A,\overline{D}_{\dot B}\}_P\\ \{\overline{D}_{\dot A},D_B\}_P & \{\overline{D}_{\dot A},\overline{D}_{\dot B}\}_P\end{pmatrix}=2i\begin{pmatrix}-\Phi_{AB} & \Phi_{A\dot B}\\ \Phi_{B\dot A} & -\overline{\Phi}_{\dot A\dot B}\end{pmatrix}. \tag{4.12}$$

The RHS of Eq. (4.12) vanishes weakly, i.e., on the constraint surface (4.5)–(4.7).

Note that the expressions (4.5)–(4.7), (4.10), and (4.11) contain only the combination $(\lambda+\rho)$ of the commuting spinors. We denote this combination by $\tilde\lambda$

$$\tilde\lambda_A=\lambda_A+\rho_A,\quad \tilde{\overline{\lambda}}_{\dot A}=\overline{\lambda}_{\dot A}+\overline{\rho}_{\dot A}, \tag{4.13}$$

while the linearly independent variables

[8]We denote the converted constraints with the same letters as the original ones.

$$\tilde{\rho}_A=\lambda_A-\rho_A, \quad \tilde{\bar{\rho}}_{\dot{A}}=\bar{\lambda}_{\dot{A}}-\bar{\rho}_{\dot{A}} \tag{4.14}$$

completely decouple and have vanishing canonical momenta (4.9). Hence, the variables (4.14) can be excluded from the consideration, since the wave functions will not depend on these variables.

C. Quantization of the converted system: equations for the wave function

Now it is straightforward to quantize the system by using the Dirac method [28]. For this purpose let us choose the (super-)Shrödinger representation for the superspace coordinates and the bosonic spinor variables

$$\hat{P}_{A\dot{B}}=-i\frac{\partial}{\partial X^{A\dot{B}}},$$

$$\hat{Z}_{AB}=-i\frac{\partial}{\partial y^{AB}},$$

$$\hat{\bar{Z}}_{\dot{A}\dot{B}}=-i\frac{\partial}{\partial \bar{y}^{\dot{A}\dot{B}}}, \tag{4.15}$$

$$\hat{P}^{A}_{\tilde{\lambda}}=-i\frac{\partial}{\partial\tilde{\lambda}^{A}}, \quad \hat{\bar{P}}^{\dot{A}}_{\tilde{\bar{\lambda}}}=-i\frac{\partial}{\partial\tilde{\bar{\lambda}}^{\dot{A}}},$$

$$\hat{\pi}_A=+i\frac{\partial}{\partial\Theta^{A}}, \quad \hat{\bar{\pi}}_{\dot{A}}=+i\frac{\partial}{\partial\bar{\Theta}^{\dot{A}}}. \tag{4.16}$$

The fermionic variables f_1 and f_2 become Clifford algebra operators

$$(\hat{f}_1)^2=\frac{1}{2}(1-a), \quad (\hat{f}_2)^2=\frac{1}{2}(1+a), \quad \{\hat{f}_1,\hat{f}_2\}=0. \tag{4.17}$$

The Grassmann parity of f_1 and f_2 must be odd because the constraints (4.10) and (4.11) should have definite parity.

Note that the linear combinations (4.3) of fermionic quantum variables f_1 and f_2 satisfy the commutation relations

$$\{\hat{S},\hat{S}\}=0, \quad \{\hat{\bar{S}},\hat{\bar{S}}\}=0, \quad \{\hat{S},\hat{\bar{S}}\}=2(1-a^2). \tag{4.18}$$

So one can choose S as an odd coordinate and $\hat{\bar{S}}$ as its momentum operator

$$\hat{\bar{S}}=2(1-a^2)\frac{\partial}{\partial S}, \quad \hat{S}=S.$$

Despite the fact that such a representation makes hermiticity condition nonmanifest, it is convenient since it simplifies the calculations and provides the possibility of treating the cases $0<a<1$ and $a>1$ on an equal footing.

After quantization the first-class constraints (4.5)–(4.8), (4.10), and (4.11) are imposed on the wave function

$$\Psi=\Psi(x^{A\dot{A}};y^{AB},\bar{y}^{\dot{A}\dot{B}};\tilde{\lambda}^{A},\tilde{\bar{\lambda}}^{\dot{A}};\Theta^{A},\bar{\Theta}^{\dot{A}},S) \tag{4.19}$$

[we recall that we have consistently removed the variables (4.14) from the consideration]. Thus, the wave function of the system satisfies the first order differential equations

$$\left(\frac{\partial}{\partial x^{A\dot{B}}}-i\tilde{\lambda}_A\tilde{\bar{\lambda}}_{\dot{B}}\right)\Psi=0, \tag{4.20}$$

$$\left(\frac{1}{2}\frac{\partial}{\partial y^{AB}}-ia\tilde{\lambda}_A\tilde{\lambda}_B\right)\Psi=0, \tag{4.21}$$

$$\left(\frac{1}{2}\frac{\partial}{\partial \bar{y}^{\dot{A}\dot{B}}}-ia\tilde{\bar{\lambda}}_{\dot{A}}\tilde{\bar{\lambda}}_{\dot{B}}\right)\Psi=0, \tag{4.22}$$

$$\left(\frac{\partial}{\partial\Theta^{A}}+i\frac{\partial}{\partial x^{A\dot{B}}}\bar{\Theta}^{\dot{B}}+\frac{i}{2}a\frac{\partial}{\partial y^{AB}}\Theta^{B}-i(f_1+if_2)\tilde{\lambda}_A\right)\Psi=0, \tag{4.23}$$

$$\left(\frac{\partial}{\partial\bar{\Theta}^{\dot{A}}}+i\frac{\partial}{\partial x^{B\dot{A}}}\Theta^{B}+\frac{i}{2}a\frac{\partial}{\partial \bar{y}^{\dot{A}\dot{B}}}\bar{\Theta}^{\dot{B}}+i(f_1-if_2)\bar{\lambda}_{\dot{A}}\right)\Psi=0. \tag{4.24}$$

The solution of Eqs. (4.20)–(4.22) is

$$\Psi=e^{i\lambda_A\bar{\lambda}_{\dot{A}}x^{A\dot{A}}+ia\lambda_A\lambda_By^{AB}+ia\bar{\lambda}_{\dot{A}}\bar{\lambda}_{\dot{B}}\bar{y}^{\dot{A}\dot{B}}}g(\tilde{\lambda}^{A},\tilde{\bar{\lambda}}^{\dot{A}};\Theta^{A},\bar{\Theta}^{\dot{A}},S). \tag{4.25}$$

Because of the constraints (4.23) and (4.24) the function g (4.25) satisfies the conditions

$$\left(\frac{\partial}{\partial\Theta^{A}}-\tilde{\lambda}_A[(\bar{\Theta}\bar{\lambda})-a(\Theta\lambda)-i(f_1+if_2)]\right)g=0, \tag{4.26}$$

$$\left(\frac{\partial}{\partial\bar{\Theta}^{\dot{A}}}-\bar{\lambda}_{\dot{A}}[(\Theta\lambda)-a(\bar{\Theta}\bar{\lambda})+i(f_1-if_2)]\right)g=0. \tag{4.27}$$

An evident consequence of Eqs. (4.26) and (4.27) is that g depends only on the composite Grassmann variables $\chi=\Theta^{B}\lambda_B$ and $\bar{\chi}=\bar{\Theta}^{\dot{B}}\bar{\lambda}_{\dot{B}}$ introduced in Eq. (3.4)

$$g(\tilde{\lambda}^{A},\tilde{\bar{\lambda}}^{\dot{A}};\Theta^{A},\bar{\Theta}^{\dot{A}},S)=g(\tilde{\lambda}^{A},\tilde{\bar{\lambda}}^{\dot{A}};\Theta^{B}\lambda_B,\bar{\Theta}^{\dot{B}}\bar{\lambda}_{\dot{B}},S). \tag{4.28}$$

Then Eqs. (4.26) and (4.27) reduce to

$$\left(\frac{\partial}{\partial\chi}-\bar{\chi}-a\chi-i(f_1+if_2)\right)g(\lambda_A,\bar{\lambda}_{\dot{A}};\chi,\bar{\chi},S)=0, \tag{4.29}$$

$$\left(\frac{\partial}{\partial\bar{\chi}}-\chi-a\bar{\chi}+i(f_1-if_2)\right)g(\lambda_A,\bar{\lambda}_{\dot{A}};\chi,\bar{\chi};S)=0. \tag{4.30}$$

D. Quantization of the converted system: dependence on the fermionic variables

To find the solution of Eqs. (4.29) and (4.30) we take their linear combinations and rewrite them in the following form:

$$\left[\sqrt{1-a^2}\left(\frac{\partial}{\partial\overline{\chi}(a)}-\chi(a)\right)-S\right]g=0, \tag{4.31}$$

$$\left[\sqrt{1-a^2}\left(\frac{\partial}{\partial\chi(a)}-\overline{\chi}(a)\right)-2(1-a^2)\frac{\partial}{\partial S}\right]g=0, \tag{4.32}$$

where $\chi(a)$ were introduced in Eqs. (3.5), (3.6) and S are defined in Eq. (4.3).

Equations (4.31) and (4.32) are easily solved in terms of the components of the superfunction $g(S)$

$$g(S)=g_0(\chi,\overline{\chi})+iSg_1(\chi,\overline{\chi}) \tag{4.33}$$

which satisfies the conditions

$$\left(\frac{\partial}{\partial\overline{\chi}(a)}-\chi(a)\right)g_0=0$$

$$\Rightarrow g_0(\chi,\overline{\chi})$$

$$=e^{-\overline{\chi}(a)\chi(a)}\Phi[\lambda_A,\overline{\lambda}_{\dot A};\chi(a)], \tag{4.34}$$

$$2\sqrt{1-a^2}g_1(\chi,\overline{\chi})=-i\left(\frac{\partial}{\partial\chi(a)}-\overline{\chi}(a)\right)g_0. \tag{4.35}$$

We see that when $a\neq 1$ the component g_1 of the superfunction (4.33) is expressed in terms of g_0 which is specified by the condition (4.34) in terms of a single independent superfield $\Phi[\lambda_A,\overline{\lambda}_{\dot A};\chi(a)]$.

Hence, the independent wave function (4.19) which describes the general solution of Eqs. (4.20)–(4.24) is

$$\Psi=e^{i\lambda_A\overline{\lambda}_{\dot A}x^{A\dot A}+ia\lambda_A\lambda_B y^{AB}+ia\overline{\lambda}_{\dot A}\overline{\lambda}_{\dot B}\overline{y}^{\dot A\dot B}-\overline{\chi}(a)\chi(a)}\Phi[\tilde{\lambda}^A,\tilde{\bar\lambda}^{\dot A};\chi(a)]. \tag{4.36}$$

At the critical value $a=1$ the result is the same though the proof is slightly changed since in such a case (as in Sec. III A 4) we should deal with a single conversion Clifford variable f instead of f_1 and f_2 (and/or S and $\overline{S}$). More precisely, in Eqs. (4.29) and (4.30) one should put $a=1$, $\overline{\chi}=\chi$, and $f_1=S=0$, and then follow the quantization prescription described in the Appendix.

One should notice that the wave function $\Phi[\tilde{\lambda}^A,\tilde{\bar\lambda}^{\dot A};\chi(a)]$ in Eq. (4.36) has exactly the same structure as in the supertwistor case (3.20), but where now $\chi(a)$ are the composite Grassmann coordinates defined by Eqs. (3.5), (3.6). We therefore conclude that the direct supertwistor quantization and the quantization with the use of conversion of the superparticle model based on the generic action (2.1) result in the same supersymmetric spectrum of the quantum states.

The supersymmetry transformations of the components of $\Phi[\tilde{\lambda}^A,\tilde{\bar\lambda}^{\dot A};\chi(a)]$ are easily derived from Eq. (4.36) using the supersymmetric variations (2.7) of the coordinates. The higher dimensional generalization of the $a=1$ model and its quantization will be the subject of the next section.

V. THE $a=1$ MODEL IN HIGHER DIMENSIONS AND INTERNAL DEGREES OF FREEDOM

A generalization of the $D=4$, $a=1$ superparticle model has been proposed in Ref. [1]. In higher space-time dimensions D we consider an extension of an $N=1$ supersymmetry algebra by tensorial central charges

$$\{Q_\alpha,Q_\beta\}=P_{\alpha\beta},\quad [P_{\alpha\beta},Q_\gamma]=0, \tag{5.1}$$

where, depending on space–time dimension D, the supercharges Q_α ($\alpha=1,\dots 2^k$) are real Majorana, or Majorana-Weyl spinors[9] and $P_{\alpha\beta}$ is a symmetric generalized ''momentum'' generator conjugate to $2^k(2^k+1)$ symmetric spin-tensor coordinates $X^{\alpha\beta}$, which can be split into the usual space-time coordinates and tensorial central charge coordinates, as we shall demonstrate below.

We assume that $P_{\alpha\beta}$ is defined by the Cartan-Penrose relation

$$P_{\alpha\beta}=\lambda_\alpha\lambda_\beta, \tag{5.2}$$

where the real bosonic spinor λ_α has the same spinor properties as the supercharge Q_α. The expression (5.2) implies the Bogomol'nyi-Prasad-Sommerfield (BPS) condition $\det P_{\alpha\beta}=0$ and can be obtained as a primary constraint from the action functional [1]

$$S=\int_{\mathcal{M}^1}\lambda_\alpha\lambda_\beta\Pi^{\alpha\beta}, \tag{5.3}$$

$$\Pi^{\alpha\beta}=dX^{\alpha\beta}-id\Theta^{(\alpha}\Theta^{\beta)}=d\tau\,\Pi^{\alpha\beta}_\tau,$$

$$\alpha=1,\dots,2^k.$$

For any value of k the model possesses 2^k global target space supersymmetries generated by Eq. (5.1)

$$\delta_{\rm SUSY}X^{\alpha\beta}=i\Theta^{(\alpha}\epsilon^{\beta)},\quad \delta_{\rm SUSY}\Theta^\alpha=\epsilon^\alpha,\quad \delta_{\rm SUSY}\lambda_\alpha=0,$$

as well as 2^k-1 κ symmetries. To show the presence of the 2^k-1 κ symmetries let us write the variation of the action (5.3)

$$\delta S=\int_{\mathcal{M}^1}(2\delta\lambda_\alpha\lambda_\beta\Pi^{\alpha\beta}+d(\lambda_\alpha\lambda_\beta)i_\delta\Pi^{\alpha\beta}$$

$$-2i\,d\Theta^\alpha\lambda_\alpha\lambda_\beta\delta\Theta^\beta)+(\lambda_\alpha\lambda_\beta i_\delta\Pi^{\alpha\beta})|_{\tau_i}^{\tau_f}, \tag{5.4}$$

where

[9] In the general case one can also consider the cases of pseudo-Majorana, simplectic-Majorana and Dirac (complex) supercharges [36]. Technical details of the extension of the results of Sec. V to arbitrary type of supercharges will be considered in another publication.

$$i_\delta \Pi^{\alpha\beta} = \delta X^{\alpha\beta} - i\,\delta\Theta^{(\alpha}\Theta^{\beta)}.$$

One can see that only one linear combination $\lambda_\beta \delta\Theta^\beta$ of 2^k independent variations of Grassmann coordinates $\delta\Theta^\beta$ is effectively involved into the variation of the action.

Hence, other 2^k-1 Grassmann coordinate variations [which do not appear in Eq. (5.4)] can be identified with the parameters of local fermionic κ–symmetry. They can be written in the form

$$\kappa^I = u^I_\beta \delta\Theta^\beta,$$

where u^I_β $(I=1,\ldots,2^k-1)$ and λ_β form a set of 2^k linearly independent bosonic spinors. Identifying the κ symmetry with the part of target space supersymmetry which is preserved by the particle or brane configuration, we claim that the model (5.3) describes the dynamics of BPS states preserving all but one target space supersymmetries in spacetime of a dimension D.

Examples. In $D=3$ (where $k=1$) the action (5.3) describes the standard massless superparticle

$$k=1 \leftrightarrow D=3: \quad X^{\alpha\beta} = X^m \gamma_m^{\alpha\beta}, \quad P^{\alpha\beta} = P^m \gamma_m^{\alpha\beta}.$$

On the other hand the case $k=1$ can be regarded as a model in a ''minimal'' $D=2+2$ superspace with self-dual tensorial central charge coordinates $X^{\alpha\beta} = y^{mn}\sigma^{\alpha\beta}_{mn}$, $y^{mn} = \frac{1}{2}\epsilon^{mnkl} y_{kl}$. The case of $k=2$ corresponds to the $D=4$, $a=1$ model considered in Secs. II–IV but written in the Majorana representation.

The construction also holds in $D=6$ where $k=3$, but here we should use the [$SU(2)$–Majorana-Weyl] ''reality'' conditions. In addition to the four-dimensional spinor index α the complex eight-component spinors Q^i_α and λ^i_α carry the $SU(2)$ index $i=1,2$ and they are the $SU(2)$ Majorana-Weyl spinors (see for details Ref. [36]). The number of tensorial central charges in this model is 30.

The case $k=4$ can be regarded as describing a $D=10$ massless superparticle with 126 composite (self-dual) tensorial central charges $Z_{m_1\cdots m_5}$ [1] (cf. with Refs. [2,22]). The real supercharges Q_α satisfy the Majorana-Weyl reality condition.

The action (5.3) with $k=5$ corresponds to a 0-superbrane model in $D=11$ superspace with 517 tensorial central charges composed from 32 components of one real bosonic Majorana spinor. In contrast to the cases of $D=3$, 4, 6, and 10, in such a model the superparticle is not massless, the mass of the 0-brane being generated dynamically in a way similar to the mechanism generating the tension of superstrings and superbranes [41].

On the other hand it is possible to use the twelve dimensional $D=2+10$ 32×32 gamma matrices to treat the $k=5$ model from the point of view of two-time physics [4]. The bosonic coordinates $X^{\alpha\beta}$ are decomposed into two-index and self-dual six-index central charge coordinates $y^{mn}, y^{m_1\cdots m_6} = 1/6!\,\epsilon^{m_1\cdots m_6 n_1\cdots n_6} y_{n_1\cdots n_6}$.

A. $OSp(1|2^k)$ supertwistor representation of the D-dimensional model

Performing the integration by parts we can rewrite the action (5.3) in the $OSp(1|2^k)$ invariant form [i.e., $OSp(1|16)$ for $D=10$ and $OSp(1|32)$ for $D=11$] in terms of a supertwistor $Y^{\mathcal{A}} = (\mu^\alpha, \zeta)$

$$S = -\int (\mu^\alpha d\lambda_\alpha + i\,d\zeta\,\zeta), \quad \alpha = 1,\ldots,2^k. \tag{5.5}$$

The generalized Penrose-Ferber correspondence between the supertwistors and the generalized superspace looks as follows

$$P_{\alpha\beta} = \lambda_\alpha\lambda_\beta, \quad \mu^\alpha = X^{\alpha\beta}\lambda_\beta - i\Theta^\alpha(\Theta^\beta\lambda_\beta), \quad \zeta = \Theta^\alpha\lambda_\alpha \tag{5.6}$$

and does not imply other constraints.

B. Quantization of the higher dimensional model with the use of conversion

The quantization of the supertwistor formulation (5.3) is straightforward and is completely analogous to the quantization of the $D=4$ model considered in Sec. III A 4 and the Appendix. The spectrum of quantum states is described by the superfield

$$\Phi = \Phi(\lambda_\alpha, \zeta) = \phi(\lambda_\alpha) + i\zeta\psi(\lambda_\alpha) \tag{5.7}$$

depending on the bosonic spinor λ_α and one Grassmann (or, equivalently, Clifford) variable ζ. For completeness we briefly describe the quantization of the higher dimensional model (5.3) with the use of conversion.

The primary constraints of the model (5.3) are

$$\Phi_{\alpha\beta} \equiv P_{\alpha\beta} - \lambda_\alpha\lambda_\beta = 0, \tag{5.8}$$

$$D_\alpha \equiv \pi_\alpha + i\Theta^\beta P_{\beta\alpha} = 0, \tag{5.9}$$

$$P^\alpha_{(\lambda)} = 0, \tag{5.10}$$

where the momenta are defined in such a way that the nonvanishing Poisson brackets have the following form:

$$[P_{\alpha\beta}, X^{\gamma\delta}]_P = 2\,\delta^{(\gamma}_{(\alpha}\delta^{\delta)}_{\beta)}, \quad \{\pi_\alpha, \Theta^\beta\}_P = \delta^\beta_\alpha,$$

$$[P^\alpha_{(\lambda)}, \lambda_\beta]_P = \delta^\alpha_\beta. \tag{5.11}$$

This set of $2^k(2^k+1)/2$ bosonic and 2^k fermionic constraints obeys the algebra

$$[\Phi_{\alpha\beta}, P^\gamma_{(\lambda)}]_P = 2\lambda_{(\alpha}\delta^\delta_{\beta)},$$

$$\{D_\alpha, D_\beta\}_P = 2iP_{\alpha\beta} \equiv 2i(\Phi_{\alpha\beta} + \lambda_\alpha\lambda_\beta), \tag{5.12}$$

$$\text{all other brackets} = 0,$$

and thus contains 2^k bosonic and 1 fermionic second class constraints. Therefore, our system with $2^k(2^k+1)/2$ bosonic and 2^k fermionic configuration space variables contain 2^k

bosonic and 1 fermionic physical degrees of freedom which can be identified with the components of $OSp(1|2^k)$ supertwistor.

Exactly as in the $D=4$ case, to perform the conversion (see Refs. [15–18]) of the second class constraints into the first class ones we introduce additional ''conversion'' degrees of freedom (two for each pair of the bosonic second class constraints and one self-conjugate fermionic variable for each fermionic second class constraint)

$$[P^{\alpha}_{(\rho)},\rho_{\beta}]_P=\delta^{\alpha}{}_{\beta},\quad \{\xi,\xi\}=-\frac{i}{2}, \tag{5.13}$$

and transform the second–class constraints into the first-class ones extending the former with the new coordinates and momenta

$$\tilde{\Phi}_{\alpha\beta}\equiv P_{\alpha\beta}-\tilde{\lambda}_{\alpha}\tilde{\lambda}_{\beta}=0, \tag{5.14}$$

$$\tilde{D}_{\alpha}\equiv D_{\alpha}+2\tilde{\lambda}_{\alpha}\xi\equiv\pi_{\alpha}+i\Theta^{\beta}P_{\beta\alpha}+2\tilde{\lambda}_{\alpha}\xi=0, \tag{5.15}$$

$$P^{\alpha}_{\tilde{\rho}}=0, \tag{5.16}$$

where

$$\tilde{\lambda}_{\alpha}=\lambda_{\alpha}+\rho_{\alpha},\quad \tilde{\rho}_{\alpha}=\lambda_{\alpha}-\rho_{\alpha}. \tag{5.17}$$

Following the Appendix we obtain the superwave function describing the first-quantized states of the model determined by the single superfield (5.7) depending on $\tilde{\lambda}_{\alpha}$ and one Grassmann variable $\chi=(\Theta\lambda)$. We have

$$\Psi[\tilde{\lambda}_a,(\Theta\tilde{\lambda})]=e^{\frac{i}{2}\tilde{\lambda}_{\alpha}\tilde{\lambda}_{\beta}X^{\alpha\beta}}[\phi(\tilde{\lambda}_{\alpha})+i(\Theta\tilde{\lambda})\psi(\tilde{\lambda}_{\alpha})]. \tag{5.18}$$

In the sector with even λ-parity of the wave function ($\Psi=\Psi_+$) the spectrum of the quantum states of the model (5.3) is described by one bosonic $\phi_+(\tilde{\lambda})$ and one fermionic $\psi_-(\tilde{\lambda})$ function, while in the λ-odd sector ($\Psi=\Psi_-$) we have the fermionic field $\phi_-(\tilde{\lambda})$ and the bosonic field $\psi_+(\tilde{\lambda})$. This is in complete correspondence with the result of the quantization of the free supertwistor model (5.5).

C. Properties of the wave function with arbitrary helicity spectrum

To clarify the meaning of the wave function (5.7) [or Eq. (5.18)], let us consider its bosonic limit at $\Theta^{\alpha}=0$

$$\Psi=e^{\frac{i}{2}\tilde{\lambda}_{\alpha}\tilde{\lambda}_{\beta}X^{\alpha\beta}}\phi(\tilde{\lambda}_{\alpha}),\quad \alpha=1,\ldots,2^k, \tag{5.19}$$

and use the decomposition of the product of the spinor representations in the basis of D-dimensional gamma matrices. For the simplest case $k=1$ ($\alpha=1,2$), where our model coincides with a $D=3$ counterpart of the usual (Ferber-Shirafuji) model [9,10], the Fierz identity reads[10]

$$D=3:\quad \lambda_{\alpha}\lambda_{\beta}=\frac{1}{2}\gamma^{a}_{\alpha\beta}(\lambda\gamma_a\lambda)=\frac{1}{2}\gamma^{a}_{\alpha\beta}P_a,$$

and we can identify the matrix coordinates $X^{\alpha\beta}$ and their momenta $P_{\alpha\beta}$ with the usual vector coordinates and momenta

$$D=3:\quad X^a=\frac{1}{2}\gamma^{a}_{\alpha\beta}X^{\alpha\beta},\quad X^{\alpha\beta}=X^a\gamma_{a\alpha\beta},$$

$$P^a=\frac{1}{2}\gamma^{a}_{\alpha\beta}P^{\alpha\beta},\quad P_{\alpha\beta}=P^a\gamma_{a\alpha\beta}.$$

Thus, in $D=3$ Eq. (5.19) describes a plane wave solution[11]

$$D=3:\quad \Psi=e^{ip_mX^m}\phi(p_m). \tag{5.20}$$

The case $k=2$, $D=4$ has been analyzed in detail in Secs. II–IV. To transform the wave function (5.19) to the wave function (4.36) (at $a=1$, and $\Theta=0$) one should perform the similarity transformation from the real Majorana to the complex Weyl representation of the $D=4$ gamma–matrices and replace the Majorana spinor by the pair of complex conjugate Weyl spinors

$$\tilde{\lambda}_{\alpha}\leftrightarrow\begin{pmatrix}\tilde{\lambda}_A\\ \tilde{\bar{\lambda}}_{\dot{A}}\end{pmatrix}. \tag{5.21}$$

In the momentum representation the wave function $\phi(\lambda_{\alpha})$ differs from the usual one given by $\phi_0(p_m)$ by the presence of additional dependence on the angle variable α which describes the common phase factor of the Weyl spinor λ_A ($\lambda_1=e^{i(\alpha+\beta)}|\lambda_1|$, $\lambda_2=e^{i(\alpha-\beta)}|\lambda_2|$) and parametrizes the one-dimensional sphere S^1

$$D=4:\quad \phi(\lambda_a)=\phi(\lambda_A,\bar{\lambda}_{\dot{B}})=\phi(p_m,\alpha),\quad \alpha\in[0,2\pi),$$

where $p_m=\frac{1}{4}\lambda\gamma_m\lambda=\frac{1}{2}\lambda^A\sigma_{A\dot{A}}\bar{\lambda}^{\dot{A}}$ (see also Ref. [10]). The additional internal momentum variable α is the only independent degree of freedom contained in the $D=4$ tensorial central charges composed of the bosonic spinor

$$\alpha\in[0,2\pi)\Leftrightarrow Z_{mn}=\frac{1}{4}\lambda\gamma_{mn}\lambda.$$

---

[10]We use the matrices $\gamma^{a}_{\alpha\beta}$ which are symmetric and obtained from standard Dirac matrices $(\gamma^a)_{\alpha}{}^{\beta}$, by lowering one of the indices with the charge conjugation matrix $C=\gamma_0=i\tau_2$, which plays the role of the metric in the spinor space.

[11]More precisely, in $D=3$ $\phi(\lambda_a)=\psi[p_m,\mathrm{sgn}(\lambda)]$, where $\mathrm{sgn}(\lambda)$ denotes the sign factor (±1) of the bosonic spinor. This is a ''parameter'' of the residual $\mathbf{Z}_2$ symmetry, whose action on λ does not change p_m.

It describes the $D=4$ helicity spectrum of the quantum states of the superparticle.

In the general case of $k>2$ with 2^k equal to the dimension of an irreducible spinor representation of $SO(1,D-1)$ in $D=3,4,6,10$ (mod 8) (i.e., $k=3,4,\ldots$) the discussion is similar. For example, in the case $k=4$, $D=10$ we can use the basis of symmetric σ matrices $\sigma_m, \sigma_{m_1\ldots m_5}$ to make the decomposition

$$\lambda_\alpha \lambda_\beta \equiv P_{\alpha\beta} = P_m \sigma^m_{\alpha\beta} + Z_{m_1\cdots m_5}\sigma^{m_1\cdots m_5}_{\alpha\beta}, \qquad (5.22)$$

where

$$P_m = \frac{1}{16}\lambda_\alpha \sigma^{\alpha\beta}_m \lambda_\beta \Rightarrow P_m P^m = 0 \qquad (5.23)$$

is an ordinary lightlike momentum vector in $D=10$ and

$$Z_{m_1\cdots m_5} = \frac{1}{16\times 5!}\lambda_\alpha \sigma^{\alpha\beta}_{m_1\cdots m_5}\lambda_\beta \qquad (5.24)$$

is the momenta canonically conjugate to the **126** tensorial central charge coordinates $y^{m_1\cdots m_5}$.

It was demonstrated that the $D=10$ model contains the local symmetries (first class constraints) and second-class constraints which reduce the number of the classical bosonic degrees of freedom to the ones described by the 16-component bosonic spinor λ_α and its momentum plus one Grassmann degree of freedom. In the quantum theory this is reflected in the dependence of the ''momentum space representation'' of the wave function on 16 bosonic spinor variables and one Grassmann variable only.

Due to the identities $(\sigma_m)_{(\alpha\beta}(\sigma^m)_{\gamma)\delta}=0$ the $D=10$ momentum (5.23) is lightlike. Hence, the tensorial central charge momenta $Z_{m_1\cdots m_5}$ contain $16-9=7$ additional degrees of freedom which are not determined by the lightlike momentum.

We now show that these additional internal degrees of freedom parametrize an S^7 sphere. For this purpose we perform a Lorentz transformation to the frame where the lightlike momentum (5.23) acquires the form

$$P_m = (p,0,0,0,0,0,0,0,0,p). \qquad (5.25)$$

Then in this frame we make an $SO(8)$ invariant split of the bosonic spinor $\tilde\lambda_\alpha$

$$\tilde\lambda_\alpha = \begin{pmatrix} \Lambda_q \\ \Sigma_{\dot q} \end{pmatrix}, \quad q=1,\ldots 8, \quad \dot q = 1,\ldots 8, \qquad (5.26)$$

and choose the $SO(8)\times SO(1,1)$ covariant representation for the D=10 σ matrices

$$\sigma^0_{\underline{\alpha\beta}} = \mathrm{diag}(\delta_{qp}, \delta_{\dot q\dot p}) = \tilde\sigma^0_{\underline{--}\,\underline{\alpha\beta}},$$

$$\sigma^9_{\underline{\alpha\beta}} = \mathrm{diag}(\delta_{qp}, -\delta_{\dot q\dot p}) = -\tilde\sigma^{9\underline{\alpha\beta}},$$

$$\sigma^i_{\underline{\alpha\beta}} = \begin{pmatrix} 0 & \gamma^i_{q\dot p} \\ \tilde\gamma^i_{\dot q p} & 0 \end{pmatrix} = -\tilde\sigma^{i\underline{\alpha\beta}}, \qquad (5.27)$$

$$\sigma^{++}_{\underline{\alpha\beta}} \equiv (\sigma^0_{--} + \sigma^9_{--})_{\underline{\alpha\beta}} = \mathrm{diag}(2\delta_{qp}, 0)$$
$$= -(\tilde\sigma^0_{--} - \tilde\sigma^9_{--})^{\underline{\alpha\beta}} = \tilde\sigma^{--\,\underline{\alpha\beta}},$$

$$\sigma^{--}_{\underline{\alpha\beta}} \equiv (\sigma^0_{--} - \sigma^9_{--})_{\underline{\alpha\beta}} = \mathrm{diag}(0, 2\delta_{\dot q\dot p})$$
$$= (\tilde\sigma^0_{--} + \tilde\sigma^9_{--})^{\underline{\alpha\beta}} = \tilde\sigma^{++\,\underline{\alpha\beta}}.$$

In the frame (5.25) the Cartan-Penrose representation (5.23) looks as follows:

$$\Lambda_q \Lambda_q = p, \quad 2\Sigma_{\dot q}\Sigma_{\dot q} = 0, \quad \Lambda_q \gamma^i_{q\dot p}\Sigma_{\dot p} = 0. \qquad (5.28)$$

The general solution of Eqs. (5.28) is

$$\Sigma_{\dot q} = 0, \quad \Rightarrow \tilde\lambda_\alpha = \begin{pmatrix} \Lambda_q \\ 0 \end{pmatrix}, \qquad (5.29)$$

and the only nonvanishing component of the momentum (5.25) is given by the norm of the $SO(8)$ spinor Λ_q

$$p = \Lambda_q \Lambda_q. \qquad (5.30)$$

The expression (5.30) is invariant under the $SO(8)$ rotations

$$\Lambda_q \to \Lambda_p S_{pq}, \quad SS^T = I.$$

But not all $SO(8)$ transformations act on Λ_q effectively. Indeed, if one fixes the $SO(8)$ gauge

$$\Lambda_q = \begin{pmatrix} \pm\sqrt{p} \\ 0 \\ \cdots \\ \cdots \\ \cdots \\ 0 \end{pmatrix} \qquad (5.31)$$

one finds that (i) this gauge is invariant under the $SO(7)$ transformations and (ii) any form of the spinor Λ_q can be obtained from Eq. (5.31) by a transformation from the coset space $SO(8)/SO(7)$ isomorphic to the sphere S^7. Thus, the 16 components of the bosonic spinor λ_α in $D=10$ can be split into (i) degrees of freedom which characterize the lightlike momentum P_m and (ii) 7 coordinates of the sphere S^7. The variables parametrizing the sphere S^7 correspond to ''helicity'' degrees of freedom of the quantum states of the massless $D=10$ superparticle.

It is worth mentioning that the appearance of extra compact dimensions in the momentum spaces of the superparticle models considered above is related to the well known fact that in $D=3$, 4, 6, and 10 the commuting spinors (twistors) with $n=2(D-2)=2$, 4, 8, and 16 components parametrize, modulo scale transformations, S^1, S^3, S^7, and S^{15} spheres, respectively. These spheres are Hopf fibrations (fiber bundles) which are associated with the division algebras

R, **C**, **H**, and **O**. Their bases are the spheres S^1, S^2, S^4, and S^8, and the fibers are Z^2, $S^1=U(1)$, $S^3=SU(2)$, and S^7, respectively. The base spheres are parametrized (up to a scaling factor which, due to the Cartan-Penrose representation, is identified with the square of spinor components) by the lightlike vectors (massless particle momenta) in $D=3$, 4, 6, and 10, respectively. We see that the fibers are extra ''momentum dimensions'' which we have in our models with central charges (at $a\neq 0$). This is the geometrical ground for the appearance of S^1 in $D=4$ and S^7 in $D=10$.

VI. CONCLUSION AND DISCUSSION

We have performed the detailed analysis and quantization of the massless superparticle model with tensorial central charges associated with twistorlike commuting spinors in space-times of dimension $D=3$, 4, 6, and 10. The physical phase space degrees of freedom of this model have a natural description in terms of supertwistors which form a fundamental representation of a corresponding maximal supergroup of conformal type underlying the dynamics of the superparticle.

A peculiarity of the $a=1$ model is that it possesses $n=2^{[D/2]}-1$ κ symmetries, while the standard massless superparticles have $n=2^{[D/2]-1}$ κ symmetries. The presence of such a large number of κ symmetries in the $a=1$ models means that the superparticle breaks only one of the $2^{[D/2]}$ supersymmetries of the target space vacuum. This results in very short two-component supermultiplets describing the quantum states of the $a=1$ superparticle, since the corresponding target space superfields depend only on one Grassmann coordinate. The existence of these short Lorentz-covariant superfields is made possible because the target superspace has been enlarged by commuting spinor coordinates, whose role is in singling out a ''small'' covariant subsuperspace in the extended target superspace. Let us compare this situation with well known cases.

In the case of the ordinary massless superparticle in $N=1$, $D=4$ superspace the quantum states of the superparticle are described by a chiral scalar superfield [42]. The chirality constraint is a consequence of first-class fermionic constraints generating κ symmetries. Consequently the chiral superfield effectively depends on only two Grassmann coordinates, which reflects the fact that the ordinary superparticle preserves half (i.e., two out of four) supersymmetries of $N=1$, $D=4$ superspace.

In the case of an $N=2$, $D=4$ superparticle in harmonic superspace [43] which also breaks half of the target space supersymmetries, $SU(2)$-harmonic variables allow one to pick a harmonic analytic subsuperspace out of the general $N=2$, $D=4$ superspace [44], and quantum states of the superparticle are described by analytic superfields which depend on four Grassmann coordinates singled out from the original eight Grassmann coordinates by the use of the harmonic variables. In the analogous way, in the case of the generalized superparticle model (2.1), (5.3) at $a=1$, when only one target space supersymmetry is broken, one finds a Lorentz covariant subsuperspace of the target superspace, which has only one Grassmann direction parametrized by the Lorentz scalar $\theta^\alpha\lambda_\alpha$. The ''short'' superfields (4.36), (5.18), which exist only due to the presence of the auxiliary spinor variable, describe the quantum states of the generalized superparticle.

We have shown that in contrast to standard superparticles the considered model possesses additional compact phase-space variables which describe helicity degrees of freedom of the superparticle and which upon quantization parametrize infinite tower of free states with arbitrary (half-)integer helicities. Due to this property it would be interesting to consider the possibility of treating our generalized superparticle model as a classical mechanics counterpart of the theory of higher-spin fields developed by Vasiliev [13]. Since the nontrivially interacting higher spin fields should live in a space-time of (anti)–de Sitter geometry a natural generalization of the results of this paper would be to consider a superparticle model on supergroup manifolds describing isometries of corresponding AdS superspaces. For $D=4$ the supergroup $OSp(1|4)$ is the isometry of a $D=4$ AdS superspace $OSp(1|4)/SO(1,3)$ which in addition to four bosonic directions has four Grassmann fermionic directions. Six bosonic coordinates corresponding to the group $SO(1,3)$ [which extends the coset superspace $OSp(1|4)/SO(1,3)$ to the supergroup manifold $OSp(1|4)$] are a non-Abelian generalization of the central charge coordinates of the $D=4$ model considered above. It appears that our model with central charges can be regarded as an appropriate truncation of the $OSp(1|4)$ model. Work in this direction is now in progress.

We should also remark that tensorial central charges are usually associated with brane charges, which are topological and take discrete (quantum) values. In contrast, in the superparticle models considered in this paper the central charges take continuous values and parametrize compact manifolds, while their Fourier conjugate coordinates are quantized.

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APPENDIX

1. Quantization of one fermionic degree of freedom by a ''half-conversion'' prescription

Here we shall present a method of quantizing a single fermionic variable alternative to that used in Sec. III A 4, but which leads to the same spectrum of quantum states. Let us convert the second-class constraint (3.34) into the first-class constraint by introducing one more Clifford-like variable ξ'

$$\{\xi',\xi'\}_P=-\frac{i}{2}. \tag{A1}$$

Using ξ' we replace Eq. (3.34) by the first class constraint

$$\mathcal{A}\equiv\xi-i\pi_{(\xi)}+2\xi'=0,\quad \{\mathcal{A},\mathcal{A}\}_P=0, \tag{A2}$$

where instead of χ_1 of Eq. (3.34) we have introduced $\xi=2\chi_1$.

Let us quantize the model using the coordinate representation for the original Grassmann variable ξ

$$\hat{\xi}=\xi,\quad \hat{\pi}_\xi=i\frac{\partial}{\partial\xi}$$

and a real 2×2 matrix representation

$$\hat{\xi}'=\frac{1}{2}\tau=\frac{1}{2}\begin{pmatrix}*&*\\*&*\end{pmatrix}\quad \tau^2=I \tag{A3}$$

for the new Clifford algebra valued variable $\hat{\xi}'$

$$\hat{\xi}'^2=1/4.$$

Then the wave function is regarded as a column

$$\Psi_a=\begin{pmatrix}\phi(\xi)\\ \psi(\xi)\end{pmatrix} \tag{A4}$$

and the quantum counterpart of the first class constraint (A2)

$$\hat{\mathcal{A}}\equiv\left(\xi-\frac{\partial}{\partial\xi}\right)\tau'+\tau \tag{A5}$$

should be imposed on the wave function

$$\hat{\mathcal{A}}_{ab}\Psi_b\equiv\left[\left(\xi-\frac{\partial}{\partial\xi}\right)\tau'_{ab}+\tau_{ab}\right]\Phi_b(\xi)=0. \tag{A6}$$

In Eqs. (A5) and (A6) the second 2×2 matrix

$$\tau'_{ab}=\begin{pmatrix}*&*\\*&*\end{pmatrix},\quad (\tau')^2=I \tag{A7}$$

was introduced. It is required to ensure the anticommutativity of the Grassmann and Clifford part of the first class constraint (A6). Indeed, let us calculate the square of the quantum constraint (A5)

$$\hat{\mathcal{A}}^2\equiv\frac{1}{2}\{\hat{\mathcal{A}},\hat{\mathcal{A}}\}=\tau^2+\left(\xi-\frac{\partial}{\partial\xi}\right)^2(\tau')^2+\left(\xi-\frac{\partial}{\partial\xi}\right)\{\tau',\tau\}. \tag{A8}$$

Since

$$\left(\xi-\frac{\partial}{\partial\xi}\right)^2=\frac{1}{2}\left\{\left(\xi-\frac{\partial}{\partial\xi}\right),\left(\xi-\frac{\partial}{\partial\xi}\right)\right\}=-1 \tag{A9}$$

and

$$\tau^2=I=(\tau')^2,$$

one easily finds that the first two terms in (A8) cancel and arrives at

$$\hat{\mathcal{A}}^2=\left(\xi-\frac{\partial}{\partial\xi}\right)\frac{1}{2}\{\tau',\tau\}. \tag{A10}$$

The last input vanishes if and only if $\{\tau',\tau\}=0$. This result cannot be reached if one chose τ' to be the unit matrix. τ and τ' can be chosen to be two Pauli matrices.

Let us stress that the necessity to introduce the second matrix τ' is a peculiarity of the quantization of the odd number of Clifford variables. τ' can be the unit matrix in the case of even number of Clifford variables (see, e.g., Ref. [45], and references therein).

To fix the representation for the matrices τ and τ' one has to note that the conservation of the Grassmann parity in the form of the first class constraint (A6) requires the following.

The components $\phi(\xi)$ and $\psi(\xi)$ of (A3) must have *different Grassmann parity*. For instance, if we choose $\phi(\xi)$ to be bosonic superfield then $\psi(\xi)$ is fermionic.

If the diagonal representation is chosen for one of the matrices, say τ', then another matrix τ is antidiagonal.

Keeping these in mind we choose

$$\tau'=\tau_3\equiv\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \tau=\tau_1\equiv\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{A11}$$

Then the quantum constraints (A6) acquire the form

$$\hat{\mathcal{A}}_{ab}\Psi_b\equiv\begin{pmatrix}(\xi-\partial/\partial\xi)&1\\1&-(\xi-\partial/\partial\xi)\end{pmatrix}\begin{pmatrix}\phi(\xi)\\ \psi(\xi)\end{pmatrix}_b$$
$$=\begin{pmatrix}(\xi-\partial/\partial\xi)\phi(\xi)+\psi(\xi)\\ \phi(\xi)-(\xi-\partial/\partial\xi)\psi(\xi)\end{pmatrix}=0, \tag{A12}$$

which splits into two equations

$$\left(\xi-\frac{\partial}{\partial\xi}\right)\phi(\xi)=-\psi(\xi), \tag{A13}$$

$$\phi(\xi)=\left(\xi-\frac{\partial}{\partial\xi}\right)\psi(\xi).$$

Using Eq. (A9), we notice that the second equation is a consequence of the first one. The first equation

$$\left(\xi-\frac{\partial}{\partial\xi}\right)\phi(\xi)=-\psi(\xi) \tag{A14}$$

expresses the fermionic superfield through the bosonic one. That is, if we write $\phi(\xi)$ in components

$$\phi(\xi)=\phi_0+i\xi\psi_1,$$

then from Eq. (A14) it follows that

$$\psi(\xi)=i\psi_1-\xi\phi_0.$$

Thus, we can represent the spectrum of states carrying one Clifford degree of freedom by one (either bosonic or fermionic) superfield $\phi(\xi)$ depending on the single *Grassmann* variable ξ

$$\xi^* = \xi, \quad \xi^2 = 0.$$

This result is in accordance with that of Sec. III A 4 (see also Ref. [27]), and both methods of quantizing a single fermionic variable result in the same field content of quantum states (one boson and one fermion).

2. Quantization of the high-dimensional model with the use of conversion

Here we present some details of getting the wave function (5.18) from the converted system of constraints (5.14), (5.15), and (5.16) describing the high-dimensional generalization of the first-quantized $a=1$ model. Let us choose the (super)coordinate representation for supercoordinates and bosonic spinors

$$\hat{P}_{\alpha\beta} = -i\frac{\partial}{\partial X^{\alpha\beta}}, \quad \hat{P}^{\alpha}_{(\tilde{\lambda})} = -i\frac{\partial}{\partial \tilde{\lambda}^{\alpha}}, \tag{A15}$$

$$\hat{\pi}_{\alpha} = i\frac{\partial}{\partial \Theta^{\alpha}}, \tag{A16}$$

and use the 2×2 matrix representation

$$\hat{\xi} = \frac{1}{2}\tau_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \{\hat{\xi}, \hat{\xi}\} = \frac{1}{2} \tag{A17}$$

for the Clifford variable ξ.

Then the wave function is a column

$$\Psi_a = \Psi_a(X^{\alpha\beta}, \tilde{\lambda}_{\beta}) = \begin{pmatrix} \phi(X^{\alpha\beta}, \tilde{\lambda}_{\beta}) \\ \psi(X^{\alpha\beta}, \tilde{\lambda}_{\beta}) \end{pmatrix} \tag{A18}$$

with the elements carrying opposite Grassmann parity (e.g., ϕ is bosonic and ψ is fermionic) and the quantum first class constraints (5.14), (5.15), (5.16) should be taken in the form

$$\hat{\tilde{\Phi}}_{\alpha\beta} = -i(\partial_{\beta\alpha} - i\tilde{\lambda}_{\alpha}\tilde{\lambda}_{\beta})I, \tag{A19}$$

$$\hat{\tilde{D}}_{\alpha} = \hat{D}_{\alpha} + \tilde{\lambda}_{\alpha}\hat{\xi} = i(\partial_{\alpha} - i\Theta^{\beta}\partial_{\beta\alpha})\tau_3 + \tilde{\lambda}_{\alpha}\tau_2. \tag{A20}$$

The incorporation of the τ_3 matrix is necessary to provide the properties of the first class constraints to form the closed algebra

$$\{\hat{\tilde{D}}_{\alpha}, \hat{\tilde{D}}_{\beta}\} = -2i\hat{\tilde{\Phi}}_{\alpha\beta}. \tag{A21}$$

This is a peculiarity of the quantization of the models with odd number of phase space Grassmann variables (see Sec. 1 of this appendix).

The further steps of the quantization procedure exactly repeat the steps of the $D=4$ case (see Sec. IV). The wave function describing the spectrum of the quantum states is

$$\tilde{\Psi}_a = e^{(i/2)\tilde{\lambda}_{\alpha}\tilde{\lambda}_{\beta}X^{\alpha\beta}} \begin{pmatrix} \Psi(\tilde{\lambda}_{\alpha}, \Theta\lambda) \\ -i[\partial/\partial(\Theta\lambda) + \Theta\lambda]\Psi(\tilde{\lambda}_{\alpha}, \Theta\lambda) \end{pmatrix}. \tag{A22}$$

As the second element in the column is expressed through the first one, we can describe the spectrum of the quantum states by the single superfield (5.18) depending on bosonic $\tilde{\lambda}_{\alpha}$ and fermionic $\chi = (\Theta\lambda)$ coordinates.

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