

Levi-Civita solutions with a cosmological constant

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The main properties of the Levi-Civita solutions with a cosmological constant are studied. In particular, it is found that some of the solutions need to be extended beyond certain hypersurfaces in order to have geodesically complete spacetimes. Some extensions are considered and found to give rise to a black hole structure but with plane symmetry. All the spacetimes that are not geodesically complete are Petrov type D , while in general the spacetimes are Petrov type I .

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I. INTRODUCTION

Recently, we studied the Levi-Civita (LC) solutions and found that the solutions have a physical meaning at least for $\sigma \in [0, 1]$, where σ is a free parameter related to the mass per unit length [1]. In this paper, we shall study the Levi-Civita solutions with a cosmological constant (LCC). This is not trivial, as the inclusion of a cosmological constant usually makes the problem considerably complicated and changes the spacetime properties dramatically.

The paper is organized as follows: In Sec. II we shall study the main properties of the LCC solutions, including their singularity behavior. We shall show that some spacetimes are not geodesically complete and need to be extended. In Sec. III we will present some extensions and show that some of the extended spacetimes have black hole structure, but with plane symmetry. To distinguish these black holes from the spherical ones, we shall refer to them as *black membranes*. To further study the LCC solutions, we devote Sec. IV to investigate their Petrov classifications, while Sec. V contains our main conclusions.

II. MAIN PROPERTIES OF THE LEVI-CIVITA SOLUTIONS WITH A COSMOLOGICAL CONSTANT

The LCC solutions are not new and were rederived several times, for example, see [2] and references therein. It can be shown that, in addition to the cosmological constant, the solutions have *only* two physically relevant parameters, similar to the LC solutions, and that, without loss of generality, they can be written in the form

$$ds^2 = Q(r)^{2/3} \{ P(r)^{-2(4\sigma^2 - 8\sigma + 1)/3A} dt^2 - P(r)^{2(8\sigma^2 - 4\sigma - 1)/3A} dz^2 - C^{-2} P(r)^{-4(2\sigma^2 + 2\sigma - 1)/3A} d\varphi^2 \} - dr^2, \quad (1)$$

where $\{x^\mu\} \equiv \{t, r, z, \varphi\}$ are the usual cylindrical coordinates, $A \equiv 4\sigma^2 - 2\sigma + 1$. The constant σ is related, but not equal, to

the mass per unit length, and C is related to the angle defects [1,3]. The functions $P(r)$ and $Q(r)$ are defined as

$$P(r) \equiv \frac{2}{\sqrt{3\Lambda}} \tan\left(\frac{\sqrt{3\Lambda}r}{2}\right), \quad Q(r) \equiv \frac{1}{\sqrt{3\Lambda}} \sin(\sqrt{3\Lambda}r). \quad (2)$$

It is easy to show that as $\Lambda \rightarrow 0$ the above solutions reduce to the LC solutions [4]. To study these solutions, it is found convenient to consider the two cases $\Lambda > 0$ and $\Lambda < 0$, separately.

A. $\Lambda > 0$

In this case, from Eq. (2) we find that, as $r \rightarrow 0$, we have $Q(r) \approx r$, $P(r) \approx r$. Then the corresponding solutions approach the LC ones. As a result, the metric (1) has the same singularity behavior as the LC ones near the axis $r=0$. In particular, for the cases $\sigma=0$ and $\sigma=1/2$, the solutions are free of spacetime singularities [1]. Thus one may consider the LCC solutions with $\sigma=0, 1/2$ as cylindrical analogues of the de Sitter solution, although there is a fundamental difference between these two cases. In the present case the Weyl tensor is different from zero, and the spacetimes are not conformally flat (cf. the discussion in Sec. IV). As a matter of fact, they are all Petrov type D . In addition to the usual three Killing vectors $\xi_{(t)} = \partial t$, $\xi_{(z)} = \partial z$, $\xi_{(\varphi)} = \partial \varphi$, the solution with $\sigma=0$ has one more Killing vector $\xi_{(0)} = t \partial z - z \partial t$, which corresponds to a Lorentz boost in the tz plane, while the solution with $\sigma=1/2$ has a fourth Killing vector given by $\xi_{(1/2)} = C^{-1} \varphi \partial z - C z \partial \varphi$, which corresponds to a rotation in the φz plane. Since in the latter case the extrinsic curvatures of the two spacelike surfaces $r=\text{const}$ and $t=\text{const}$ are identically zero, it is difficult to consider this spacetime as having cylindrical symmetry. Instead, one may extend the φ coordinate from the range $[0, 2\pi]$ to the range $(-\infty, +\infty)$, so the resulting spacetime has plane symmetry [1].

On the other hand, Eq. (2) shows that the solutions usually are also singular on the hypersurface $r=r_g \equiv \pi/\alpha$,

where $\alpha \equiv (3|\Lambda|)^{1/2}$. To study the singular behavior of the solutions near this hypersurface, we use the relations $Q(r) \approx R$, $P(r) \approx R^{-1}$, as $r \rightarrow r_g$, where $R \equiv r - r_g$. Substituting these expressions into Eq. (1), we find

$$ds^2 \approx R^{4(4\sigma^2 - 5\sigma + 1)/3A} dt^2 - R^{-4(2\sigma^2 - \sigma - 1)/3A} dz^2 - C^{-2} R^{2(8\sigma^2 + 2\sigma - 1)/3A} d\varphi^2 - dr^2 \quad (R \approx 0). \quad (3)$$

The corresponding Kretschmann scalar is given by

$$R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = \frac{64(\sigma - 1)^2(2\sigma + 1)^2(4\sigma - 1)^2}{27A^3R^4} \quad (R \approx 0), \quad (4)$$

which is always singular as $R \rightarrow 0$, except for the cases where $\sigma = -1/2, 1/4, 1$. It can be shown that all 14 scalars built from the Riemann tensor have the same properties. Therefore, in the cases $\sigma = -1/2, 1/4, 1$ the singularities on the hypersurface $r = r_g$ are coordinate ones, and to have the corresponding spacetimes geodesically complete, the solutions need to be extended beyond this surface. Note that, similar to the solutions with $\sigma = 0, 1/2$, all these three solutions are Petrov type *D*. Moreover, in addition to the usually three Killing vectors, they also have a fourth Killing vector, given, respectively, by $\xi_{(-1/2)} = C^{-1}\varphi\partial z - Cz\partial\varphi$, $\xi_{(1/4)} = C^{-1}\varphi\partial t - Ct\partial\varphi$, and $\xi_{(1)} = z\partial t - t\partial z$. Using the same arguments as those given for the solution with $\sigma = 1/2$, the solution with $\sigma = -1/2$ can be also considered as representing plane symmetry.

Combining the analysis of the singular behavior of the solutions on the axis and on the hypersurface $r = r_g$, we can see that all the solutions are singular on both of the two surfaces, except for the ones with $\sigma = -1/2, 0, 1/4, 1/2, 1$, which are the only solutions that are Petrov type *D* (see the discussion given in Sec. IV). These singularities make a physical interpretation of the solutions very difficult. A possible way to circumvent these difficulties is to cut the spacetimes along the hypersurface $r = r_\Sigma < r_g$ and then join the part $r < r_\Sigma$ with an asymptotically de Sitter region, while considering the singularities on the axis as representing matter sources [3]. On the other hand, the solutions with $\sigma = 0, 1/2$ are free of spacetime singularities on the axis, but do have them on the hypersurface $r = r_g$. To give a meaningful physical interpretation of these solutions, one may take $r = r_g$ as the symmetry axis, and then extend the spacetimes beyond $r = 0$. When $\sigma = -1/2, 1/4, 1$, the corresponding solutions are singular on the axis, but free of spacetime singularities on the hypersurface $r = r_g$. Thus we need to extend the spacetimes beyond this surface. We shall leave these considerations to the next section.

B. $\Lambda < 0$

As $r \rightarrow 0$, the functions $Q(r)$ and $P(r)$ have the same asymptotical behavior as those given in the last case. As a result, in both of the two cases the solutions have the same singularity behavior as the LC ones near the axis $r = 0$; that is, they are all singular, except for the cases $\sigma = 0$ and $\sigma = 1/2$.

On the other hand, Eq. (2) shows that in the present case $Q(r)$ and $P(r)$ are monotonically increasing functions of r and are positive for any given $r > 0$, in contrast with the case $\Lambda > 0$, where they are periodic functions [cf. Eq. (2)]. When $r \rightarrow +\infty$, we find $Q(r) \approx e^{\alpha r}/(2\alpha)$, $P(r) \approx 2/\alpha$, and the corresponding metric, after t and z are rescaled, takes the form

$$ds^2 \approx C_0 e^{2\alpha r/3} (dt^2 - dz^2 - C^{-2} d\varphi^2) - dr^2 \quad (r \rightarrow +\infty), \quad (5)$$

where C_0 is a positive constant. This is exactly the anti-de Sitter spacetime, but written in horospherical coordinates [5]. Since the metric does not depend on the parameter σ , we conclude that *all the LCC solutions with negative cosmological constant are asymptotically anti-de Sitter*.

III. SOLUTIONS REPRESENTING BLACK MEMBRANES

As shown in the last section, the solutions with $\sigma = \pm 1/2$ both for $\Lambda > 0$ and $\Lambda < 0$ have a fourth Killing vector $\xi = C^{-1}\varphi\partial z - Cz\partial\varphi$, which represents the rotation invariant in the $z\varphi$ planes, or in other words, the extrinsic curvature of the planes is identically zero. This property makes these two-dimensional planes more likely to have plane symmetry than a cylindrical one [1]. Then, the ranges of r and φ should be extended to $-\infty < r, \varphi < +\infty$. In the following we shall denote such an extended coordinate φ by Y . Once this is done, we can see that the spacetime is not geodesically complete. In particular, the solutions with $\sigma = 1/2$ for both $\Lambda > 0$ and $\Lambda < 0$ are not singular on the hypersurface $r = 0$ and need to be extended beyond it, while the one with $\sigma = -1/2$ and $\Lambda > 0$ is not singular on the hypersurface $r = r_g$ and needs to be extended beyond this surface, too. In the following, we shall consider these cases separately.

Case (α): $\sigma = 1/2, \Lambda > 0$. In this case, making the coordinate transformations

$$T = \frac{2t}{3}, \quad X = \cos^{2/3}\left(\frac{\alpha r}{2}\right), \quad Y = \frac{\alpha\varphi}{3C}, \quad Z = \frac{\alpha z}{3}, \quad (6)$$

we find that the corresponding solution can be written in the form

$$ds_{\sigma=1/2}^2 = \frac{9}{\alpha^2} \{f(X)dT^2 - f^{-1}(X)dX^2 - X^2(dY^2 + dZ^2)\} \quad (\Lambda > 0), \quad (7)$$

where $f(X)$ is defined as

$$f(X) \equiv \frac{1}{X} - X^2. \quad (8)$$

From Eq. (6) we can see that the region $0 \leq r \leq r_g$ is mapped onto the region $0 \leq X \leq 1$, and the point $r = r_g$, where the spacetime is singular, is mapped onto the point $X = 0$. Extending X to the range $(-\infty, +\infty)$, we find that in the extended spacetime two new regions $X > 1$ and $X < 0$ are included. The curvature singularity at $X = 0$ divides the whole

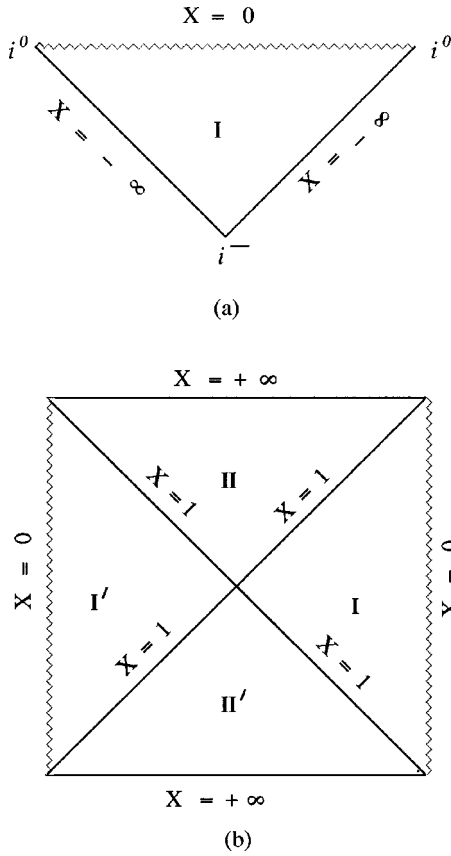


FIG. 1. The Peurose diagram for the case $\sigma = \pm 1/2$, $\Lambda > 0$. (a) $X \leq 0$, (b) $X \geq 0$. Each point actually represents a plane. The lines $X=0$ represent spacetime singularities, while the lines $X=1$ represent Cauchy horizons. As $|X| \rightarrow +\infty$, the spacetimes are asymptotically de Sitter.

spacetime into two unconnected regions $X \geq 0$ and $X \leq 0$. In the region $X \leq 0$, the function $f(X)$ is always negative and the X coordinate is timelike. Then, the spacetime is essentially time dependent and the singularity at $X=0$ is spacelike and naked. As $X \rightarrow -\infty$, the metric is asymptotically de Sitter [6]:

$$ds_{\sigma=1/2}^2 \approx d\tilde{T}^2 - e^{2a\tilde{T}/3}(dX^2 + dY^2 + dZ^2) \quad (X \rightarrow -\infty), \quad (9)$$

where $T = e^{\alpha\tilde{T}/3}$ and X, Y, Z have been rescaled. The corresponding Penrose diagram is given by Fig. 1(a).

When $X \geq 0$, $f(X)$ is greater than zero for $0 \leq X < 1$ and less than zero for $X > 1$; that is, X is spacelike when $0 \leq X < 1$ and timelike when $X > 1$. On the hypersurface $X=0$ it becomes null, which represents a horizon. Since the spacetime singularity at $X=0$ now is timelike, the horizon is actually a Cauchy horizon. As $X \rightarrow +\infty$, the spacetime is also asymptotically de Sitter and approaches the same form as that given by Eq. (9). The corresponding Penrose diagram is given by Fig. 1(b).

Case (β): $\sigma = -1/2$, $\Lambda > 0$. In this case, the spacetime is singular at $r=0$ and is free of curvature singularity at r

$=r_g$. Thus, to have a geodesically complete spacetime, we need to extend the solution beyond the hypersurface $r=r_g$. To make such an extension, we can introduce a new coordinate X by $X = \sin^{2/3}(\alpha r/2)$ and rescale the coordinates t, z , and φ then, we will find that the corresponding metric takes the same form as that given by Eq. (7). This is not expected. As we know, in the limit $\Lambda \rightarrow 0$ the solution with $\sigma = 1/2$ approaches the Rindler space [7], which represents a uniformly gravitational field and is free of any kind of spacetime curvature singularities, while the one with $\sigma = -1/2$ is the static Taub solution with plane symmetry [8] and is singular on the hypersurface $X=0$. The total mass of the Taub spacetime is negative, while the one of Rindler is not [9]. However, the presence of the cosmological constant makes up these differences and turns the two spacetimes into being identical.

Case (γ): $\sigma = 1/2$, $\Lambda < 0$. In this case, the spacetime is free of curvature singularity for $0 \leq r < +\infty$ and needs to be extended beyond the hypersurface $r=0$. Similar to the last two cases, introducing the new coordinate X as $X = \cosh^{2/3}(\alpha r/2)$ and rescaling the coordinates t, z, φ , the corresponding metric can be written in the form

$$ds_{\sigma=1/2}^2 = \frac{9}{\alpha^2} \{ -f(X)dT^2 + f^{-1}(X)dX^2 - X^2(dY^2 + dZ^2) \} \quad (\Lambda < 0), \quad (10)$$

where $f(X)$ is given by Eq. (8). From the expression of X we can see that the region $0 \leq r < +\infty$ is mapped onto the region $1 \leq X < +\infty$. The region $X < 1$ is an extended region. After the extension, a spacetime curvature singularity appears at $X=0$, which divides the whole X axis into two parts $X \leq 0$ and $X \geq 0$. It can be shown that, unlike the case $\Lambda > 0$, now the spacetime is static in the region $X \leq 0$ and the curvature singularity at $X=0$ is timelike and naked. As $X \rightarrow -\infty$, the spacetime is asymptotically anti-de Sitter spacetime [5]:

$$ds_{\sigma=1/2}^2 \approx \frac{9}{\alpha^2 \tilde{X}^2} (dT^2 - d\tilde{X}^2 - dY^2 - dZ^2) \quad (X \rightarrow -\infty), \quad (11)$$

where $\tilde{X} = 1/X$. The corresponding Penrose diagram is given by Fig. 2(a).

In the region $X \geq 0$, the spacetime singularity at $X=0$ becomes spacelike. Except for this curvature singularity, there is a coordinate one located at $X=1$. This coordinate singularity actually represents an event horizon. As shown in the last section, the spacetime is asymptotically anti-de Sitter ($X \rightarrow +\infty$). The corresponding Penrose diagram is given by Fig. 2(b). This is the black hole solution with plane symmetry found recently by Cai and Zhang with vanishing electromagnetic charge [10].

Case (δ): $\sigma = -1/2$, $\Lambda < 0$. In this case, a spacetime singularity appears at $r=0$, and the region $0 \leq r < +\infty$ is geodesically complete. However, since in this case the solution has also plane symmetry, hence the range of r should be taken as $-\infty < r < +\infty$. Then one may ask, what is the physical interpretation of the spacetime in the region $r \leq 0$? To an-

answer this question, let us introduce a new coordinate $X = -\sinh^{2/3}(\alpha r/2)$ and rescale the other three; then, we will find that the metric takes the same form as that given by Eq. (10). From the expression for X we can see that the region $0 \leq r < +\infty$ now is mapped onto the region $-\infty < X \leq 0$, while the region $-\infty < r \leq 0$ is mapped onto the region $0 \leq X < +\infty$. In the region $0 \leq r < +\infty$ the solution represents a static spacetime with a naked singularity located at $r=0$. The spacetime is asymptotically anti-de Sitter, and the corresponding Penrose diagram is given by Fig. 2(a). In the region $-\infty < r \leq 0$ the solution represents a black hole solution with plane symmetry, and the corresponding Penrose diagram is given by Fig. 2(b).

IV. PETROV CLASSIFICATION OF THE SOLUTIONS

To further study the LCC solutions, we shall consider their Petrov classifications in this section. Choosing a null tetrad $e_{(\alpha)}^\mu = \{l^\mu, n^\mu, m^\mu, \bar{m}^\mu\}$ as

$$\begin{aligned}
 l^\mu &= \frac{1}{\sqrt{2}} \{(g_{tt})^{1/2} \delta_t^\mu + \delta_r^\mu\}, \\
 n^\mu &= \frac{1}{\sqrt{2}} \{(g_{tt})^{1/2} \delta_t^\mu - \delta_r^\mu\}, \\
 m^\mu &= \frac{1}{\sqrt{2}} \{(-g_{zz})^{1/2} \delta_z^\mu + i(-g_{\varphi\varphi})^{1/2} \delta_\varphi^\mu\}, \\
 \bar{m}^\mu &= \frac{1}{\sqrt{2}} \{(-g_{zz})^{1/2} \delta_z^\mu - i(-g_{\varphi\varphi})^{1/2} \delta_\varphi^\mu\},
 \end{aligned} \tag{12}$$

where the metric coefficients can be read off directly from Eq. (1), we find that the nonvanishing components of the Ricci and Weyl tensors are given by

$$\begin{aligned}
 R &= 4\Lambda, \\
 \Psi_0 &\equiv -C_{\mu\nu\lambda\delta} l^\mu m^\nu l^\lambda m^\delta \\
 &= -\frac{\Lambda(4\sigma-1)}{4D^2 \cos^2 \theta \sin^2 \theta} [D \cos^2 \theta + 2\sigma^2 - \sigma - 1], \\
 \Psi_2 &\equiv -\frac{1}{2} C_{\mu\nu\lambda\delta} l^\mu m^\nu \bar{m}^\lambda n^\delta \\
 &= -\frac{\Lambda}{12D^2 \cos^2 \theta \sin^2 \theta} [D(8\sigma^2 - 4\sigma - 1) \cos^2 \theta \\
 &\quad - 32\sigma^3(\sigma-1) + 6\sigma^2 - 7\sigma + 1], \\
 \Psi_4 &\equiv -C_{\mu\nu\lambda\delta} l^\mu m^\nu l^\lambda m^\delta = \Psi_0,
 \end{aligned} \tag{13}$$

where $\theta \equiv \sqrt{3}\Lambda r/2$. Note that the above expressions are valid for any Λ , including $\Lambda=0$. When $\Lambda < 0$ the function θ becomes imaginary, and the trigonometric functions become

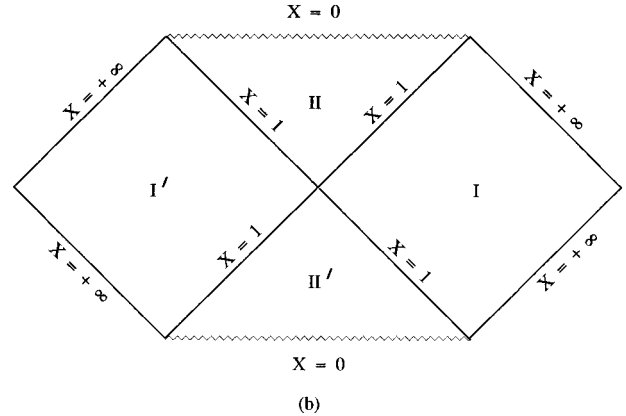
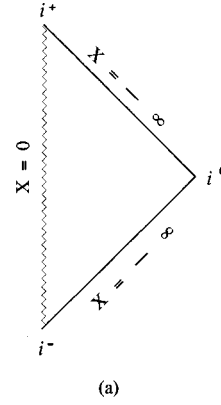


FIG. 2. The Penrose diagram for the cases $\sigma = \pm 1/2$, $\Lambda < 0$. (a) $X \leq 0$, (b) $X \geq 0$. The lines $X=0$ represent spacetime singularities, while the lines $X=1$ represent event horizons. $\Lambda s|X| \rightarrow +\infty$, the spacetimes are asymptotically anti-de Sitter.

hyperbolic functions. Since Ψ_0 , Ψ_2 , and Ψ_4 are the only components of the Weyl tensor different from zero, it can be shown that the metric in general is Petrov type I [4], unless (i) $\Psi_0=0, \Psi_2 \neq 0$; (ii) $\Psi_0 = \pm 3\Psi_2 \neq 0$. In the last two cases, the solutions are Petrov type D . Further specialization $\Psi_0 = \Psi_4 = \Psi_2 = 0$ leads to Petrov type O solutions. However, the last case holds only when $\Lambda=0$ and $\sigma=0, 1/2$. That is, all the solutions with $\Lambda \neq 0$ are either Petrov type I or D . From Eq. (13) we find that the condition $\Psi_0=0$ and $\Psi_2 \neq 0$ yields $\sigma=1/4$, while the one $\Psi_0 = \pm 3\Psi_2 \neq 0$ yields $\sigma = -1/2, 0, 1/2, 1$. Thus *all the solutions with $\Lambda \neq 0$ are Petrov type I , except for the ones with $\sigma = -1/2, 0, 1/4, 1/2, 1$, which are Petrov type D* . In the latter cases, all of the solutions have an additional Killing vector [cf. Sec. II]. Since conformally flat solutions are necessarily Petrov type O , we conclude that all the solutions with $\Lambda \neq 0$ are not conformally flat, and the de Sitter and anti-de Sitter solutions are not particular cases of the LCC solutions.

It is interesting to note that if we introduce a new parameter τ by $\sigma = 1/4 + \tau$, we find that the metric can be obtained from the one with $\sigma = 1/4 - \tau$ following the change $t = iC^{-1}\varphi'$, $\varphi = iCt'$. This indicates some kind of symmetries with respect to the solution $\sigma = 1/4$. The study of the Ricci and Weyl tensors using the null tetrad defined by Eq. (12) will make this symmetry clear. For any given τ , we find

$$\begin{aligned}
R^+(r, \tau) &= R^-(r, \tau), & \Psi_0^+(r, \tau) &= -\Psi_0^-(r, \tau), \\
\Psi_2^+(r, \tau) &= \Psi_2^-(r, \tau), & \Psi_4^+(r, \tau) &= -\Psi_4^-(r, \tau),
\end{aligned}
\tag{14}$$

where quantities with “+” denote the ones calculated from the metric with $\sigma=1/4+\tau$ and the quantities with “-” denote the ones calculated from the metric with $\sigma=1/4-\tau$. The above relations are valid even for $\Lambda=0$. From Eq. (14) we can see that, for any given τ , the solution with $\sigma=1/4+\tau$ and the one with $\sigma=1/4-\tau$ have the same Petrov classification. For example, the solution with $\sigma=0$ and the one with $\sigma=1/2$ all belong to Petrov type D when $\Lambda \neq 0$ and to Petrov type O when $\Lambda=0$.

V. CONCLUSIONS

In this paper, we have studied the main properties of the Levi-Cività solutions with a cosmological constant and found that, among other things, some solutions need to be extended beyond certain hypersurfaces in order to obtain geodesically complete spacetimes. We have considered some

extensions for the case where the solutions have a rotating Killing vector in the $z\varphi$ plane and found that some of the extensions give rise to black hole structures, but with plane symmetry, *black membranes*. It is interesting to note that these structures exist even in the range $-\infty < r \leq 0$. This naturally raises the question, what kind of spacetimes do the general solutions represent in this region? This problem is currently under our investigation.

To further study the solutions, we have also considered their Petrov classifications and found that all solutions that are not geodesically complete, including the ones that represent black membranes, are Petrov type D , while in general they are Petrov type I . As we know, Kerr-Newmann solutions are Petrov type D , too. So it would be very interesting to show that all black hole solutions with plane or cylindrical symmetry are Petrov type D .

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