

Kerr-Schild approach to the boosted Kerr solution

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Using a complex representation of the Debney-Kerr-Schild solutions and the Kerr theorem we analyze the boosted Kerr geometries and give the exact and explicit expressions for the metrics, the principal null congruences, the coordinate systems and the location of the singularities for an arbitrary value and orientation of the boost with respect to the angular momentum. In the limiting, ultrarelativistic case we obtain lightlike solutions possessing diverging and twisting principal null congruences and having, contrary to the known pp -wave limiting solutions, a nonzero value of the total angular momentum. The implications of the above results in various related fields are discussed.

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I. INTRODUCTION

Recently, boosted black-hole solutions attracted renewed interest in connection with numerical simulations of black-hole interactions [1]. On the other hand, the problem of finding the ultrarelativistic limit of exact, particlelike solutions of the Einstein field equations received considerable attention in connection with nontrivial gravitational effects which are expected to occur in the interparticle interactions at extreme energies due to the presence of gravitational shock waves [2,3].

First results in this field were obtained by Aichelburg and Sexl [4], who considered the behavior of the Schwarzschild metric under the ultrarelativistic boost. Because of spherical symmetry the results are in this case independent of the direction of the boost.

A similar treatment in the case of the Kerr geometry, which can be considered as a model of a spinning particle in general relativity, has to take into account the orientation of the angular momentum with respect to the boost. Unlike the simplest twist-free case the problem exhibits extra difficulties which enforce quite complicated and refined methods of analysis and usually lead to very complicated or approximate expressions before taking the ultrarelativistic limit [5–7]. In particular, difficulties appear when ultrarelativistic limits are involved, due to the singular character of Lorentz transformations at $v=c$. We should note that this singularity *a priori* can lead to different limiting results depending on the performed limiting procedure.

The approach which we are going to formulate here deals with a class of *sourceless* gravitational solutions of the Kerr-Schild class and it is based on the Debney-Kerr-Schild (DKS) formalism [8] and on the Kerr theorem [9–11]. It gives the possibility of obtaining exact and explicit expressions for the boosted Kerr geometry by arbitrary values and orientations of the boost with respect to the angular momen-

tum. As a result, in the general cases of the boost, including the ultrarelativistic cases, we determine the exact expressions for the metric, coordinate system, principal null congruence, and location of singularity. In the ultrarelativistic cases this method leads to a DKS class of solutions possessing diverging and twisting principal null congruences contrary to the pp -wave limiting solutions which possess zero total angular momentum [7].

The paper is organized as follows. First, we briefly recall the DKS formalism and the Kerr theorem in a form, in which the Kerr solution is represented as being generated by a complex source. This approach was initiated by Lind and Newman [12,13] and was considered in the DKS formalism in [14] (a more complete description of this approach and of its geometrical basis can be found in [15]). Using this representation, the boosted Kerr solutions can be constructed simply by considering straight lines in complexified Minkowski space as complex world lines of the sources. In this way, we obtain explicit expressions for the metric and the singular regions in the most representative cases. We discuss various physical applications and some unusual features in the limiting behavior of the Kerr singular ring.

II. THE DKS FORMALISM AND THE KERR THEOREM

In the notation we follow the work of Debney, Kerr, and Schild [8] (see also [9,16] for a review on DKS solutions). In the four-dimensional space time with signature $(-+++)$, let e_1, e_2, e_3, e_4 be a null tetrad satisfying

$$g_{ab} = e_a^\mu e_{b\mu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (1)$$

The vectors e^3 and e^4 are real null vectors, while e^1 and e^2 are complex conjugates. The general Kerr-Schild metric can be written as

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$$g_{\mu\nu} = \eta_{\mu\nu} + 2he^3 e^3_{\mu} e^3_{\nu}, \quad (2)$$

where h is a scalar function and the principal null direction e^3 is null also with respect to the auxiliary Minkowski space with metric

$$\eta = dx^2 + dy^2 + dz^2 - dt^2 = 2dudv + 2d\bar{\zeta}d\zeta.$$

In the above formula, the null coordinates $\zeta, \bar{\zeta}, u, v$ are related to the Cartesian coordinates by

$$\begin{aligned} 2^{1/2}\zeta &= x + iy, & 2^{1/2}\bar{\zeta} &= x - iy, \\ 2^{1/2}u &= z + t, & 2^{1/2}v &= z - t. \end{aligned} \quad (3)$$

A general field of null directions in Minkowski space can be defined by

$$e^3 = du + \bar{Y}d\zeta + Yd\bar{\zeta} - Y\bar{Y}dv, \quad (4)$$

where $Y(x)$ is a complex function. The Kerr theorem gives a rule to construct the geodesic, shear free congruences: the general geodesic, shear-free null congruence in Minkowski space is defined by a function Y which is a solution of the equation

$$F(\lambda_1, \lambda_2, Y) = 0, \quad (5)$$

where F is an arbitrary analytic function of the *projective twistor coordinates*

$$\lambda_1 = \zeta - Yv, \quad \lambda_2 = u + Y\bar{\zeta}, \quad Y \quad (6)$$

[the above parameters can be written in twistor notation $(\mu^A, \psi_{\dot{A}})$, $\mu^A = x^\mu \sigma_\mu^{A\dot{A}} \psi_{\dot{A}}$ as $(\lambda_1, \lambda_2, Y, 1) = (\mu^0, \mu^1, \psi_{\dot{0}}, \psi_{\dot{1}})/\psi_{\dot{1}}$].

The Kerr theorem also allows us to obtain other important parameters of the solution. In particular, the quantity

$$\tilde{r} := -dF/dY \quad (7)$$

is a complex radial distance, which is connected with the complex representation of the Kerr solution, as will be explained below.

The singularities of the metric can be defined as the caustics of the congruence given by the system of equations

$$F = 0, \quad dF/dY = 0. \quad (8)$$

The Kerr solution belongs to the subclass of solutions having singularities contained in a bounded region [14,17]. In this case the function F must be at most quadratic in Y :

$$F \equiv a_0 + a_1 Y + a_2 Y^2 + (qY + c)\lambda_1 - (pY + \bar{q})\lambda_2, \quad (9)$$

where the coefficients c and p are real constants and $a_0, a_1, a_2, q, \bar{q}$, are complex constants. The solutions of Eqs. (8) can be found in this case in explicit form. It can be shown that they correspond to the Kerr solution up to a Lorentz boost and a shift of the origin.

III. CONGRUENCES GENERATED BY A COMPLEX WORLD LINE

Our approach is based on the complex world-line representation of the Kerr solution which was initiated by Newman and Lind [12,13]. It allows us to represent the Kerr solution as a retarded-time field, generated by a complex source propagating along a complex world line. The structure of this representation in DKS formalism gives a very convenient form of the function F , which turns out to be dependent on the coordinates of the world line [14,15]. For the aim of convenience we give here a short review of this approach.

Let $x_0^\mu(\tau)$ be a complex world line parametrized by a complex time parameter $\tau = t + i\sigma$. The coordinates of this world line are complex, $x_0(\tau) = (\zeta_0, \bar{\zeta}_0, u_0, v_0) \in CM^4$, so that $\bar{\zeta}_0$ and ζ_0 are not necessarily complex conjugates.

The function F can be expressed in the form

$$F \equiv (\lambda_1 - \lambda_1^0) \hat{K} \lambda_2 - (\lambda_2 - \lambda_2^0) \hat{K} \lambda_1, \quad (10)$$

where the twistor components with zero indices

$$\lambda_1^0(\tau) = \zeta_0(\tau) - Yv_0(\tau), \quad \lambda_2^0(\tau) = u_0(\tau) + Y\bar{\zeta}_0(\tau), \quad (11)$$

denote the values of λ_1 and λ_2 on the points of the complex world line $x_0(\tau)$, while \hat{K} is a Killing vector of the solution, whose action on a scalar f is defined by

$$\hat{K}f = \dot{x}_0^\mu(\tau) \partial_{\mu} f, \quad (12)$$

(a dot denotes derivative with respect to τ).

It has been shown [15] that the form of F given by Eq. (10) is equivalent to Eq. (9). In this representation, the Kerr congruence can be described via a retarded-time construction (in particular, the Schwarzschild and the Kerr metrics correspond to the world line of a particle at rest at the origin and to the world line of a particle at rest ‘‘at a distance ia from the origin,’’ respectively [13]). This one-to-one correspondence between straight lines in complex Minkowski space and the class of the DKS solutions having singularities contained in a bounded region allows us to boost the Kerr solution *via* the DKS formalism. Indeed the complex Minkowski space inherits the linear structure of the real Poincaré group, and therefore the boosted Kerr metrics are the real slices of the boosted solutions in CM^4 . This is readily seen observing that, in spite of the complex character of the world line, the vector of its four velocity is actually real and has to follow the real Lorentz transformations. Therefore, the boost yields a complex world line with a real four velocity obtained by the corresponding Lorentz transformation from the rest four velocity.

It is worth noticing that a very similar situation occurs in the case of the Maxwell equations. Indeed applying a complex boost to the Coulomb field one obtains the general boosted (Lienard-Wiechert) field [12]. Actually Newman

was the first to suggest, on the base of this analogy, that Kerr-Schild boosted solutions could eventually be found in this way.

To obtain the general case of a solution with a boost we consider, therefore, a straight, complex world line with three velocity \vec{V} in CM^4

$$x_0^\mu(\tau) = x_0^\mu(0) + \xi^\mu \tau; \quad \xi^\mu = (1, \vec{V}). \quad (13)$$

Writing the function F in the form

$$F = AY^2 + BY + C, \quad (14)$$

where

$$\begin{aligned} A &= (\bar{\zeta} - \zeta_0) \dot{v}_0 - (v - v_0) \dot{\zeta}_0; \\ B &= (u - u_0) \dot{v}_0 + (\zeta - \zeta_0) \dot{\zeta}_0 - (\bar{\zeta} - \bar{\zeta}_0) \dot{\zeta}_0 - (v - v_0) \dot{u}_0; \\ C &= (\zeta - \zeta_0) \dot{u}_0 - (u - u_0) \dot{\zeta}_0, \end{aligned} \quad (15)$$

from $F=0$ we obtain the following explicit solutions for the function $Y(x)$:

$$Y_{1,2} = (-B \pm \Delta)/2A, \quad (16)$$

where $\Delta = (B^2 - 4AC)^{1/2}$. On the other hand, from Eqs. (7) and (14) one obtains

$$Y = -(B + \tilde{r})/2A, \quad (17)$$

and consequently

$$\tilde{r} = \mp (B^2 - 4AC)^{1/2}. \quad (18)$$

This relation reflects the ‘‘twofoldedness’’ of the Kerr geometry: the complex radial coordinate \tilde{r} can be expressed as $r + ia \cos \theta$ and the double sign corresponds to a transition from the ‘‘positive’’ r sheet of the metric to the ‘‘negative’’ one where $r \leq 0$.

In the DKS notation the metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + (m/P^3)(Z + \bar{Z})e_\mu^3 e_\nu^3, \quad (19)$$

where

$$P = \dot{x}_0^\mu(\tau) e_\mu^3. \quad (20)$$

The field e^3 can be normalized by introducing $l^\mu = e^{3\mu}/P$ so that $\dot{x}_0^\mu l_\mu = 1$, and this yields the following, equivalent form of the metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + m(P^{-1}Z + P^{-1}\bar{Z})l_\mu l_\nu. \quad (21)$$

The complex radial distance (7) is given by

$$PZ^{-1} = \tilde{r}, \quad (22)$$

and the twistor parameters λ_1 and λ_2 may be represented in the form

$$\lambda_1 = x^\mu e_\mu^1, \quad \lambda_2 = x^\mu (e_\mu^3 - \bar{Y} e_\mu^1) \quad (23)$$

(the explicit form of the DKS tetrad e^a is given in the Appendix).

IV. BOOSTED KERR SOLUTION: EXAMPLES AND BEHAVIOR OF SINGULARITIES

To analyze the DKS boosted Kerr solutions we represent the world line in the form $x_0^\mu(\tau) = \{\tau, \vec{x}_0(0) + \vec{V}\tau\}$, and observe that the complex parameter τ may be always chosen in such a way that $\text{Re } \tau$ corresponds to the ‘‘real time’’ t (see [13] for details on complex Minkowski space). The complex initial displacement can be decomposed as $\vec{x}_0(0) = \vec{c} + i\vec{d}$, where \vec{c} and \vec{d} are real three-vectors with respect to the space $O(3)$ rotation. The real part \vec{c} defines the initial shift of the solution, and the imaginary part \vec{d} defines the size and the position of the singular ring as well as the corresponding angular momentum. It can be easily shown that in the rest frame, when $\vec{V} = 0$, $\vec{d} = \vec{d}_0$, the singular ring lies in the plane orthogonal to \vec{d} and has a radius $a = |\vec{d}_0|$. The corresponding angular momentum is $\vec{J} = m\vec{d}_0$.

In the case of a boost orthogonal to the direction of \vec{d} , this vector is not altered by Lorentz contraction ($\vec{d} = \vec{d}_0$, $|\vec{d}| = a$), while if \vec{d} and \vec{V} are collinear we have

$$\vec{d}_0 = \vec{d} / \sqrt{1 - |\vec{V}|^2}. \quad (24)$$

This shows that the parameter a coincides with its rest value a_0 if \vec{d} and \vec{V} are orthogonal, while

$$a_0 = a / \sqrt{1 - |\vec{V}|^2}, \quad (25)$$

if \vec{V} and \vec{d} are collinear.

In order to calculate the parameters A, B, C it is convenient to express the complex world line in null coordinates

$$\begin{aligned} 2^{1/2} \zeta_0 &= x_0 + iy_0, & 2^{1/2} \bar{\zeta}_0 &= x_0 - iy_0, \\ 2^{1/2} u_0 &= z_0 + t_0, & 2^{1/2} v_0 &= z_0 - t_0. \end{aligned} \quad (26)$$

The Killing vector of the solution will then be

$$\xi^\mu = 2^{-1/2} \{\dot{u}_0 - \dot{v}_0, \dot{\zeta}_0 + \dot{\bar{\zeta}}_0, -i(\dot{\zeta}_0 + \dot{\bar{\zeta}}_0), \dot{u}_0 + \dot{v}_0\}, \quad (27)$$

while the function P takes the form

$$P = e_\mu^3 \dot{x}_0^\mu = \dot{u}_0 + \bar{Y} \dot{\zeta}_0 + Y \dot{\bar{\zeta}}_0 - Y \bar{Y} \dot{v}_0. \quad (28)$$

The complex radial coordinate $\tilde{r} \equiv PZ^{-1}$ is given by Eq. (18). As for the standard Kerr solution, one can represent \tilde{r} as a ‘‘sum’’ of the real radial distance r and an angular coordinate. Then Eq. (18) can be used to fix the relation between the polar coordinates r, θ, ϕ and the null Cartesian coordinates (26) through the expressions (15) for the coefficients A, B, C . Because of the formula (7), the singular regions are defined by the zeros of the function \tilde{r} . In what

follows, we present some examples of boosted Kerr solutions and then discuss the general features exhibited by them.

Example I. A spinning particle moves with the speed of light in the positive direction of the z axis, and the three vector $\vec{d}=(0,0,a)$ is also directed along the z axis. The complex world line is, therefore, given by $t_0(\tau)=\tau$, $z_0(\tau)=ia+\tau$, $x_0(\tau)=y_0(\tau)=0$. In null coordinates this yields

$$\sqrt{2}u_0=z_0+\tau=ia+2\tau, \quad \sqrt{2}v_0=z_0-\tau=ia, \quad \xi_0=\bar{\xi}_0=0,$$

so that $\dot{u}_0=\sqrt{2}$, $\dot{v}_0=0$, $\dot{\xi}_0=\dot{\bar{\xi}}_0=0$, and

$$u-u_0=(z-ia+t-2\tau)/\sqrt{2},$$

$$v-v_0=(z-ia-t)/\sqrt{2},$$

$$\xi-\xi_0=\xi, \quad \bar{\xi}-\bar{\xi}_0=\bar{\xi}.$$

Formula (15) implies that the coefficients A, B, C are given by

$$A=0, \quad B=t-z+ia, \quad C=x+iy.$$

As a result the function F acquires the form

$$F=x+iy-Y(z-ia-t), \quad (29)$$

and the solution of the equation $F=0$ is

$$Y=(x+iy)/(z-ia-t), \quad (30)$$

so that

$$\tilde{r}=-dF/dY=z-ia-t, \quad (31)$$

and, therefore, the metric has no singularities [there is no real solution to the system of Eqs. (8)]. On the other hand, setting $a=0$, we obtain the case of spinless particle, and a moving singular plane is placed at $z=t$. Therefore, there is no smooth limit as $a \rightarrow 0$.

Example II. A spinning particle moves with the speed of light in the positive direction of the x axis, orthogonal to the three vector \vec{d} which defines the direction and the value of the angular momentum $\vec{J}=m(0,0,a)$, $a=|\vec{d}|$. We have the complex world line $t_0(\tau)=x_0(\tau)=\tau$, $y_0(\tau)=0$, $z_0(\tau)=ia$. Correspondingly, the world line in null coordinates is

$$\sqrt{2}u_0=ia+\tau, \quad \sqrt{2}v_0=ia-\tau, \quad \sqrt{2}\xi_0=\tau, \quad \sqrt{2}\bar{\xi}_0=\tau,$$

and the velocities are $\sqrt{2}\dot{u}_0=1$, $\sqrt{2}\dot{v}_0=-1$, $\sqrt{2}\dot{\xi}_0=1$, $\sqrt{2}\dot{\bar{\xi}}_0=1$. We have, therefore,

$$\sqrt{2}(u-u_0)=z+t-ia-\tau, \quad \sqrt{2}(v-v_0)=z-ia-t+\tau,$$

$$\sqrt{2}(\xi-\xi_0)=x+iy-\tau, \quad \sqrt{2}(\bar{\xi}-\bar{\xi}_0)=x-iy-\tau,$$

and the coefficients A, B, C take the form

$$A=(-x+iy-z+t+ia)/2, \quad B=ia+iy-z; \quad (32)$$

$$C=(x+iy-z-t+ia)/2.$$

The function $Y(x)$ takes the form

$$Y=(x-t-z+iy+ia)/(x-t+z-iy-ia). \quad (33)$$

The function $\tilde{r} \equiv PZ^{-1}$ takes the form

$$PZ^{-1}=-dF/dY=x-t. \quad (34)$$

This solution is, therefore, singular: there is a moving singular plane placed at $x=t$.

Example III. The absence of a smooth limit in the first example may be better understood considering the general case in which the value of the velocity is arbitrary as well as its direction with respect to the angular momentum. Without loss of generality, we can consider the boost performed with a parameter α in the z direction ($\alpha=v_z/c$), and a parameter β in the x direction ($\beta=v_x/c$), while the angular momentum is defined by $\vec{d}=(0,0,a)$. Denoting $w^2=\alpha^2+\beta^2$ the following general formula for the coordinate relations can be obtained:

$$(x-\beta t)\sqrt{1-\alpha^2}+iy\sqrt{1-w^2}=(r+ia\sqrt{1-\beta^2})e^{i\phi}\sin\theta, \quad (35)$$

$$z-\alpha t=-r\cos\theta/\sqrt{1-\beta^2}.$$

The singular region $r=0, \cos\theta=0$ is placed on the plane $z=\alpha t$ and is described by

$$(1-\alpha^2)(x-\beta t)^2+(1-w^2)y^2=a^2(1-\beta^2). \quad (36)$$

Let us first consider the cases corresponding to the two examples above. If $\beta=0$ (the boost in the direction of the angular momentum) the singularity is a ring of radius $a_0=a/\sqrt{1-\alpha^2}$ located on the moving plane $z=-\alpha t$. In this case, the coordinate relations are the following:

$$x+iy=(r+ia)e^{i\phi}\sin\theta/\sqrt{1-\alpha^2}, \quad (37)$$

$$z-\alpha t=r\cos\theta. \quad (38)$$

The radius of the ring grows as α increases, and for $\alpha \rightarrow 1$ the singularity goes at infinity. At first sight, this result looks strange. However, it is easy to check that the radius of ring is a_0 , the rest value of a . The position of singular region becomes a ring of constant radius a_0 if we keep $a_0=\text{const}$.

However, keeping $a=\text{const} \neq 0$ and taking the limit $\alpha=1$ we obtain that the singular region is going to infinity, so there will not be a singularity in the finite region, in agreement with the results of Example I. We will discuss this situation later in connection with the problem of renormalization of parameters.

In the case $\alpha=0$, the boost is performed with a speed orthogonal to the direction of angular momentum. The singular region is a moving ring oblate in the x direction with a Lorentz factor $\sqrt{1-\beta^2}$ and located on the plane $z=0$. In this case the coordinate relations are

$$(x-\beta t)/\sqrt{1-\beta^2}+iy=(r/\sqrt{1-\beta^2}+ia)e^{i\phi}\sin\theta, \quad (39)$$

$$z=r\cos\theta/\sqrt{1-\beta^2}. \quad (40)$$

As in the previous example, the limit $\beta \rightarrow 1$ is not smooth since the singular region is a line parallel to the y axis placed at $x=t$, $z=0$.

In the general case described by Eq. (37) the singularity is a moving ring on the $z=\alpha t$ plane, deformed in the x direction by a factor $\sqrt{(1-\beta^2)/(1-\alpha^2)}$ and in the y direction by a factor $\sqrt{(1-\beta^2)/(1-w^2)}$. The ultrarelativistic limit corresponds to $w=1$ and the singular region is a couple of straight lines parallel to the y axis. Therefore, we can conclude that the nonsmoothness and the noncommutativeness of the limiting procedure is a general feature of the boosted Kerr solutions. Another peculiarity, which can be seen by the analysis of the above examples is a nontrivial coordinate dependence of the function Y which forms the principal null congruence. As a consequence the congruence itself acquires a nontrivial coordinate dependence and a nonzero expansion θ and twist ω . This property is conserved even in the ultrarelativistic limit. For instance, in the case of Example I the expansion and twist of the congruence are defined by $Z=\theta+i\omega$ [8] and are given by $Z/P=-(dF/dY)^{-1}=(z-t+ia)/[(z-t)^2+a^2]$. One sees that there is no singularity in this case, and expansion tends to zero only at the $z=t$ plane where the twist takes the constant value $1/a$.

V. CONCLUDING REMARKS

There are three different physical situations which should be described by the boosted Kerr solution. The first is connected with the original Aichelburg-Sexl problem, namely the description of the gravitational field of lightlike particles with or without spin. For this case the problem of ‘‘renormalization’’ of the parameters of the solution has been discussed by many authors [5,7]. Indeed the lightlike particle must have an infinitely small rest mass in such a way that the boosted momentum will be finite. Similarly, renormalization of other parameters, such as charge and angular momentum, has been discussed [6,7], and there is not yet a unique agreement concerning this renormalization procedure. The above considerations on the behavior of the singular ring under the boost in the orthogonal direction suggest, however, that the physical most satisfactory way to perform the renormalization in this case should be to keep $J=ma=\text{const}$. In fact, in this way the projection of the angular momentum on the direction of the boost is invariant with respect to the value of the boost. One can come to this conclusion also by considering spinning particles in a quantum context, since the projection of the spin on the boost direction is the helicity which indeed must be considered as a constant. In terms of the rest values a_0 and m_0 we have to set $J=ma=m_0a_0=\text{const}$. As far as $m=m_0/\sqrt{1-v^2}$ by the boost, this yields $a_0=a/\sqrt{1-v^2}$.

Therefore, for finite values of a the rest value a_0 and the location of the singularity *tend to infinity* in the ultrarelativistic limit, explaining the results of Example III, and clarifying the absence of the singularity in the ultrarelativistic limit of Example I.

One should note that, keeping the value $a_0=\text{const}$ during the limit, one enforces a fixed size of the singular ring, so that putting in the limit $m_0=0$, one obtains in fact a limiting

twist-free solution with $J=m_0a_0 \rightarrow 0$. This corresponds to the known results on pp -wave limiting metrics [5–7,18,19] with a finite size of the singular ring and, correspondingly, vanishing total angular momentum [7].

These arguments are not valid for Example II since the corresponding projection of angular momentum is initially zero, and we have here a twist-free solution with a finite location of singularity.

The second application, which has led to the recently renewed interest in this problem, consists in modeling the gravitational field of elementary particles with finite rest mass under the boost. A specific feature of this case is that the rest mass m_0 as well as the projection of the angular momentum $J=m_0a_0=ma$ have to be kept constant. This leads to $m_0=\text{const}$, and consequently we obtain a finite position of singularity which is determined by the value of a_0 .¹ However, the parameter of the solution a has to be scaled by the boost as $a=a_0\sqrt{1-v^2}$.

In both problems described above one deals with naked singularities rather than black holes, since the values of mass, spin, and charge of elementary particles typically correspond to this kind of solution.

The third physical problem is connected with astrophysical applications [1] of the boosted black-hole solutions. In this case also the behavior of the horizon and of the ergosphere under the boost are of interest. A simple analysis using the above suggested coordinates shows that the horizon as well as the ergosphere are simply given by the known formulas for the Kerr case where m must be the relativistic mass parameter.

The method proposed here allows us to describe in *explicit* form the metric and the behavior of the singular region of the Kerr solution under arbitrary boost and with arbitrary orientations of the angular momentum. In particular, we have shown that the Kerr theorem automatically allows us to obtain the exact form of the boosted solution in an asymptotically flat coordinate system and the equations describing the singularities in these coordinates. The ultrarelativistic limit is a singular point of the Lorentz transformations, and we have obtained a quite general picture of the nonsmoothness and noncommutativeness of the limits $a \rightarrow 0$, $v \rightarrow 1$, and $r \rightarrow 0$. The method shows that lightlike limits of the Kerr geometry exist which belong to the DKS class and have twisting principal null congruences and nonzero total angular momentum J .

We want to stress that the present work does not give a definitive answer to the ultrarelativistic limit of the Kerr solution. Our aim here was rather to provide a unifying framework to be used to investigate various limits of the Kerr geometry. All such limits are mathematically valid, but correspond to different physical situations and therefore to different behaviors of the physical quantities like mass and an-

¹One sees that singular ring is not subjected to Lorentz contraction in this case since it lies in the plane orthogonal to the boost direction.

gular momentum, as well as to different locations and behavior of the singularities.

The present results can be easily extended also to the boost of the Kerr-Newman solution and of the Kerr-Sen [20] solution, generalizing the Kerr solution to low-energy string theory. It was shown in [21] that one of the principal null congruences retains its properties to be geodesic and shear free, and that the Kerr theorem remains valid for the Kerr-Sen solution too [22].

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APPENDIX

Let e^a be a null tetrad and define the Ricci rotation coefficients as

$$\Gamma_{bc}^a = -e_{\mu;\nu}^a e_b^\mu e_c^\nu.$$

The principal null congruence has the e^3 direction as the tangent. It will be geodesic if and only if $\Gamma_{424}=0$, and shear free if and only if $\Gamma_{422}=0$ (the corresponding complex conjugate terms are $\Gamma_{414}=0$ and $\Gamma_{411}=0$). The null tetrad e_a^μ can be completed as follows:

$$\begin{aligned} e^1 &= d\zeta - Y dv, \\ e^2 &= d\bar{\zeta} - \bar{Y} dv, \\ e^4 &= dv - h e^3. \end{aligned} \tag{A1}$$

The inverse tetrad has the form

$$\begin{aligned} \partial_1 &= \partial_\zeta - \bar{Y} \partial_u, \\ \partial_2 &= \partial_{\bar{\zeta}} - Y \partial_u, \\ \partial_3 &= \partial_u - h \partial_4, \end{aligned} \tag{A2}$$

$$\partial_4 = \partial_v + Y \partial_\zeta + \bar{Y} \partial_{\bar{\zeta}} - Y \bar{Y} \partial_u.$$

It was shown in [8] that

$$\Gamma_{42} = \Gamma_{42a} e^a = -dY - h Y_{,4} e^4.$$

The congruence e^3 is geodesic if $\Gamma_{424} = -Y_{,4}(1-h) = 0$, and is shear free if $\Gamma_{422} = -Y_{,2} = 0$. Thus, the function Y with the conditions

$$Y_{,2} = Y_{,4} = 0,$$

defines a shear free and geodesic congruence.

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