PHYSICAL REVIEW D, VOLUME 61, 044009

Holography based on noncommutative geometry and the AdS/CFT correspondence

Zhe Chang
Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, Föhringer Ring 6, D-80805 München, Germany
and Institute of High Energy Physics, Academia Sinica, P.O. Box 918(4), 100039 Beijing, China

(Received 11 August 1999; published 25 January 2000)

Exponential regularization of orthogonal and anti–de Sitter (AdS) space is presented based on noncommutative geometry. We show that an adequately adopted noncommutative deformation of geometry makes the holography of higher dimensional quantum theory of gravity and lower dimensional theory possible. We present detailed calculations for the counting of the observable degrees of freedom of a quantum system of gravity in the bulk of noncommutative space $\mathrm{SO}_q(3)$ and the classical limit of its boundary surface S^2. Taking the noncommutivity effect into account, we get the desired form of entropy for our world, which is consistent with the physical phenomena associated with gravitational collapse. Conformally invariant symmetry is obtained for the equivalent theory of the quantum gravity living on the classical limit of the boundary of the noncommutative AdS space. This is the basis of the AdS/CFT correspondence in string theory.

PACS number(s): 04.50.+h, 11.25.Hf

I. INTRODUCTION

By making use of the idea that a quantum system of gravity may possess far less degrees of freedom than usually expected for a $(3+1)$-dimensional field theory [1,2], it was suggested recently that there is an AdS conformal field theory (CFT) correspondence [3–5]: string theory (M theory) on a background of the form $\mathrm{AdS}_d\times \mathrm{M}_{D-d}$ is dual to a conformal field theory living on the spacetime boundary. Here AdS_d is the d-dimensional anti–de Sitter (AdS) space, and M_{D-d} is a compact space of dimension $D-d$ with $D=10$ for string theory ($D=11$ for M theory). A thoroughly studied example is the correspondence between the type-IIB string theory on the background $\mathrm{AdS}_5\times \mathrm{S}^5$ and the four-dimensional $\mathcal{N}=4$ SU(N) super Yang-Mills theory [3]. Here the correspondence goes as follows: for each field Φ_i there is a corresponding local operator $\mathcal{O}_i$ in the conformal field theory and the precise relation between string theory in the bulk and field theory on the boundary is

$$Z_{\mathrm{eff}}(\Phi_i)=e^{iS_{\mathrm{eff}}(\Phi_i)}=\langle Te^{i\int_{\mathcal{B}}\Phi_{bi}\mathcal{O}^i}\rangle, \tag{1}$$

where S_{eff} is the effective action in the bulk and Φ_{bi} is the field Φ_i restricted to the boundary. In the large-N limit, the string theory is weakly coupled and supergravity is a good approximation to it. By using this correspondence we have a precise recipe expressing correlation functions of the $\mathcal{N}=4$ super Yang-Mills theory in four dimensions in terms of the effective action of tree approximation to supergravity in the bulk.

At first sight, it seems very strange that quantum theories in different spacetime dimensions could ever be equivalent in any sense. The key to understanding this equivalence is the fact that the theory in a higher dimension is always a quantum theory of gravity. For these theories 't Hooft [1] has introduced the concepts of holography based on the phenomenological study of the black hole theory. It is believed that these concepts are generic properties of the quantum system of gravity [1,2]. The AdS/CFT correspondence is just an example of the realization of the holography on the quantum theory of gravity. From the point of view of general relativity, gravity is nothing but spacetime geometry. To check the Maldacena conjecture [3], one has to start from the geometric description for the quantum theory of gravity. Because holography is a generic property of the quantum gravity, in principle, it should be deduced naturally from geometric properties of spacetime. 't Hooft showed that simple regularization of spacetime cannot give a correct account of observable degrees of freedom for quantum gravity. On the other hand, the understanding of physics at the Planck scale also indicates that the small scale structure of spacetime might not be adequately described by classical continuum geometry. Thus, new geometry should be appeared in quantum gravity. It has for a long time been suspected that the noncommutative spacetime might be a realistic picture of how spacetime behaves near the Planck scale [6,7]. Strong quantum fluctuations of gravity may make points fuzzy (in space). The noncommutative geometry [8,9] is a promising candidate for the quantum theory of gravity. We think that the holography can be obtained explicitly from the noncommutative geometry picture of quantum gravity and show the AdS/CFT correspondence by demonstrating conformally invariant symmetry on the boundary surface of noncommutative AdS space.

In this paper, we present a kind of special regularization with exponentially increasing spacetime cutoff for both orthogonal and AdS space based on noncommutative geometry. We argue that the same minimal cutoff for any geometry is the Planck scale l_p. The most direct and obvious physical cutoff is from the formation of microscopic black holes when enough energy are accumulated into a small region [1]. We show that an adequately adopted noncommutative deformation of geometry makes the holography of a higher dimensional quantum system of gravity and lower dimensional theory possible. As an example, detailed calculations are carried out for the counting of observable degrees of freedom of quantum gravity in the bulk of noncommutative space $\mathrm{SO}_q(3)$ and the classical limit of its boundary surface S^2. Results show that a very small (may be $<10^{-15}$) displacement of the noncommutative deformation parameter from its classical value 1 reduces sharply the entropy of the quantum system of gravity. The desired entropy expression $S=4\pi M^2+C$ of the universe can be deduced naturally. Conformally invariant symmetry is obtained for the equivalent theory of quantum gravity. This is the basis of the AdS/CFT correspondence in nonperturbative string theory and M theory.

This paper is organized as follows. In Sec. II, we discuss the noncommutative orthogonal space $\mathrm{SO}_q(3)$. The algebra formed by the coordinates and derivatives is decoupled into three independent subalgebras by the introduction of a new set of variables. Quantum coherent states are constructed as a reference for investigating representations of the spacetime algebra. The structure of the Hilbert space shows clearly that the noncommutative spacetime is discretely latticed with exponentially increasing space distances. The noncommutative deformation parameter is determined by an algebraic equation. The noncommutative space $\mathrm{SO}_q(3)$ has the same entropy or observable degrees of freedom as the classical S^2 surface. Section III is devoted to the study of noncommutative AdS_{2n} space. A conjugate operation is set up for the noncommutative AdS space. This conjugation has an induced counterpart for the set of decoupled coordinates and derivatives. The Hilbert space of the noncommutative AdS_{2n} space is also constructed based on quantum coherent states. The discrete lattice structure of the noncommutative AdS_{2n} with exponentially increasing space distances is obtained. Holography makes the quantum system of gravity on the noncommutative AdS_{2n} space equivalent to the conformally invariant quantum theory living on the classical limit of its boundary surface. This is crucial for the AdS/CFT correspondence of string theory and M theory. By almost the same procedures, the properties of the noncommutative AdS_{2n-1} space are shown in Sec. IV. Some concluding remarks are given in Sec. V.

II. NONCOMMUTATIVE GEOMETRY AND HOLOGRAPHY

We begin by discussing the quantum space $\mathrm{SO}_q(3)$ with coordinates $x^i(i=-,0,+)$. The commutation relations [10] among coordinates x^i are

$$\begin{aligned} x^-x^0&=qx^0x^-, \quad x^0x^+=qx^+x^0, \\ x^+x^- - x^-x^+&=(q^{1/2}-q^{-1/2})x^0x^0. \end{aligned} \tag{2}$$

The algebra formed by derivatives is of the form

$$\begin{aligned} \partial_-\partial_0&=q^{-1}\partial_0\partial_-, \quad \partial_0\partial_+=q^{-1}\partial_+\partial_0, \\ \partial_-\partial_+ - \partial_+\partial_-&=(q^{1/2}-q^{-1/2})\partial_0\partial_0. \end{aligned} \tag{3}$$

The action of the derivatives on the coordinates is standard [11] and is given as follows:

$$\begin{aligned} &\partial_-x^-=1+q^2x^-\partial_-+\lambda qx^0\partial_0+\lambda(q-1)x^+\partial_+, \qquad (4)\\ &\partial_-x^0=qx^0\partial_- - q^{1/2}\lambda x^+\partial_0, \quad \partial_-x^+=x^+\partial_-, \\ &\partial_0x^-=qx^-\partial_0-q^{1/2}\lambda x^0\partial_+, \\ &\partial_0x^0=1+qx^0\partial_0+q\lambda x^+\partial_+, \quad \partial_0x^+=qx^+\partial_0, \\ &\partial_+x^-=x^-\partial_+, \quad \partial_+x^0=qx^0\partial_+, \quad \partial_+x^+=1+q^2x^+\partial_+. \end{aligned}$$

It is convenient to introduce two dilatation operators [11]

$$\begin{aligned} \mu_+&=1+q\lambda x^+\partial_+, \\ \Lambda&=1+q\lambda\sum_{j=0,\pm} x^j\partial_j+q^3\lambda^2\left(q^{-1/2}x^-x^+ +\frac{q}{1+q}x^0x^0\right) \\ &\quad\times\left(q^{-1/2}\partial_+\partial_- +\frac{q}{1+q}\partial_0\partial_0\right). \end{aligned} \tag{5}$$

These dilatation operators obey the following relations:

$$\begin{aligned} \mu_+x^+&=q^2x^+\mu_+, \quad \mu_+\partial_+=q^{-2}\partial_+\mu_+, \\ \Lambda x^k&=q^2x^k\Lambda, \quad \Lambda\partial_k=q^{-2}\partial_k\Lambda, \quad \text{for } k=0,\pm. \end{aligned} \tag{6}$$

The real form $\mathrm{SO}_q(3,R)$ [denoted as $\mathrm{SO}_q(3)$ whenever no confusion arises] of the noncommutative space $\mathrm{SO}_q(3)$ is obtained by a consistent conjugation:

$$\begin{aligned} \overline{x^i}&=C_{ji}x^j, \\ \overline{\partial_i}&=-q^{-2}C_{ij}\Lambda^{-1}\left(q^{-1/2}[\partial_+\partial_-,x^j]+\frac{q}{1+q}[\partial_0\partial_0,x^j]\right), \end{aligned} \tag{7}$$

where the metric C_{ij} is of the form

$$C=\begin{pmatrix} & & q^{-1/2} \\ & 1 & \\ q^{1/2} & & \end{pmatrix}.$$

We remark that throughout this paper we limit our analysis to the case of real q.

By making use of the dilatation operators μ_+ and Λ, we introduce a new set of coordinates and derivatives

$$\begin{aligned} \mathcal{X}^-&=\Lambda^{-1/2}\mu_+^{-1/2}\left[x^-+q^{3/2}\lambda\left(q^{-1/2}x^-x^+ +\frac{q}{1+q}x^0x^0\right)\partial_+\right], \\ \mathcal{D}_-&=q^{-1}\Lambda^{-1/2}\mu_+^{-1/2}\Bigg[\partial_-+q^{3/2}\lambda \\ &\quad\times\left(q^{-1/2}\partial_+\partial_- +\frac{q}{1+q}\partial_0\partial_0\right)x^+\Bigg], \\ \mathcal{X}^0&=\mu_+^{-1/2}x^0, \quad \mathcal{D}_0=\mu_+^{-1/2}\partial_0, \\ \mathcal{X}^+&=x^+, \quad \mathcal{D}_+=\partial_+. \end{aligned} \tag{8}$$

In terms of these new variables, the commutation relations among coordinates and derivatives of the noncommutative space $\mathrm{SO}_q(3)$ are changed to

$$\mathcal{D}_-\mathcal{X}^- - \mathcal{X}^-\mathcal{D}_- = \mu_-^{-1}, \quad \mu_-\mathcal{X}^- = q^2\mathcal{X}^-\mu_-,$$

$$\mu_-\mathcal{D}_- = q^{-2}\mathcal{D}_-\mu_-, \quad \mu_-^{-1} \equiv 1 + (q^{-2}-1)\mathcal{X}^-\mathcal{D}_-,$$

$$\mathcal{D}_0\mathcal{X}^0 - \mathcal{X}^0\mathcal{D}_0 = \mu_0^{1/2}, \quad \mu_0\mathcal{X}^0 = q^2\mathcal{X}^0\mu_0,$$

$$\mu_0\mathcal{D}_0 = q^{-2}\mathcal{D}_0\mu_0, \quad \mu_0^{1/2} \equiv 1 + (q-1)\mathcal{X}^0\mathcal{D}_0,$$

$$\mathcal{D}_+\mathcal{X}^+ - \mathcal{X}^+\mathcal{D}_+ = \mu_+, \quad \mu_+\mathcal{X}^+ = q^2\mathcal{X}^+\mu_+,$$

$$\mu_+\mathcal{D}_+ = q^{-2}\mathcal{D}_+\mu_+, \quad \mu_+^{1/2} = 1 + (q^2-1)\mathcal{X}^+\mathcal{D}_+,$$

$$[\mathcal{X}^i,\mathcal{X}^j] = 0, \quad [\mathcal{D}_i,\mathcal{D}_j] = 0, \quad [\mathcal{D}_i,\mathcal{X}_j] = 0,$$

$$[\mu_i,\mathcal{X}^j] = 0, \quad [\mu_i,\mathcal{D}_j] = 0, \quad \text{for } i \neq j. \tag{9}$$

In terms of the independent operators $\mathcal{X}^j$ and $\mathcal{D}_j$ the noncommutative surface S_q^2 is of the form

$$\frac{q^{-1}}{1+q}\mathcal{X}^0\mathcal{X}^0 + q^{-1/2}\Lambda^{1/2}\mu_+^{-1/2}\mathcal{X}^+\mathcal{X}^- = R^2. \tag{10}$$

In the limit $q \to 1$, S_q^2 reduces to the familiar S^2 surface with radius R.

The conjugate operation on $\mathcal{X}^j$ and $\mathcal{D}_j$ is induced by the operation on x^j and ∂_j [Eq. (7)],

$$\overline{\mathcal{X}^-} = \left[\overline{x^-} + q^{3/2}\lambda\overline{\partial_+}\left(q^{-1/2}\overline{x^+}\,\overline{x^-} + \frac{q}{1+q}\overline{x^0}\,\overline{x^0}\right)\right]$$
$$\times\overline{\mu_+}^{-1/2}\overline{\Lambda}^{-1/2},$$

$$\overline{\mathcal{D}_-} = q^{-1}\left[\overline{\partial_-} + q^{3/2}\lambda\overline{x^+}\left(q^{-1/2}\overline{\partial_-}\,\overline{\partial_+} - \frac{q}{1+q}\overline{\partial_0}\,\overline{\partial_0}\right)\right]$$
$$\times\overline{\mu_+}^{-1/2}\overline{\Lambda}^{-1/2},$$

$$\overline{\mathcal{X}^0} = \overline{x^0}\,\overline{\mu_+}^{-1/2}, \quad \overline{\mathcal{D}_0} = \overline{\partial_0}\,\overline{\mu_+}^{-1/2},$$

$$\overline{\mathcal{X}^+} = \overline{x^+}, \quad \overline{\mathcal{D}_+} = \overline{\partial_+}. \tag{11}$$

Thus, we conclude that the quantum Heisenberg-Weyl algebra corresponding to the noncommutative space $\mathrm{SO}_q(3)$ can be decoupled into three independent subalgebras. One can then investigate properties of the noncommutative space by constructing Hilbert spaces of the three quantum subalgebras.

For the quantum algebra $\mathcal{A}_-$

$$\mathcal{D}_-\mathcal{X}^- - \mathcal{X}^-\mathcal{D}_- = \mu_-^{-1}, \quad \mu_-\mathcal{X}^- = q^2\mathcal{X}^-\mu_-,$$

$$\mu_-\mathcal{D}_- = q^{-2}\mathcal{D}_-\mu_-, \quad \mu_-^{-1} \equiv 1 + (q^{-2}-1)\mathcal{X}^-\mathcal{D}_-, \tag{12}$$

we construct a quantum coherent state $|z\rangle_-$ as follows:

$$|z\rangle_- = \exp_{q^2}\left(-\frac{1}{2}|q^{-2}z|\right)\sum_{m=0}^{\infty}\frac{(-q^{-2}z)^m}{[m]_{q^2}!}(\mathcal{D}_-)^m|0\rangle_-, \tag{13}$$

where we have used the notation of q exponential:

$$\exp_q(x) \equiv \sum_{n=0}^{\infty}\frac{x^n}{[n]_q!},$$

$$[n]_q! = [n]_q[n-1]_q\cdots[1]_q, \quad [n]_q = \frac{q^n-1}{q-1}.$$

The reference state $|0\rangle_-$ is chosen such that $\mathcal{X}^-|0\rangle_- = 0$.

As in the classical case, the coordinate $\mathcal{X}^-$ is diagonal in the quantum coherent state basis

$$\mathcal{X}^-|z\rangle_- = z|z\rangle_-. \tag{14}$$

The value of z may be interpreted as position of an indispersive wave pocket [12]. Here we should notice that z can be of any complex number because we are working on a general quantum orthogonal space. The complex values of the quantum coherent state parameter is also consistent with the conjugate operation on the noncommutative space.

Denoting the quantum coherent state as

$$|0,z\rangle_- \equiv |z\rangle_-,$$

we can construct a representation for the quantum algebra $\mathcal{A}_-$ based on the quantum coherent state as the following:

$$\mathcal{X}^-|n,z\rangle_- = q^{2n}z|n,z\rangle_-,$$

$$\mathcal{D}_-|n,z\rangle_- = -q^{-1-2n}\lambda^{-1}z^{-1}|n+1,z\rangle_-,$$

$$\mu_-|n,z\rangle_- = |n-1,z\rangle_-. \tag{15}$$

From the Hilbert space representation of the quantum algebra $\mathcal{A}_-$, the coordinates of the noncommutative orthogonal space is discretely latticed with exponentially increasing space distances. In fact, this is in agreement with the discrete difference representation of quantum derivatives:

$$\mathcal{D}f(\mathcal{X}) = \frac{f(q^2\mathcal{X}) - f(\mathcal{X})}{(q^2-1)\mathcal{X}}.$$

Similarly, we can also construct the reference states $|0,z\rangle_0$ and $|0,z\rangle_+$ as

$$|0,z\rangle_0 = \exp_{q^{-1}}\left(-\frac{1}{2}|qz|\right)\sum_{m=0}^{\infty}\frac{(-qz)^m}{[m]_{q^{-1}}!}(\mathcal{D}_0)^m|0\rangle_0,$$

$$\mathcal{X}^0|0\rangle_0 = 0,$$

$$|0,z\rangle_+ = \exp_{q^{-2}}\left(-\frac{1}{2}|q^2z|\right)\sum_{m=0}^{\infty}\frac{(-q^2z)^m}{[m]_{q^{-2}}!}(\mathcal{D}_+)^m|0\rangle_+,$$

$$\mathcal{X}^+|0\rangle_+ = 0. \tag{16}$$

By using these states the corresponding representations of the quantum algebras $\mathcal{A}_0$

$$\mathcal{D}_0\mathcal{X}^0-\mathcal{X}^0\mathcal{D}_0=\mu_0^{1/2}, \quad \mu_0\mathcal{X}^0=q^2\mathcal{X}^0\mu_0,$$

$$\mu_0\mathcal{D}_0=q^{-2}\mathcal{D}_0\mu_0, \quad \mu_0^{1/2}\equiv 1+(q-1)\mathcal{X}^0\mathcal{D}_0, \tag{17}$$

and $\mathcal{A}_+$

$$\mathcal{D}_+\mathcal{X}^+-\mathcal{X}^+\mathcal{D}_+=\mu_+, \quad \mu_+\mathcal{X}^+=q^2\mathcal{X}^+\mu_+,$$

$$\mu_+\mathcal{D}_+=q^{-2}\mathcal{D}_+\mu_+, \quad \mu_+\equiv 1+(q^2-1)\mathcal{X}^+\mathcal{D}_+, \tag{18}$$

are given as follows:

$$\mathcal{X}^0|n,z\rangle_0=q^n z|n,z\rangle_0,$$

$$\mathcal{D}_0|n,z\rangle_0=\frac{q^{1-n}}{q-1}z^{-1}|n-1,z\rangle_0,$$

$$\mu_0|n,z\rangle_0=|n-1,z\rangle_0, \tag{19}$$

for the algebra $\mathcal{A}_0$, and

$$\mathcal{X}^+|n,z\rangle_+=q^{2n}z|n,z\rangle_+,$$

$$\mathcal{D}_+|n,z\rangle_+=q^{1-2n}\lambda^{-1}z^{-1}|n-1,z\rangle_+,$$

$$\mu_+|n,z\rangle_+=|n-1,z\rangle_+, \tag{20}$$

for the algebra $\mathcal{A}_+$. It has been strongly argued [1] that the most direct and obvious physical cutoff of spacetime is from the formation of microscopic black holes, as soon as too much energy would be accumulated into too small a region. Thus, from a physical point of view, the black holes should provide a natural cutoff all by themselves. The cutoff distance scale is the Planck scale. Because of this origin of spacetime cutoff, for any geometry we are working on, there should be a universal minimal cutoff l_p. For the classical geometry, the spacetime regularization is equal distance latticed and there is one degree of freedom per Planck area. However, from the above discussion of noncommutative geometry, the spacetime is discretely latticed with exponentially increasing space distances. Thus, much less information can be stored in the noncommutative geometry. In fact, this may be the origin of the holography for the quantum system of gravity.

If the assumed minimal cutoff for the noncommutative space $SO_q(3)$ with radius R is the Planck scale l_p, it is not difficult to count the degrees of freedom $\mathcal{N}_{\text{bulk}}$,

$$\mathcal{N}_{\text{bulk}}\approx\sum_{i=1}^{N}\frac{(q^{2i}l_p)^2}{(q^{2i}l_p-q^{2(i-1)}l_p)^2}, \quad q^{2N}=R,$$

$$=\frac{q^4\ln\left(\frac{R}{l_p}\right)}{2(q^2-1)^2\ln q}. \tag{21}$$

By taking the deformation parameter of the noncommutative space q to be determined by the following algebraic equation:

$$q^{-4}(q^2-1)^2\ln q=\frac{\ln\left(\frac{R}{l_p}\right)}{8\pi\left(\frac{R}{l_p}\right)^2}, \tag{22}$$

we can check that $\mathcal{N}_{\text{bulk}}$ is equal to the degrees of freedom on the classical limit of its boundary surface with radius R, $\mathcal{N}_{\text{boundary}}$,

$$\mathcal{N}_{\text{boundary}}=\frac{4\pi R^2}{l_p^2}. \tag{23}$$

Then one can write down the entropy of our world at the Planck scale as

$$S=4\pi M^2+C, \tag{24}$$

where M is the mass of the world (black hole) in natural units and C is a constant entropy which is not determined. This is exactly what was called the holography or dimension reduction in the quantum theory of gravity by 't Hooft. At the Planck scale, our world is not $3+1$ dimensional. Rather, the observable degrees of freedom can best be described as if they were Boolean variables defined on a two-dimensional lattice, evolving with time. It is now clear that the exact meaning of the holography can be interpreted as follows: the quantum theory of gravity in higher dimensional noncommutative space is equivalent to the theory living on the classical limit of spacetime boundary. This gives a reasonable picture for the 't Hooft's holography.

III. NONCOMMUTATIVE AdS$_{2n}$ SPACE AND EXPONENTIAL REGULARIZATION

The noncommutative AdS$_{2n}$ space is defined as the $2n$-dimensional noncommutative real hyperboloid embedded in a $(2n+1)$-dimensional space with coordinates x^i ($i=-n,-n+1,\dots,-1,0,1,\dots,n$),

$$\frac{1}{1+q^{2n-1}}C_{ij}x^ix^j=-\frac{1}{a^2},$$

$$\overline{x^i}=C_{ji}M_{jk}x^k, \tag{25}$$

where $\rho_{-i}=i-\frac{1}{2}$, $\rho_0=0$, $\rho_i=-i+\frac{1}{2}$, and the metric $C_{ij}=q^{-\rho_i}\delta_{i,-j}$ and

$$M=\begin{pmatrix} 1 & & & & & & \\ & -1 & & & & & \\ & & -1 & & & & \\ & & & \cdots & & & \\ & & & & -1 & & \\ & & & & & -1 & \\ & \multicolumn{5}{c}{\underbrace{\hspace{10em}}_{2n-1}} & \\ & & & & & & 1\end{pmatrix}.$$

It is easy to check that, in the limit $q\rightarrow 1$, the noncommutative AdS$_{2n}$ space reduces to the familiar AdS$_{2n}$ space

$$\eta_{ab}x^a x^b=-\frac{1}{a^2}$$

in R^{2n+1} with Cartesian coordinates x^a. Here

$$\eta_{ab}=\mathrm{diag}(-1,\ \underbrace{1,\ \cdots,\ 1,}_{2n-1}\ -1)$$

is the flat spacetime metric.

For the noncommutative AdS$_{2n}$ space, the commutation relations (in components) among coordinates are given as follows:

$$x^i x^j = q x^j x^i \quad \text{for } i<j \quad \text{and } i\neq -j,$$

$$\begin{aligned} x^i x^{-i} &= x^{-i}x^i + \lambda q^{i-3/2}L_{i-1} \\ &= q^{-2}x^{-i}x^i+\lambda q^{i-3/2}L_i \quad \text{for } i>0, \end{aligned} \tag{26}$$

where we have used the notation for intermediate lengths

$$L_i=\sum_{k=1}^{i} q^{\rho_k}x^{-k}x^k+\frac{q}{1+q}x^0x^0.$$

By making use of the intermediate Laplacians

$$\Delta_i=\sum_{k=1}^{i} q^{\rho_k}\partial_k\partial_{-k}+\frac{q}{1+q}\partial_0\partial_0,$$

the algebra satisfied by the derivatives can be written compactly as

$$\partial_i\partial_j=q^{-1}\partial_j\partial_i \quad \text{for } i<j \quad \text{and } i\neq -j, \tag{27}$$

$$\begin{aligned} \partial_{-i}\partial_i &= \partial_i\partial_{-i}+\lambda q^{i-3/2}\Delta_{i-1} \\ &= q^{-2}\partial_i\partial_{-i}+\lambda q^{i-3/2}\Delta_i \quad \text{for } i>0. \end{aligned}$$

The commutation relations among the coordinates and derivatives are as follows:

$$\partial_{-i}x^i=x^i\partial_{-i} \quad \text{for } i\neq 0,$$

$$\partial_i x^j = q x^j\partial_i \quad \text{for } j>-i, \text{ and } j\neq i,$$

$$\partial_i x^j = q x^j\partial_i - q\lambda q^{-\rho_j-\rho_k}x^{-i}\partial_{-j}$$

$$\text{for } j<-i, \quad \text{and } i\neq j,$$

$$\partial_j x^j = 1+q^2x^j\partial_j+q\lambda\sum_{k>j}x^k\partial_k-q^{1-2\rho_j}\lambda x^{-j}\partial_{-j}$$

$$\text{for } j<0,$$

$$\partial_0 x^0 = 1+qx^0\partial_0+q\lambda\sum_{k>0}x^k\partial_k,$$

$$\partial_j x^j = 1+q^2x^j\partial_j+q\lambda\sum_{k>j}x^k\partial_k \quad \text{for } j>0. \tag{28}$$

As in our previous discussions, we introduce the dilatation operator Λ_m $(0<m\leqslant n)$ as follows:

$$\Lambda_m=1+q\lambda E_m+q^{2m+1}\lambda^2L_m\Delta_m, \quad E_m=\sum_{j=-m}^{m}x^j\partial_j.$$

These dilatation operators satisfy

$$\Lambda_m x^k=q^2x^k\Lambda_m, \quad \Lambda_m\partial_k=q^{-2}\partial_k\Lambda_m \quad \text{for } |k|\leqslant m. \tag{29}$$

The noncommutative AdS$_{2n}$ space is also accompanied with the conjugation [13]

$$\begin{aligned} \overline{x^i} &= C_{ji}M_{jk}x^k, \\ \overline{\partial_i} &= -q^{-2}\Lambda_n^{-1}C_{ij}M_{jk}[\Delta_n,x^k]. \end{aligned} \tag{30}$$

For $k>0$, using the notations

$$\begin{aligned} y^{-k} &= x^{-k}+q^{k+1/2}\lambda L_k\partial_{+k}, \\ \delta_{-k} &= \partial_{-k}+q^{k+1/2}\lambda\Delta_k x^{+k}, \end{aligned}$$

we can construct the following set of independent bases [11] on the noncommutative AdS$_{2n}$ space:

$$\mathcal{X}^n=x^n, \tag{31}$$

$$\mathcal{D}_n=\partial_n,$$

$$\mathcal{X}^{+j}=\mu_n^{-1/2}\mu_{n-1}^{-1/2}\cdots\mu_{j+1}^{-1/2}x^{+j} \quad \text{for } n>j\geqslant 0,$$

$$\mathcal{D}_{+j}=\mu_n^{-1/2}\mu_{n-1}^{-1/2}\cdots\mu_{j+1}^{-1/2}\partial_{+j},$$

$$\mathcal{X}^{-j}=\mu_n^{-1/2}\mu_{n-1}^{-1/2}\cdots\mu_{j+1}^{-1/2}\Lambda_{+j}^{-1/2}\mu_{+j}^{-1/2}y^{-j},$$

$$\mathcal{D}_{-j}=q^{-1}\mu_n^{-1/2}\mu_{n-1}^{-1/2}\cdots\mu_{j+1}^{-1/2}\Lambda_{+j}^{-1/2}\mu_{+j}^{-1/2}\delta_{-j},$$

$$\mathcal{X}^{-n}=\Lambda_{+n}^{-1/2}\mu_{+n}^{-1/2}y^{-n},$$

$$\mathcal{D}_{-j}=q^{-1}\Lambda_{+n}^{-1/2}\mu_{+n}^{-1/2}\delta_{-n},$$

where $(\mu_{\pm i})^{\pm 1}=\mathcal{D}_{\pm i}\mathcal{X}^{\pm i}-\mathcal{X}^{\pm i}\mathcal{D}_{\pm i}$ and $\mu_0^{1/2}=\mathcal{D}_0\mathcal{X}^0-\mathcal{X}^0\mathcal{D}_0$.

We note that the μ_i's satisfy simple commutation relations with the new variables $\mathcal{X}^j$ and $\mathcal{D}_j$,

$$[\mu_i,\mu_j]=0,$$

$$\mu_i\mathcal{X}^j=\mathcal{X}^j\mu_i\begin{cases}q^2 & \text{for } i=j,\\ 1 & \text{for } i\neq j,\end{cases}$$

$$\mu_i\mathcal{D}^j=\mathcal{D}^j\mu_i\begin{cases}q^{-2} & \text{for } i=j,\\ 1 & \text{for } i\neq j.\end{cases}$$

With the new basis of coordinates and derivatives on the noncommutative AdS$_{2n}$ space, it is not difficult to show that

$$\mathcal{D}_{-k}\mathcal{X}^{-k}=1+q^{-2}\mathcal{X}^{-k}\mathcal{D}_{-k}\quad\text{for } k>0,$$

$$\mathcal{D}_0\mathcal{X}^0=1+q\mathcal{X}^0\mathcal{D}_0,$$

$$\mathcal{D}_{+k}\mathcal{X}^{+k}=1+q^2\mathcal{X}^{+k}\mathcal{D}_{+k},$$

$$[\mathcal{D}_i,\mathcal{D}_j]=0,\quad[\mathcal{X}^i,\mathcal{X}^j]=0,$$

$$\mathcal{D}_i\mathcal{X}^j=\mathcal{X}^j\mathcal{D}_i\quad\text{for } i\neq j. \tag{32}$$

And the noncommutative AdS$_{2n}$ space in terms of the $\mathcal{X}^i$ and $\mathcal{D}_i$ is of the form

$$\sum_{j=1}^{n}q^{\rho_j-2(n-j)}\Lambda_j^{1/2}\mu_j^{-1/2}\mathcal{X}^j\mathcal{X}^{-j}+\frac{q^{-2n+1}}{1+q}\mathcal{X}^0\mathcal{X}^0=-\frac{1}{a^2}. \tag{33}$$

The conjugate operation on the independent set of operators $\mathcal{X}^j$ and $\mathcal{D}_j$ is deduced from the conjugate operation on x^i and ∂_i [Eq. (30)], and we have the following explicit results:

$$\overline{\mathcal{X}}^{n}=\overline{x}^{n}, \tag{34}$$

$$\overline{\mathcal{D}}_n=\overline{\partial}_n,$$

$$\overline{\mathcal{X}}^{+j}=\overline{x}^{+j}\overline{\mu}_{j+1}{}^{-1/2}\overline{\mu}_{j+2}{}^{-1/2}\cdots\overline{\mu}_n{}^{-1/2}\quad\text{for } n>j\geqslant 0,$$

$$\overline{\mathcal{D}}_{+j}=\overline{\partial}_{+j}\overline{\mu}_{j+1}{}^{-1/2}\overline{\mu}_{j+2}{}^{-1/2}\cdots\overline{\mu}_n{}^{-1/2},$$

$$\overline{\mathcal{X}}^{-j}=\overline{y}^{-j}\overline{\mu}_{+j}{}^{-1/2}\overline{\Lambda}_{+j}{}^{-1/2}\overline{\mu}_{j+1}{}^{-1/2}\overline{\mu}_{j+2}{}^{-1/2}\cdots\overline{\mu}_n{}^{-1/2},$$

$$\overline{\mathcal{D}}_{-j}=q^{-1}\overline{\delta}_{-j}\overline{\mu}_{+j}{}^{-1/2}\overline{\Lambda}_{+j}{}^{-1/2}\overline{\mu}_{j+1}{}^{-1/2}\times\overline{\mu}_{j+2}{}^{-1/2}\cdots\overline{\mu}_n{}^{-1/2},$$

$$\overline{\mathcal{X}}^{-n}=\overline{y}^{-n}\overline{\mu}_{+n}{}^{-1/2}\overline{\Lambda}_{+n}{}^{-1/2},$$

$$\overline{\mathcal{D}}_{-j}=q^{-1}\overline{\delta}_{-n}\overline{\mu}_{+n}{}^{-1/2}\overline{\Lambda}_{+n}{}^{-1/2}.$$

Thus, the quantum Heisenberg-Weyl algebra corresponding to the noncommutative AdS$_{2n}$ space can be decoupled into $(2n+1)$-independent subalgebras.

For the quantum algebra $\mathcal{A}_{-k}$ $(0<k\leqslant n)$

$$\mathcal{D}_{-k}\mathcal{X}^{-k}-\mathcal{X}^{-k}\mathcal{D}_{-k}=\mu_{-k}^{-1},\quad\mu_{-k}\mathcal{X}^{-k}=q^2\mathcal{X}^{-k}\mu_{-k},$$

$$\mu_{-}\mathcal{D}_{-k}=q^{-2}\mathcal{D}_{-k}\mu_{-k},$$

$$\mu_{-}^{-1}\equiv 1+(q^{-2}-1)\mathcal{X}^{-k}\mathcal{D}_{-k}, \tag{35}$$

we can construct the quantum coherent state $|0,z\rangle_{-k}$ as

$$|0,z\rangle_{-k}=\exp_{q^2}\left(-\frac{1}{2}|q^{-2}z|\right)\times\sum_{m=0}^{\infty}\frac{(-q^{-2}z)^m}{[m]_{q^2}!}(\mathcal{D}_{-k})^m|0\rangle_{-k},$$

$$\mathcal{X}^{-k}|0,z\rangle_{-k}=z|0,z\rangle_{-k}, \tag{36}$$

where the reference state $|0\rangle_{-k}$ was chosen such that $\mathcal{X}^{-k}|0\rangle_{-k}=0$.

From the coherent state $|0,z\rangle_{-k}$, we can construct a representation for the quantum algebra $\mathcal{A}_{-k}$ as follows:

$$\mathcal{X}^{-k}|n,z\rangle_{-}=q^{2n}z|n,z\rangle_{-k},$$

$$\mathcal{D}_{-k}|n,z\rangle_{-}=-q^{-1-2n}\lambda^{-1}z^{-1}|n+1,z\rangle_{-k},$$

$$\mu_{-k}|n,z\rangle_{-}=|n-1,z\rangle_{-k}. \tag{37}$$

The quantum coherent state which corresponds to the quantum algebras $\mathcal{A}_0$,

$$\mathcal{D}_0\mathcal{X}^0-\mathcal{X}^0\mathcal{D}_0=\mu_0^{1/2},\quad\mu_0\mathcal{X}^0=q^2\mathcal{X}^0\mu_0,$$

$$\mu_0\mathcal{D}_0=q^{-2}\mathcal{D}_0\mu_0,\quad\mu_0^{1/2}\equiv 1+(q-1)\mathcal{X}^0\mathcal{D}_0, \tag{38}$$

is

$$|0,z\rangle_0=\exp_{q^{-1}}\left(-\frac{1}{2}|qz|\right)\sum_{m=0}^{\infty}\frac{(-qz)^m}{[m]_{q^{-1}}!}(\mathcal{D}_0)^m|0\rangle_0,$$

$$\mathcal{X}^0|0\rangle_0=0,$$

$$\mathcal{X}^0|0,z\rangle_0=z|0,z\rangle_0. \tag{39}$$

By using this state we can construct the representation of the quantum algebra $\mathcal{A}_0$ and the result is

$$\mathcal{X}^0|n,z\rangle_0=q^n z|n,z\rangle_0, \tag{40}$$

$$\mathcal{D}_0|n,z\rangle_0=\frac{q^{1-n}}{q-1}z^{-1}|n-1,z\rangle_0,$$

$$\mu_0|n,z\rangle_0=|n-1,z\rangle_0 .$$

Quite similarly, we can also construct the quantum algebra $\mathcal{A}_{+k}$ in terms of a reference state. The quantum algebra $\mathcal{A}_{+k}$ is

$$\mathcal{D}_{+k}\mathcal{X}^{+k}-\mathcal{X}^{+k}\mathcal{D}_{+k}=\mu_{+k}, \quad \mu_{+k}\mathcal{X}^{+k}=q^2\mathcal{X}^{+k}\mu_{+k},$$

$$\mu_{+k}\mathcal{D}_{+k}=q^{-2}\mathcal{D}_{+k}\mu_{+k}, \quad \mu_{+k}\equiv 1+(q^2-1)\mathcal{X}^{+k}\mathcal{D}_{+k}, \tag{41}$$

and the quantum coherent state $|0,z\rangle_{+k}$ is given by

$$|0,z\rangle_{+k}=\exp_{q^{-2}}\left(-\frac{1}{2}|q^2z|\right)\sum_{m=0}^{\infty}\frac{(-q^2z)^m}{[m]_{q^{-2}}!}(\mathcal{D}_{+k})^m|0\rangle_{+k},$$

$$\mathcal{X}^{+k}|0\rangle_{+k}=0,$$

$$\mathcal{X}^{+k}|0,z\rangle_{+k}=z|0,z\rangle_{+k}. \tag{42}$$

The representation of the quantum algebra $\mathcal{A}_{+k}$ is given as follows:

$$\mathcal{X}^{+k}|n,z\rangle_{+k}=q^{2n}z|n,z\rangle_{+k},$$

$$\mathcal{D}_{+k}|n,z\rangle_{+k}=q^{1-2n}\lambda^{-1}z^{-1}|n-1,z\rangle_{+k},$$

$$\Lambda_{+k}|n,z\rangle_{+k}=|n-1,z\rangle_{+k}. \tag{43}$$

All these, Eqs. (37), (40), and (43), give a complete representation for the Hilbert space of the noncommutative AdS_{2n} space. The above results show that the noncommutative AdS_{2n} space is discretely latticed with exponentially increasing space distances. The minimal cutoff induced by the quantum gravity itself is the Planck scale l_p. As in the case of noncommutative orthogonal space, the exponential regularization may effectively reduce the amount of observable degrees of freedom of the noncommutative AdS_{2n} space; even one cannot enumerate it exactly because it is infinite [14]. An adequately adopted noncommutative deformation parameter q (it may be even closer to 1 than the case of limited geometry) can give the equal of the entropy of the quantum system of gravity in the bulk of noncommutative AdS_{2n} space and that on the classical limit of its boundary. The commutative boundary is equal distance lattice regularized and possesses conformally invariant symmetry. This is crucial for the AdS/CFT correspondence.

IV. NONCOMMUTATIVE AdS_{2n-1} SPACE WITH EXPONENTIAL REGULARIZATION

Now we extend our discussion to the noncommutative AdS_{2n-1} space. For this space, commutation relations among coordinates x^i $(i=-n,-n+1,\dots,-2,-1,+1,+2,\dots,n)$ are given as follows:

$$x^ix^j=qx^jx^i \quad \text{for } i<j \quad \text{and} \quad i\neq -j,$$

$$x^ix^{-i}=x^{-i}x^i+\lambda q^{i-2}L_{i-1}$$
$$=q^{-2}x^{-i}x^i+\lambda q^{i-2}L_i \quad \text{for } i>0, \tag{44}$$

where we have used the notation for intermediate lengths

$$L_i=\sum_{k=1}^{i}q^{\rho_k}x^{-k}x^k, \quad \rho_{-k}=k-1, \quad \rho_k=-k+1.$$

By making use of the intermediate Laplacians

$$\Delta_i=\sum_{k=1}^{i}q^{\rho_k}\partial_k\partial_{-k},$$

the algebra satisfied by the derivatives can be written compactly as

$$\partial_i\partial_j=q^{-1}\partial_j\partial_i \quad \text{for } i<j \quad \text{and} \quad i\neq -j,$$

$$\partial_{-i}\partial_i=\partial_i\partial_{-i}+\lambda q^{i-2}\Delta_{i-1}$$
$$=q^{-2}\partial_i\partial_{-i}+\lambda q^{i-2}\Delta_i \quad \text{for } i>0. \tag{45}$$

The commutation relations among the coordinates and derivatives can also be derived and are given as follows:

$$\partial_{-i}x^i=x^i\partial_{-i} \quad \text{for } i\neq 0,$$

$$\partial_ix^j=qx^j\partial_i \quad \text{for } j>-i, \text{ and } j\neq i,$$

$$\partial_ix^j=qx^j\partial_i-q\lambda q^{-\rho_j-\rho_k}x^{-i}\partial_{-j} \quad \text{for } j<-i, \quad \text{and } i\neq j,$$

$$\partial_jx^j=1+q^2x^j\partial_j+q\lambda\sum_{k>j}x^k\partial_k-q^{1-2\rho_j}\lambda x^{-j}\partial_{-j} \quad \text{for } j<0,$$

$$\partial_jx^j=1+q^2x^j\partial_j+q\lambda\sum_{k>j}x^k\partial_k \quad \text{for } j>0. \tag{46}$$

As in Sec. III, we introduce the following dilatation operators Λ_m $(0<m\leqslant n)$:

$$\Lambda_m=1+q\lambda E_m+q^{2m}\lambda^2(1+q^{2n-2})L_m\Delta_m,$$

$$E_m=\sum_{j=-m}^{m}x^j\partial_j.$$

They satisfy the following relations:

$$\Lambda_mx^k=q^2x^k\Lambda_m, \quad \Lambda_m\partial_k=q^{-2}\partial_k\Lambda_m \quad \text{for } 0<|k|\leqslant m. \tag{47}$$

The noncommutative AdS_{2n-1} space is accompanied with the conjugation

$$\overline{x^i}=C_{ji}N_{jk}x^k,$$

$$\overline{\partial_i}=-q^{-2}\Lambda_n^{-1}C_{ij}N_{jk}[\Delta_n,x^k], \tag{48}$$

where

$$N=\begin{pmatrix} 1 & & \\ & \underbrace{\begin{matrix} -1 & & & & \\ & -1 & & & \\ & & \cdots & & \\ & & & -1 & \\ & & & & -1 \end{matrix}}_{2n-2} & \\ & & 1 \end{pmatrix}.$$

For $k>0$, introducing the notations

$$y^{-k}=x^{-k}+q^{k+1/2}\lambda(1+q^{2n-2})L_k\partial_{+k},$$

$$\delta_{-k}=\partial_{-k}+q^{k+1/2}\lambda\Delta_k x^{+k},$$

we can construct the following new set of variables for the coordinates and derivatives:

$$\begin{aligned}
&\mathcal{X}^n=x^n,\\
&\mathcal{D}_n=\partial_n,\\
&\mathcal{X}^{+j}=\mu_n^{-1/2}\mu_{n-1}^{-1/2}\cdots\mu_{j+1}^{-1/2}x^{+j} \quad \text{for } n>j>0,\\
&\mathcal{D}_{+j}=\mu_n^{-1/2}\mu_{n-1}^{-1/2}\cdots\mu_{j+1}^{-1/2}\partial_{+j},\\
&\mathcal{X}^{-j}=\mu_n^{-1/2}\mu_{n-1}^{-1/2}\cdots\mu_{j+1}^{-1/2}\Lambda_{+j}^{-1/2}\mu_{+j}^{-1/2}y^{-j},\\
&\mathcal{D}_{-j}=q^{-1}\mu_n^{-1/2}\mu_{n-1}^{-1/2}\cdots\mu_{j+1}^{-1/2}\Lambda_{+j}^{-1/2}\mu_{+j}^{-1/2}\delta_{-j},\\
&\mathcal{X}^{-n}=\Lambda_{+n}^{-1/2}\mu_{+n}^{-1/2}y^{-n},\\
&\mathcal{D}_{-j}=q^{-1}\Lambda_{+n}^{-1/2}\mu_{+n}^{-1/2}\delta_{-n},
\end{aligned}\tag{49}$$

where $(\mu_{\pm i})^{\pm 1}=\mathcal{D}_{\pm i}\mathcal{X}^{\pm i}-\mathcal{X}^{\pm i}\mathcal{D}_{\pm i}$.

Commutation relations among these new coordinates and derivatives on the noncommutative AdS_{2n-1} space are as follows:

$$\begin{aligned}
&\mathcal{D}_{-k}\mathcal{X}^{-k}=1+q^{-2}\mathcal{X}^{-k}\mathcal{D}_{-k} \quad \text{for } k>0,\\
&\mathcal{D}_{+k}\mathcal{X}^{+k}=1+q^{2}\mathcal{X}^{+k}\mathcal{D}_{+k},\\
&[\mathcal{D}_i,\mathcal{D}_j]=0, \quad [\mathcal{X}^i,\mathcal{X}^j]=0,\\
&\mathcal{D}_i\mathcal{X}^j=\mathcal{X}^j\mathcal{D}_i \quad \text{for } i\neq j.
\end{aligned}\tag{50}$$

The noncommutative AdS_{2n-1} space in terms of the $\mathcal{X}^i$ and $\mathcal{D}_i$ is of the form

$$\sum_{j=1}^{n} q^{\rho_j-2(n-j)}\Lambda_j^{1/2}\mu_j^{-1/2}\mathcal{X}^j\mathcal{X}^{-j}=-\frac{1}{a^2}.\tag{51}$$

The conjugate operation on the operators $\mathcal{X}^j$ and $\mathcal{D}_j$ is induced by the conjugation on x^i and ∂_i and the explicit results are calculated to be

$$\begin{aligned}
&\overline{\mathcal{X}^n}=\overline{x^n},\\
&\overline{\mathcal{D}_n}=\overline{\partial_n},\\
&\overline{\mathcal{X}^{+j}}=\overline{x^{+j}}\,\overline{\mu_{j+1}}^{-1/2}\overline{\mu_{j+2}}^{-1/2}\cdots\overline{\mu_n}^{-1/2} \quad \text{for } n>j>0,\\
&\overline{\mathcal{D}_{+j}}=\overline{\partial_{+j}}\,\overline{\mu_{j+1}}^{-1/2}\overline{\mu_{j+2}}^{-1/2}\cdots\overline{\mu_n}^{-1/2},\\
&\overline{\mathcal{X}^{-j}}=\overline{y^{-j}}\,\overline{\mu_{+j}}^{-1/2}\overline{\Lambda_{+j}}^{-1/2}\overline{\mu_{j+1}}^{-1/2}\overline{\mu_{j+2}}^{-1/2}\cdots\overline{\mu_n}^{-1/2},\\
&\overline{\mathcal{D}_{-j}}=q^{-1}\overline{\delta_{-j}}\,\overline{\mu_{+j}}^{-1/2}\overline{\Lambda_{+j}}^{-1/2}\overline{\mu_{j+1}}^{-1/2}\overline{\mu_{j+2}}^{-1/2}\cdots\overline{\mu_n}^{-1/2},\\
&\overline{\mathcal{X}^{-n}}=\overline{y^{-n}}\,\overline{\mu_{+n}}^{-1/2}\overline{\Lambda_{+n}}^{-1/2},\\
&\overline{\mathcal{D}_{-j}}=q^{-1}\overline{\delta_{-n}}\,\overline{\mu_{+n}}^{-1/2}\overline{\Lambda_{+n}}^{-1/2}.
\end{aligned}\tag{52}$$

Then, the quantum Heisenberg-Weyl algebra corresponds to the noncommutative AdS_{2n-1} space is decoupled into $2n$ independent subalgebras. In the following we will give their representations in terms of quantum coherent states.

The quantum algebra $\mathcal{A}_{-k}$ $(n\geqslant k>0)$ is

$$\begin{aligned}
&\mathcal{D}_{-k}\mathcal{X}^{-k}-\mathcal{X}^{-k}\mathcal{D}_{-k}=\mu_{-k}^{-1}, \quad \mu_{-k}\mathcal{X}^{-k}=q^2\mathcal{X}^{-k}\mu_{-k},\\
&\mu_{-}\mathcal{D}_{-k}=q^{-2}\mathcal{D}_{-k}\mu_{-k}, \quad \mu_{-}^{-1}\equiv 1+(q^{-2}-1)\mathcal{X}^{-k}\mathcal{D}_{-k}.
\end{aligned}\tag{53}$$

For these algebras, we choose the quantum coherent state $|0,z\rangle_{-k}$ as follows:

$$\begin{aligned}
&|0,z\rangle_{-k}=\exp_{q^2}\left(-\frac{1}{2}|q^{-2}z|\right)\sum_{m=0}^{\infty}\frac{(-q^{-2}z)^m}{[m]_{q^2}!}(\mathcal{D}_{-k})^m|0\rangle_{-k},\\
&\mathcal{X}^{-k}|0\rangle_{-k}=0,\\
&\mathcal{X}^{-k}|0,z\rangle_{-k}=z|0,z\rangle_{-k}.
\end{aligned}\tag{54}$$

From the coherent state $|0,z\rangle_{-k}$, we can construct a representation for the quantum algebra $\mathcal{A}_{-k}$ as

$$\begin{aligned}
&\mathcal{X}^{-k}|n,z\rangle_{-}=q^{2n}z|n,z\rangle_{-k},\\
&\mathcal{D}_{-k}|n,z\rangle_{-}=-q^{-1-2n}\lambda^{-1}z^{-1}|n+1,z\rangle_{-k},\\
&\mu_{-k}|n,z\rangle_{-}=|n-1,z\rangle_{-k}.
\end{aligned}\tag{55}$$

Quite similarly, we can construct the reference state $|0,z\rangle_{+k}$ $(n\geqslant k>0)$ as

$$|0,z\rangle_{+k}=\exp_{q^{-2}}\left(-\frac{1}{2}\left|q^2z\right|\right)\sum_{m=0}^{\infty}\frac{(-q^2z)^m}{[m]_{q^{-2}}!}(\mathcal{D}_{+k})^m|0\rangle_{+k},\tag{56}$$

$$\mathcal{X}^{+k}|0\rangle_{+k}=0,$$

$$\mathcal{X}^{+k}|0,z\rangle_{+k}=z|0,z\rangle_{+k}.$$

The corresponding representation of the quantum algebras $\mathcal{A}_{+k}$

$$\mathcal{D}_{+k}\mathcal{X}^{+k}-\mathcal{X}^{+k}\mathcal{D}_{+k}=\mu_{+k},\quad \mu_{+k}\mathcal{X}^{+k}=q^2\mathcal{X}^{+k}\mu_{+k},$$

$$\mu_{+k}\mathcal{D}_{+k}=q^{-2}\mathcal{D}_{+k}\mu_{+k},\quad \mu_{+k}\equiv 1+(q^2-1)\mathcal{X}^{+k}\mathcal{D}_{+k},\tag{57}$$

is given as follows:

$$\mathcal{X}^{+k}|n,z\rangle_{+k}=q^{2n}z|n,z\rangle_{+k},$$

$$\mathcal{D}_{+k}|n,z\rangle_{+k}=q^{1-2n}\lambda^{-1}z^{-1}|n-1,z\rangle_{+k},$$

$$\Lambda_{+k}|n,z\rangle_{+k}=|n-1,z\rangle_{+k}.\tag{58}$$

As in the noncommutative AdS_{2n} space, the noncommutative AdS_{2n-1} space is also discretely latticed with exponentially increasing space distances. The minimal cutoff induced by the quantum gravity itself is the Planck scale l_p. The exponential regularization effectively reduces degrees of freedom in the noncommutative AdS_{2n-1} space. A very small amount of displacement of the noncommutative deformation parameter q from unity gives rise to the equal of the entropy of the quantum theory of gravity in the bulk of noncumulative AdS_{2n-1} space and that on the commutative limit of its boundary surface. The commutative boundary is regularized by an equal distance lattice and possesses a conformally invariant symmetry. Thus, the equivalent theory living on the spacetime boundary of the quantum system of gravity on the background of noncommutative AdS space is a conformal field theory. As we said in the last section, this is the basis for the AdS/CFT correspondence.

V. CONCLUDING REMARKS

In this paper, by constructing Hilbert space with quantum coherent states as reference ones, we presented a kind of special regularization with exponentially increasing spacetime cutoff for both orthogonal and AdS space based on noncommutative geometry. We argued that there must be a universal minimal cutoff for any geometry which is given by the Planck scale l_p. The most direct and obvious physical cutoff is from the formation of microscopic black holes as soon as enough energy are accumulated into a small region. We have obtained results which show a very small ($<10^{-15}$) displacement of the noncommutative deformation parameter from its classical value ($q=1$) reduces sharply the entropy of quantum system of gravity. The noncommutative deformation parameter q was determined by a well-defined algebraic equation. The noncommutative space $\mathrm{SO}_q(3)$ with such a deformation parameter have the same entropy or degrees of freedom as the classical S^2 surface. This is the so-called holography for quantum theory of gravity. The holography makes the quantum theory of gravity on the noncommutative AdS_d space equivalent to the conformally invariant quantum field theory living on the classical limit of its boundary. This is the basis of the AdS/CFT correspondence of string theory and M theory. Here we should stress that the proper geometry for quantum gravity may be noncommutative. Classical continuum geometry is not suitable for quantum gravity. This is in agreement with the long-time speculation that the small scale structure of spacetime may not be adequately described by classical continuum geometry and the noncommutative spacetime might be a realistic picture of how spacetime behaves near the Planck scale. Strong quantum fluctuations of gravity at this spacetime scale may make points fuzzy (in space). All of the strangeness about the quantum theories in different spacetime dimensions could ever be equivalent are coming from the noncommutative geometry description of the quantum gravity. The exact form of the AdS/CFT correspondence is concerned with noncommutative AdS space and the classical limit of its boundary surface (on which the conformal field theory lives). This suggests that the gravity-gauge theory connection should be of the following form: string theory or M theory on the noncommutative background of the form $\mathrm{AdS}_d^q\times \mathrm{M}_{D-d}^q$ is dual to a conformal field theory living on the classical limit of spacetime boundary. For the type-IIB string theory on the noncommutative background $\mathrm{AdS}_5^q\times \mathrm{S}_q^5$, the spectra can be compared with low-order correlation functions of the $3+1$-dimensional $\mathcal{N}=4$ SU(N) super Yang-Mills theory. Only both in the large-N limit and the commutative limit, is string theory weakly coupled and supergravity a good approximation. Therefore, in fact as ’t Hooft suspected [1], nature is much more crazy at the Planck scale than even string theorists could have imagined. Formalisms of string theory (gravity) on noncommutative geometry have to be constructed to gain insight into the unification of gravity and quantum mechanics.

ACKNOWLEDGMENTS

I would like to thank Professor J. Wess for introducing the problem and for enlightening discussions. I am also grateful to H. Steinacker for valuable discussions. After the initial version of the paper was submitted, a communication from A. Jevicki drew my attention to [15], in which the noncommutative effect in the AdS/CFT correspondence was discussed from different point of view. The author is indebted to Professor C. J. Zhu for a critical reading of the manuscript. This work was supported in part by the National Science Foundation of China under Grant No. 19625512.

---

[1] G. ’t Hooft, in *Salamfest 1993*, P284, gr-qc/9310026.
[2] L. Susskind, J. Math. Phys. **36**, 6377 (1995).
[3] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).
[4] S. Gubser, I. Klebanov, and A. Polyakov, Phys. Lett. B **428**, 105 (1998).
[5] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).

[6] J. Wess, Int. J. Mod. Phys. A **12**, 4997 (1997).

[7] J. Madore, ''Gravity on fuzzy spacetime,'' ESI report 478 1997, gr-qc/9709002.

[8] J. Wess and B. Zumino, Nucl. Phys. B (Proc. Suppl.) **18B**, 302 (1990).

[9] A. Connes, *Noncommutative Geometry* (Academic, New York, 1994).

[10] L.D. Faddeev, N.Yu. Reshetikhin, and L.A. Takhtajan, Leningrad Math. J. **1**, 193 (1990).

[11] O. Ogievetsky, Lett. Math. Phys. **24**, 245 (1992).

[12] J. Klauder and B. Skagerstam, *Coherent States, Applications in Physics and Mathematical Physics* (World Scientific, Singapore, 1985).

[13] Z. Chang, ''Quantum Anti-de Sitter space,'' MPI-PhT 99-15, hep-th/9904091.

[14] L. Susskind and E. Witten, ''The holographic bound in Anti−de Sitter space,'' SU-ITP-98-39, IASSNS-98-44, hep-th/9805114.

[15] A. Jevicki and S. Ramgoolam, J. High Energy Phys. **04**, 032 (1999).