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$N=4$ supersymmetric multidimensional quantum mechanics, partial SUSY breaking, and superconformal quantum mechanics

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The multidimensional $N=4$ supersymmetric (SUSY) quantum mechanics (QM) is constructed using the superfield approach. As a result, the component form of the classical and quantum Lagrangian and Hamiltonian is obtained. In the SUSY QM considered, both classical and quantum $N=4$ algebras include central charges, and this opens various possibilities for partial supersymmetry breaking. It is shown that quantum-mechanical models with one-quarter, one-half, and three-quarters of unbroken (broken) supersymmetries can exist in the framework of the multidimensional $N=4$ SUSY QM, while the one-dimensional $N=4$ SUSY QM, constructed earlier, admits only one half or total supersymmetry breakdown. We illustrate the constructed general formalism, as well as all possible cases of partial SUSY breaking taking as an example a direct multidimensional generalization of the one-dimensional $N=4$ superconformal quantum-mechanical model. Some open questions and possible applications of the constructed multidimensional $N=4$ SUSY QM to the known exactly integrable systems and problems of quantum cosmology are briefly discussed.

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I. INTRODUCTION

Supersymmetric (SUSY) quantum mechanics (QM), first introduced in Refs. [1] and [2] for the $N=2$ case, turns out to be a convenient tool for investigating problems of supersymmetric field theories, since it provides a simple and, at the same time, quite adequate understanding of various phenomena arising in relativistic theories.

The important question of all modern theories of fundamental interactions, including superstring and M theory, is the problem of the spontaneous breakdown of supersymmetry. Supersymmetry, as the fundamental symmetry of the nature, if it exists, has to be the spontaneously broken at low energies since particles with all equal quantum numbers, except the spin, are not observed experimentally. Several (rather different) mechanisms of spontaneous breakdown of supersymmetry have been proposed in particle physics in order to resolve this problem. One of them is to add to the supersymmetric Lagrangian, the so-called D, or F terms, which are invariant under supersymmetry transformations but break supersymmetry spontaneously due to nonzero vacuum expectation values or, alternatively, introduce into the theory some soft breaking mass terms ''by hand;'' the latter procedure does not spoil the nonrenormalization theorem of the supersymmetric field theories and was successfully applied to construct the minimal supersymmetric extension of the standard model (see [3] and references therein). The next mechanism of SUSY breaking is the dynamical (nonperturbative) breakdown of supersymmetry, caused by instantons (see, for example, [4] and references therein). In this case, the energy of tunneling between topologically distinct vacua produces an energy shift from the zero level, hence leading to the spontaneous breakdown of supersymmetry. And, finally, the mechanism of partial spontaneous breaking of the $N=2$ supersymmetry in the field theory was recently proposed in [5]. This mechanism is based on the inclusion into the Lagrangian of two types of the Fayet-Iliopoulos terms, electric and magnetic, and it leads to the corresponding modification of the $N=2$ SUSY algebra [6].

The problem of spontaneous breakdown of supersymmetry could be investigated in the framework of the supersymmetric quantum mechanics as well. The conjecture that supersymmetry can be spontaneously broken by instantons [1, 2] was investigated in detail by several authors for the case of $N=2$ SUSY QM [7–9]. However, the most physically interesting case is provided by the $N=4$ supersymmetric quantum mechanics since it can be applied to the description of the systems resulting from the ''realistic'' $N=1$ supersymmetric field theories (including supergravity) in four ($D=4$) dimensions after the dimensional reduction to one dimension (see, for example, [10]).

The one-dimensional $N=4$ SUSY QM was constructed in [11–13]. Partial breaking of supersymmetry, caused by the presence of the central charges in the corresponding superalgebra, was also discussed in [13]. It was the first example of partial breaking of supersymmetry in the framework of SUSY QM and the corresponding mechanism is in full analogy with that in [5] in the field theory. The main point is that the presence of central charges in the superalgebra allows the partial supersymmetry breakdown, whereas according to Witten's theorem [1], no partial supersymmetry breakdown is possible if the SUSY algebra includes no central charges.

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The main goal of our paper is further generalization of the construction, proposed in [13] to the multidimensional case and investigation of partial breaking of the supersymmetry under consideration.

Consideration of the supersymmetric algebra with central charges is of particular importance for several reasons. First, it provides a good tool to study dyon solutions of quantum field theory since in such theories the mass and electric and magnetic charges turns out to be the central charges [14]. Second, the central charges produce the rich structure of supersymmetry breaking. Namely, it is possible to break part of all supersymmetries retaining all others exactly [15]. In fact, the invariance of a state with respect to the supersymmetry transformation means saturation of the Bogomol'nyi bound, and this situation takes place in the $N=2$ and $N=4$ supersymmetric Yang-Mills theory [16,17] as well as in the theories of extended supergravity [18].

The investigation of supersymmetric properties of branes the in M theory has also revealed that partial breakdown of supersymmetry takes place. Namely, the ordinary branes break half of the supersymmetries, while ''intersecting'' and rotating branes can leave only 1/4, 1/8, 1/16 or 1/32 of the supersymmetries unbroken [19].

The main characteristic features of partial SUSY breaking in the field theories with the extended supersymmetry can be revealed in supersymmetric QM, since in both cases partial supersymmetry breakdown is provided by the central charges in the SUSY algebra. Therefore, the detailed study of partial supersymmetry breakdown in supersymmetric quantum mechanics can lead to the deeper understanding of an analogous effect in supersymmetric field theories.

The known examples of the breakdown of supersymmetries in the supersymmetric quantum mechanics are the cases where either all supersymmetries are broken (exact) or only half of them are broken (exact) [13]. In this paper, we demonstrate the possibility of three-quarters or one-quarter supersymmetry breakdown in the framework of the multidimensional $N=4$ SUSY QM. The later case (3/4 of the supersymmetries are exact) has not been observed before in SUSY QM (in the specific $N=4$ supergravity model it was observed in [20]) and seems to be quite interesting by itself even without specifying the physical origin of the phenomenon.

The paper is organized as follows. In Sec. II, we present a formal construction of the $N=4$ multidimensional supersymmetric quantum mechanics: classical and quantum Hamiltonian and Lagrangian, SUSY transformations, algebra of supercharges and so on. In Sec. III, partial supersymmetry breaking is investigated and all possible cases of the partial SUSY breakdown are listed. In Sec. IV, we give an exactly solvable example which illustrates main properties of the introduced formal constructions. This example is interesting by itself since we consider the multidimensional generalization of the $N=4$ superconformal quantum mechanics [13], [21] which is naturally related to the extremal Reissner-Nordström (RN) black holes in the ''near horizon'' limit and anti–de Sitter-conformal field theory correspondence [22]. In Sec. V, we conclude with some open questions and further perspectives.

II. D-DIMENSIONAL $N=4$ SUSY QUANTUM MECHANICS

In this section, we describe a general formalism of the D-dimensional ($D\geqslant 1$) $N=4$ supersymmetric quantum mechanics, starting with the superfield approach and concluding with the component form of the desired Lagrangian and Hamiltonian.

Consider $N=4$ SUSY transformations

$$\delta t=\frac{i}{2}(\epsilon^a\bar{\theta}_a+\bar{\epsilon}_a\theta^a),$$
$$\delta\bar{\theta}_a=\bar{\epsilon}_a,$$
$$\delta\theta^a=\epsilon^a, \tag{2.1}$$

in the superspace spanned by the even coordinate t and mutually complex-conjugated odd coordinates θ^a and $\bar{\theta}_a$. The parameters of $N=4$ SUSY transformations ϵ^a and $\bar{\epsilon}_a$ are complex conjugate to each other as well.[1] The generators of the above supersymmetry transformations

$$Q_a=\frac{\partial}{\partial\theta^a}+\frac{i}{2}\bar{\theta}_a\frac{\partial}{\partial t},\quad \bar{Q}^a=\frac{\partial}{\partial\bar{\theta}_a}+\frac{i}{2}\theta^a\frac{\partial}{\partial t}, \tag{2.2}$$

along with the time translation operator $H=i\partial/\partial t$ obey the following (anti)commutation relations:

$$\{Q_a,\bar{Q}^b\}=\delta_a^b H,$$
$$[H,Q_a]=[H,\bar{Q}^a]=0. \tag{2.3}$$

The automorphism group for a given algebra is SO(4) $=$SU(2)$\times$SU(2) and the generators of the $N=4$ SUSY transformations are in the spinor representation of one of the SU(2) groups.

The next step is to construct irreducible representations of the algebra (2.3). The usual way of doing this is to use the supercovariant derivatives

$$D_a=\frac{\partial}{\partial\theta^a}-\frac{i}{2}\bar{\theta}_a\frac{\partial}{\partial t},\quad \bar{D}^a=\frac{\partial}{\partial\bar{\theta}_a}-\frac{i}{2}\theta^a\frac{\partial}{\partial t}, \tag{2.4}$$

and impose some constraints on the general superfield. Hereafter we deal with the superfield Φ^i $(i=1,\dots,D)$ subjected to the following constraints:

$$[D_a,\bar{D}^a]\Phi^i=-4m^i,$$
$$D^aD_a\Phi^i=-2n^i,$$
$$\bar{D}_a\bar{D}^a\Phi^i=-2\bar{n}^i, \tag{2.5}$$

[1] Our conventions for spinors are as follows: $\theta_a=\theta^b\varepsilon_{ba}$, $\theta^a=\varepsilon^{ab}\theta_b$, $\bar{\theta}_a=\bar{\theta}^b\varepsilon_{ba}$, $\bar{\theta}^a=\varepsilon^{ab}\bar{\theta}_b$, $\bar{\theta}_a=(\theta^a)^*$, $\bar{\theta}^a=-(\theta_a)^*$, $(\theta\theta)\equiv\theta^a\theta_a=-2\theta^1\theta^2$, $(\bar{\theta}\bar{\theta})\equiv\bar{\theta}_a\bar{\theta}^a=(\theta\theta)^*$, $\varepsilon^{12}=1$, $\varepsilon_{12}=1$.

where m^i are real constants, while n^i and $\overline{n^i}$ are mutually complex-conjugated constants. Such constraints for the case of the one-dimensional $N=4$ SUSY QM were considered first in [13] as a minimal consistent generalization of the analogous constraints of [11] and [21] with a vanishing right-hand side. The presence of the additional arbitrary parameters leads, as we shall see below, to the considerably richer structure of the theory. The explicit form of the superfield Φ^i is the following:

$$\Phi^i=\phi^i+\theta^a\overline{\psi^i_a}-\overline{\theta}_a\psi^{ia}+\theta^a B_a^{bi}\overline{\theta}_b+m^i(\theta\overline{\theta})+\frac{1}{2}n^i(\theta\theta)+\frac{1}{2}\overline{n^i}(\overline{\theta\theta})+\frac{i}{4}(\theta\theta)\overline{\theta}_a\dot{\overline{\psi}}^{ai}-\frac{i}{4}(\overline{\theta\theta})\theta^a\dot{\psi}^i_a+\frac{1}{16}(\theta\theta)(\overline{\theta\theta})\ddot{\phi}^i \tag{2.6}$$

$(\dot{}\equiv\partial_t)$. Note that in the case when all the constants m^i, n^i, and $\overline{n^i}$ are equal to zero, the superfield (2.6) represents D "trivial" copies of the superfield Φ, given in [11], which describes the irreducible representation of the one-dimensional $N=4$ SUSY QM. The latter superfield contains one bosonic field ϕ, four fermionic fields ψ^a and $\overline{\psi}_a$, and three auxiliary bosonic fields $B_a^b=(\sigma_I)_a^b B^I$ where $(\sigma_I)_a^b$ $(I=1,2,3)$ are ordinary Pauli matrices.

Another irreducible representation of the algebra (2.3) can be constructed after making the appropriate generalization of the constraints given in [12]:

$$(\varepsilon^{ac}D_c\overline{D}^b+\varepsilon^{bc}D_c\overline{D}^a)\Phi=0. \tag{2.7}$$

The technique of constructing $N=4$ SUSY invariant Lagrangians is absolutely the same for both the cases and, therefore, we shall not consider the second one separately.

The components of the superfield (2.6) transform under the $N=4$ transformations as follows:

$$\delta\phi^i=\epsilon^a\overline{\psi^i_a}-\overline{\epsilon}_a\psi^{ai},$$

$$\delta\psi^{ai}=\epsilon^b B_b^{ai}+\frac{i}{2}\epsilon^a\dot{\phi}^i+\epsilon^a m^i-\overline{\epsilon^a}\overline{n^i},$$

$$\delta B_b^{ai}=-\frac{i}{2}\epsilon_b\dot{\overline{\psi}}^{ai}-\frac{i}{2}\epsilon^a\dot{\overline{\psi^i_b}}-\frac{i}{2}\overline{\epsilon^a}\dot{\psi}^i_b-\frac{i}{2}\overline{\epsilon}_b\dot{\psi}^{ai}. \tag{2.8}$$

Now one can write down the most general form of the Lagrangian which is invariant under the above-mentioned $N=4$ SUSY transformations

$$L=-8\left(\int d^2\theta d^2\overline{\theta}(A(\Phi^i))+\frac{1}{16}\lambda^a_{bi}B_a^{bi}\right), \tag{2.9}$$

where $A(\Phi^i)$ is an arbitrary function of the superfield Φ^i called the superpotential. The second term is the Fayet-Iliopoulos term and $\lambda^a_{bi}=(\sigma_I)^a_b\Lambda^I_i$ are just constants. The expression for the Lagrangian (2.9) is the most general one in the sense that any other $N=4$ SUSY invariant terms added will lead with necessity to higher derivatives in the component form.

After the integration with respect to the Grassmanian coordinates θ^a and $\overline{\theta_a}$, one obtains the component form of the Lagrangian (2.9):

$$L=K-V, \tag{2.10}$$

where

$$K=\frac{1}{2}\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}\dot{\phi}^i\dot{\phi}^j+i\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}(\overline{\psi^i_a}\dot{\psi}^{aj}+\psi^{ai}\dot{\overline{\psi^j_a}}), \tag{2.11}$$

and

$$V=2\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}(m^im^j+n^i\overline{n^j})-\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}B_b^{ai}B_a^{bj}+\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k}(2\overline{\psi^i_a}\psi^{aj}m^k+\psi^{ai}\psi^j_an^k+\overline{\psi^i_a}\overline{\psi^{aj}}\overline{n^k})-\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^p}(\overline{\psi^i_a}\psi^{bj}+\overline{\psi}^{ib}\psi^j_a)B_b^{ap}+\frac{1}{2}\frac{\partial^4 A}{\partial\phi^i\partial\phi^j\partial\phi^k\partial\phi^l}(\overline{\psi^i_a}\overline{\psi^{ja}})(\psi^{bk}\psi^l_b)+\frac{1}{2}\lambda^a_{bi}B_a^{bi}. \tag{2.12}$$

Expressing the auxiliary field B_a^{bi} in terms of the physical fields

$$B_b^{ai}=\left(\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}\right)^{-1}\left(\frac{1}{4}\lambda^a_{bj}-\frac{1}{2}\frac{\partial^3 A}{\partial\phi^j\partial\phi^k\partial\phi^p}\times(\overline{\psi^k_a}\psi^{bp}+\overline{\psi}^{bk}\psi^p_a)\right), \tag{2.13}$$

using its equation of motion and inserting it back into the Lagrangian (2.10), one obtains a final form of the potential term

$$V=\frac{1}{16}\lambda^a_{bi}\lambda^b_{aj}\left(\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}\right)^{-1}+2\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}(m^i m^j+n^i\overline{n^j})+\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k}(2\overline{\psi^i_a}\psi^{aj}m^k+\psi^{ai}\psi^j_a n^k+\overline{\psi^i_a}\,\overline{\psi^{aj}}\,\overline{n^k})$$
$$-\frac{1}{2}\lambda^a_{bp}\left(\frac{\partial^2 A}{\partial\phi^p\partial\phi^k}\right)^{-1}\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k}\overline{\psi^i_a}\psi^{bj}+\frac{1}{2}\frac{\partial^4 A}{\partial\phi^i\partial\phi^j\partial\phi^k\partial\phi^l}(\overline{\psi^i_a}\,\overline{\psi^{ja}})(\psi^{bk}\psi^l_b)$$
$$-\left(\frac{\partial^2 A}{\partial\phi^p\partial\phi^q}\right)^{-1}\left(\frac{\partial^3 A}{\partial\phi^i\partial\phi^k\partial\phi^p}\frac{\partial^3 A}{\partial\phi^q\partial\phi^j\partial\phi^l}+\frac{1}{2}\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^p}\frac{\partial^3 A}{\partial\phi^q\partial\phi^k\partial\phi^l}\right)\overline{\psi^j_a}\,\overline{\psi^k_b}\psi^{bl}\psi^{ai}, \tag{2.14}$$

where the identity

$$\left(\frac{\partial^2 A}{\partial\phi^p\partial\phi^q}\right)^{-1}\frac{\partial^3 A}{\partial\phi^i\partial\phi^k\partial\phi^p}\frac{\partial^3 A}{\partial\phi^q\partial\phi^j\partial\phi^l}\overline{\psi^i_a}\,\overline{\psi^{al}}\psi^{jb}\psi^k_b$$
$$=\left(\frac{\partial^2 A}{\partial\phi^p\partial\phi^q}\right)^{-1}\frac{\partial^3 A}{\partial\phi^i\partial\phi^k\partial\phi^p}\frac{\partial^3 A}{\partial\phi^q\partial\phi^j\partial\phi^l}$$
$$\times(\overline{\psi^i_a}\psi^{aj}\overline{\psi^l_b}\psi^{bk}+\overline{\psi^i_a}\psi^{ak}\overline{\psi^l_b}\psi^{bj}) \tag{2.15}$$

was used.

The formulas given above can be rewritten in a different and more natural form using the geometrical notation. Let us introduce the metric of some ‘‘target’’ manifold in the following way:

$$g_{ij}=\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}, \tag{2.16}$$

along with the corresponding Christoffel connection and the Riemann curvature

$$\Gamma^i_{jk}=\frac{1}{2}\frac{\partial^3 A}{\partial\phi^p\partial\phi^j\partial\phi^k}\left(\frac{\partial^2 A}{\partial\phi^p\partial\phi^i}\right)^{-1}, \tag{2.17}$$

$$R_{ij,kl}=\frac{1}{4}\left(\frac{\partial^2 A}{\partial\phi^p\partial\phi^q}\right)^{-1}\times\left(\frac{\partial^3 A}{\partial\phi^i\partial\phi^l\partial\phi^p}\frac{\partial^3 A}{\partial\phi^q\partial\phi^j\partial\phi^k}-\frac{\partial^3 A}{\partial\phi^i\partial\phi^k\partial\phi^p}\frac{\partial^3 A}{\partial\phi^q\partial\phi^j\partial\phi^l}\right). \tag{2.18}$$

Now the Lagrangian (2.10), rewritten in terms of these geometric quantities looks as follows:

$$K=\frac{1}{2}g_{ij}\dot\phi^i\dot\phi^j+ig_{ij}(\overline{\psi^i_a}\dot\psi^{aj}+\psi^{ai}\dot{\overline{\psi^j_a}}), \tag{2.19}$$

and

$$V=\frac{1}{16}\lambda^a_{bi}\lambda^b_{aj}g^{ij}+2g_{ij}(m^i m^j+n^i\overline{n^j})+4\overline{\psi^i_a}\psi^a_j D_i m^j$$
$$+2\psi^{ai}\psi_{aj}D_i n^j+2\overline{\psi^i_a}\,\overline{\psi^a_j}D_i\overline{n^j}+\overline{\psi^i_a}\psi^{bj}D_i\lambda^a_{bj}$$
$$+(D_i\Gamma_{jkl}+R_{ik,lj})\overline{\psi^i_a}\,\overline{\psi^{aj}}\psi^{bk}\psi^l_b+R_{jl,ki}\overline{\psi^i_a}\psi^{aj}\overline{\psi^k_b}\psi^{bl}, \tag{2.20}$$

where D_i is a standard covariant derivative defined with the help of the introduced Christoffel connection (2.17). Using the Noether theorem technique, one can find the classical expressions for the conserved supercharges, corresponding to the SUSY transformations (2.8)

$$\overline{Q}_a=\overline{\psi^i_a}p_i-2i\overline{\psi^i_a}m^j\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}+2i\psi^i_a n^j\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}$$
$$+\frac{i}{2}\overline{\psi^i_c}\,\overline{\psi^{cj}}\psi^k_a\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k}-\frac{1}{2}i\lambda^c_{ai}\overline{\psi^i_c}, \tag{2.21}$$

$$Q^b=\psi^{ib}p_i+2i\psi^{bi}m^j\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}+2i\overline{\psi^{bi}}\,\overline{n^j}\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}$$
$$+\frac{i}{2}\overline{\psi^{bi}}\psi^{cj}\psi^k_c\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k}+\frac{1}{2}i\lambda^b_{di}\psi^{di}. \tag{2.22}$$

These formulas for the conserved supercharges complete the classical description of the desired $N=4$ SUSY multidimensional mechanics, and now to quantize it we should analyze its constraints.

Following the standard procedure of quantization of the system with bosonic and fermionic degrees of freedom [23], we introduce the canonical Poisson brackets

$$\{\phi^i,p_j\}=\delta^i_j,\quad\{\psi^{ai},p_{(\psi),bj}\}=-\delta^a_b\delta^i_j,$$
$$\{\overline{\psi^i_a},p^b_{(\overline{\psi}),j}\}=-\delta^a_b\delta^i_j, \tag{2.23}$$

where p_i, $p_{(\psi),ai}$, and $p^a_{(\overline{\psi}),i}$ are the momenta conjugated to ϕ^i, ψ^{ai}, and $\overline{\psi^i_a}$. From the explicit form of the momenta

$$p_i=g_{ij}\dot\phi^i, \tag{2.24}$$

$$p_{(\psi),ai}=-ig_{ij}\bar{\psi}^j_a,\quad p^a_{(\bar{\psi}),i}=-ig_{ij}\psi^{aj},\tag{2.25}$$

with the metric g_{ij} given by Eq. (2.16), one can conclude that the system possesses the second-class fermionic constraints

$$\chi_{(\psi),ai}=p_{(\psi),ai}+ig_{ij}\bar{\psi}^j_a,\ \text{and}\ \chi^a_{(\bar{\psi}),i}=p^a_{(\bar{\psi}),i}+ig_{ij}\psi^{aj},\tag{2.26}$$

since

$$\{\chi^a_{(\bar{\psi}),i},\chi_{(\psi),bj}\}=-2ig_{ij}\delta^a_b.\tag{2.27}$$

Therefore, the quantization has to be done using the Dirac brackets defined for any two functions V_a and V_b as

$$\{V_a,V_b\}_{\text{Dirac}}=\{V_a,V_b\}-\{V_a,\chi_c\}\frac{1}{\{\chi_c,\chi_d\}}\{\chi_d,V_b\}.\tag{2.28}$$

As a result, we obtain the following Dirac brackets for the canonical variables:

$$\{\phi^i,p_j\}_{\text{Dirac}}=\delta^i_j,$$
$$\{\psi^{ai},\bar{\psi}^j_b\}_{\text{Dirac}}=-\frac{i}{2}\delta^a_b\left(\frac{\partial^2A}{\partial\phi^i\partial\phi^j}\right)^{-1}=-\frac{i}{2}\delta^a_bg^{ij},$$
$$\{\psi^{ai},p_j\}_{\text{Dirac}}=-\frac{1}{2}\psi^{ap}\frac{\partial^3A}{\partial\phi^p\partial\phi^m\partial\phi^j}\left(\frac{\partial^2A}{\partial\phi^m\partial\phi^i}\right)^{-1}$$
$$=-\psi^{ak}\Gamma^i_{jk},$$
$$\{\bar{\psi}^i_a,p_j\}_{\text{Dirac}}=-\frac{1}{2}\bar{\psi}^p_a\frac{\partial^3A}{\partial\phi^p\partial\phi^m\partial\phi^j}\left(\frac{\partial^2A}{\partial\phi^m\partial\phi^i}\right)^{-1}$$
$$=-\bar{\psi}^k_a\Gamma^i_{jk},\tag{2.29}$$

and, finally,

$$\{p_i,p_j\}_{\text{Dirac}}=-\frac{i}{2}\left(\frac{\partial^2A}{\partial\phi^p\partial\phi^q}\right)^{-1}\left(\frac{\partial^3A}{\partial\phi^i\partial\phi^k\partial\phi^p}\frac{\partial^3A}{\partial\phi^q\partial\phi^j\partial\phi^l}-\frac{\partial^3A}{\partial\phi^i\partial\phi^l\partial\phi^p}\frac{\partial^3A}{\partial\phi^q\partial\phi^j\partial\phi^k}\right)\bar{\psi}^k_a\psi^{al}$$
$$=2iR_{ij,kl}\bar{\psi}^k_a\psi^{al}.\tag{2.30}$$

The classical Hamiltonian, obtained after the usual Legendre transformation from the Lagrangian (2.10), has the form

$$H_{\text{class}}=\frac{1}{2}\left(\frac{\partial^2A}{\partial\phi^i\partial\phi^j}\right)^{-1}p_ip_j+V.\tag{2.31}$$

The supercharges and the Hamiltonian form the following $N=4$ SUSY algebra with respect to the introduced Dirac brackets

$$\{\bar{Q}_a,Q^b\}_{\text{Dirac}}=-i\delta^b_aH_{\text{class}}-i\lambda^b_{ai}m^i,$$
$$\{\bar{Q}_a,\bar{Q}_b\}_{\text{Dirac}}=-i\lambda_{abi}n^i,$$
$$\{Q^a,Q^b\}_{\text{Dirac}}=i\lambda^{ab}_i\bar{n}^i.\tag{2.32}$$

Note the appearance of the central charges in the algebra. This fact is extremely important especially for the investigation of partial supersymmetry breaking, given in the next section.

Replacing the Dirac brackets by (anti)commutators using the rule

$$i\{,\}_{\text{Dirac}}=\{,\},\tag{2.33}$$

one obtains the quantum algebra

$$\{\bar{Q}_a,Q^b\}=\delta^b_aH_{\text{quant}}+\lambda^b_{ai}m^i,$$
$$\{\bar{Q}_a,\bar{Q}_b\}=\lambda_{abi}n^i,\quad\{Q^a,Q^b\}=-\lambda^{ab}_i\bar{n}^i,\tag{2.34}$$

under the definite choice of operator ordering in the supercharges (2.21), (2.22) and, in the Hamiltonian (2.31),

$$\bar{Q}_a=\bar{\psi}^i_aR_i-2i\bar{\psi}^i_am^j\frac{\partial^2A}{\partial\phi^i\partial\phi^j}+2i\psi^i_an^j\frac{\partial^2A}{\partial\phi^i\partial\phi^j}-\frac{1}{2}i\lambda^c_{ai}\bar{\psi}^i_c,\tag{2.35}$$

$$Q^b=L_i\psi^{bi}+2i\psi^{bi}m^j\frac{\partial^2A}{\partial\phi^i\partial\phi^j}+2i\bar{\psi}^{bi}\bar{n}^j\frac{\partial^2A}{\partial\phi^i\partial\phi^j}+\frac{1}{2}i\lambda^b_{di}\psi^{di},\tag{2.36}$$

$$H_{\rm quant}=\frac{1}{2}L_i\left(\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}\right)^{-1}R_j+\frac{1}{16}\lambda^a_{bi}\lambda^b_{aj}\left(\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}\right)^{-1}+2\frac{\partial^2 A}{\partial\phi^i\partial\phi^j}(m^im^j+n^i\overline{n^j})+\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k}([\overline{\psi}^i_a\psi^{aj}]m^k+\psi^{ai}\psi^j_a n^k$$
$$+\overline{\psi}^i_a\overline{\psi}^{aj}\overline{n^k})-\frac{1}{4}\lambda^a_{bp}\left(\frac{\partial^2 A}{\partial\phi^p\partial\phi^k}\right)^{-1}\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k}[\overline{\psi}^i_a,\psi^{bj}]+\frac{1}{2}\frac{\partial^4 A}{\partial\phi^i\partial\phi^j\partial\phi^k\partial\phi^l}(\overline{\psi}^i_a\overline{\psi}^{ja})(\psi^{bk}\psi^l_b)$$
$$-\left(\frac{\partial^2 A}{\partial\phi^p\partial\phi^q}\right)^{-1}\left(\frac{\partial^3 A}{\partial\phi^i\partial\phi^k\partial\phi^p}\frac{\partial^3 A}{\partial\phi^q\partial\phi^j\partial\phi^l}+\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^p}\frac{\partial^3 A}{\partial\phi^q\partial\phi^k\partial\phi^l}\right)\overline{\psi}^j_a\overline{\psi}^k_b\psi^{bl}\psi^{ai}, \tag{2.37}$$

where

$$L_i=p_i+i\overline{\psi}^j_a\psi^{ak}\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k}-\frac{i}{2}\left(\frac{\partial^2 A}{\partial\phi^j\partial\phi^k}\right)^{-1}\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k},$$
$$R_i=p_i-i\overline{\psi}^j_a\psi^{ak}\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k}+\frac{i}{2}\left(\frac{\partial^2 A}{\partial\phi^j\partial\phi^k}\right)^{-1}\frac{\partial^3 A}{\partial\phi^i\partial\phi^j\partial\phi^k}. \tag{2.38}$$

The momentum operators are Hermitian with respect to the integration measure $d^D\phi\sqrt{|\det(\partial^2 A/\partial\phi^i\partial\phi^j)|}$ if they have the following form:

$$p_i=-i\frac{\partial}{\partial\phi^i}-\frac{i}{4}\frac{\partial}{\partial\phi^i}\ln(|\det g_{ik}|)-2i\omega_{i\alpha\beta}\overline{\psi}^\alpha_a\psi^{a\beta}, \tag{2.39}$$

with the new fermionic variables $\overline{\psi}^\alpha_a$ and $\psi^{a\beta}$ connected with the old ones via the tetrad e^α_i $(e^\alpha_i e^\beta_j\eta_{\alpha\beta}=g_{ij})$

$$\overline{\psi}^\alpha_a=e^\alpha_i\overline{\psi}^i_a \quad\text{and}\quad \psi^\alpha_a=e^\alpha_i\psi^i_a, \tag{2.40}$$

and $\omega_{i\alpha\beta}$ in Eq. (2.39) is the corresponding spin connection. Therefore, the quantum supercharges (2.35), (2.36) are mutually Hermitian conjugated and the resulting quantum Hamiltonian $H_{\rm quant}$ is a Hermitian self-adjoint operator as well.

As a result, Eqs. (2.34)–(2.40) completely describe the general formalism of the $N=4$ SUSY D-dimensional quantum mechanics, and this provides the basis for the analysis of its main properties.

III. PARTIAL SUSY BREAKING

Let us investigate in detail the question of partial supersymmetry breakdown in the framework of the constructed $N=4$ SUSY QM in an arbitrary D number of dimensions. As it has been mentioned in the introduction, the problem of partially broken supersymmetry is very important for applications in supergravity, superstring theories and in the M theory as well, and the supersymmetric quantum mechanics turns out to be an adequate tool for investigating of the corresponding problems in supersymmetric field theories.

We shall see that in contrast with the one-dimensional $N=4$ SUSY QM, the multidimensional one provides also possibilities when either only one-quarter of all supersymmetries is exact (for $D\geqslant 2$), or one quarter of all supersymmetries is broken (for $D\geqslant 3$).

In order to study partial SUSY breaking it is convenient to introduce a new set of real-valued supercharges

$$S^a=\overline{Q}_a+Q^a, \tag{3.1}$$

$$T^a=i(\overline{Q}_a-Q^a). \tag{3.2}$$

In the above equations the SU(2) covariance is obviously damaged. This is the price we pay for passing to the real-valued supercharges. However, for a further discussion the loss of the covariance does not cause any problems. The label ‘‘a’’ has now to be considered as just the number of supercharges denoted by S and T.

The new supercharges form the following $N=4$ superalgebra with the central charges

$$\{S^a,S^b\}=H(\delta^a_b+\delta^b_a)+(\lambda^a_{bi}+\lambda^b_{ai})m^i+(\lambda_{abi}n^i-\lambda^{ab}_i\overline{n^i}), \tag{3.3}$$

$$\{T^a,T^b\}=H(\delta^a_b+\delta^b_a)+(\lambda^a_{bi}+\lambda^b_{ai})m^i-(\lambda_{abi}n^i-\lambda^{ab}_i\overline{n^i}), \tag{3.4}$$

$$\{S^a,T^b\}=i(\lambda^a_{bi}-\lambda^b_{ai})m^i+i(\lambda_{abi}n^i+\lambda^{ab}_i\overline{n^i}), \tag{3.5}$$

where $\lambda^a_{bi}=(\sigma_I)^a_b\Lambda^I_i$ and Λ^I_i are real parameters.

The algebra (3.3)–(3.5) is still nondiagonal. However, some particular choices of the constant parameters m^i, n^i, and Λ^I_i bring the algebra to the standard form, i.e., to the form when the right-hand side of Eq. (3.5) vanishes and the right-hand sides of Eqs. (3.3) and (3.4) are diagonal with respect to the indices ‘‘a’’ and ‘‘b.’’

Now we consider several cases separately.

A. Four supersymmetries exact and four supersymmetries broken

If we set equal to zero all central charges appearing in the algebra, then no partial breakdown of supersymmetry is possible. In this case, all supersymmetries are exact if the energy of the ground state is zero; otherwise all of them are broken. This statement is obviously independent of the number of dimensions D.

B. Two supersymmetries exact

The case of partial supersymmetry breakdown, when the half of supersymmetries are exact, have been considered earlier [13] in the framework of one-dimensional $N=4$ SUSY QM, but we shall describe it for completeness as well. Consider the one-dimensional $(D=1)$ $N=4$ SUSY QM and put all the constants entering into the right-hand sides of Eqs. (3.3)–(3.5) equal to zero, except

$$m^1 \text{ and } \Lambda_1^3. \tag{3.6}$$

Then, the algebra (3.3)–(3.5) takes the form

$$\{S^1,S^1\}=2H+2m^1\Lambda_1^3,$$

$$\{S^2,S^2\}=2H-2m^1\Lambda_1^3,$$

$$\{T^1,T^1\}=2H+2m^1\Lambda_1^3,$$

$$\{T^2,T^2\}=2H-2m^1\Lambda_1^3. \tag{3.7}$$

It means that if the energy of the ground state is equal to $m^1\Lambda_1^3$ and the last-mentioned product is positive, then S^2 and T^2 supersymmetries are exact, while the other two are broken. If $m^1\Lambda_1^3$ is negative, then S^1 and T^1 supersymmetries are exact provided the energy of the ground state is equal to $-m^1\Lambda_1^3$.

C. One supersymmetry exact

The case of the three-quarters breakdown of supersymmetry is possible if the dimension of $N=4$ SUSY QM is at least two $(D\geqslant 2)$. Indeed, for $D=2$ let us keep the following set of parameters nonvanished:

$$\Lambda_1^3,\Lambda_2^1,m^1 \text{ and } \mathrm{Re}(n^2)\equiv N^2. \tag{3.8}$$

Then, one obtains

$$\{S^1,S^1\}=2H+2m^1\Lambda_1^3-2\Lambda_2^1N^2,$$

$$\{S^2,S^2\}=2H-2m^1\Lambda_1^3+2\Lambda_2^1N^2,$$

$$\{T^1,T^1\}=2H+2m^1\Lambda_1^3+2\Lambda_2^1N^2,$$

$$\{T^2,T^2\}=2H-2m^1\Lambda_1^3-2\Lambda_2^1N^2. \tag{3.9}$$

A further choice

$$m^1\Lambda_1^3=\Lambda_2^1N^2 \tag{3.10}$$

leads to the case when only the T^2 supersymmetry is exact, while all others are broken if the energy of the ground state is equal to $2m^1\Lambda_1^3$, and $m^1\Lambda_1^3>0$. If $m^1\Lambda_1^3$ is negative, then T^1 is exact provided the energy of the ground state is equal to $-m^1\Lambda_1^3$.

D. Three supersymmetries exact

The situation of the one-quarter breakdown of supersymmetry can exist, if we add to the consideration one more dimension, i.e., consider the three-dimensional $D=3$ $N=4$ supersymmetric quantum mechanics.

Keeping the following set of the parameters nonvanished

$$\Lambda_1^3,\Lambda_2^1,\Lambda_3^2,m^1,N^2 \text{ and } \mathrm{Im}(n^3)\equiv M^3, \tag{3.11}$$

we have

$$\{S^1,S^1\}=2H+2m^1\Lambda_1^3-2\Lambda_2^1N^2-2\Lambda_3^2M^3,$$

$$\{S^2,S^2\}=2H-2m^1\Lambda_1^3+2\Lambda_2^1N^2-2\Lambda_3^2M^3,$$

$$\{T^1,T^1\}=2H+2m^1\Lambda_1^3+2\Lambda_2^1N^2+2\Lambda_3^2M^3,$$

$$\{T^2,T^2\}=2H-2m^1\Lambda_1^3-2\Lambda_2^1N^2+2\Lambda_3^2M^3. \tag{3.12}$$

If

$$m^1\Lambda_1^3=\Lambda_2^1N^2, \tag{3.13}$$

$$\Lambda_2^1N^2=-\Lambda_3^2M^3, \tag{3.14}$$

and

$$m^1\Lambda_1^3<0, \tag{3.15}$$

then T^2 supersymmetry is broken, while all others are exact under the condition that the energy of the ground state is equal to $-m^1\Lambda_1^3$. If the last-mentioned product is positive, then T^2 supersymmetry is exact, while all others are broken provided that the energy of the ground state is $3m^1\Lambda_1^3$ and we arrive at the three-dimensional generalization of case **C**.

Obviously, when considering the three-dimensional $N=4$ SUSY QM, one can either keep the parameters (3.8) under the condition (3.10), or the parameters (3.6), or set all of them equal to zero and, therefore, obtain all particular cases of spontaneous breakdown of supersymmetry discussed earlier. It is also obvious that all these cases can be obtained from the higher dimensional $(D\geqslant 3)$ $N=4$ supersymmetric quantum mechanics.

To summarize this section one should note that according to the given general analysis of partial SUSY breaking in the $N=4$ multidimensional SUSY QM, there exist possibilities of constructing the models with $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ supersymmetries unbroken, as well as models with totally broken or totally unbroken supersymmetries. However, the answer to the question which of these possibilities can be realized for the considered system, crucially depends on the form of the chosen superpotential and on the imposed boundary conditions of the quantum-mechanical problem.

IV. EXPLICIT EXAMPLE

For a better illustration of the ideas of the previous section it is useful to consider a particular choice of the superpotential $A(\Phi^i)$. As it has been mentioned before, to consider all possible cases of partial supersymmetry breakdown, the minimal amount of the superfields needed is three. Therefore, let us take three superfields of the type (2.6) and choose the constants $m^i, n^i, \overline{n^i}$ and Λ_i^I in accordance with expressions (3.6), (3.8), and (3.11).

A simple and at the same time interesting example is the case when the superpotential is the direct sum of terms, each being a function of only one superfield. This gives the possibility of the considerable simplification of the classical and quantum Hamiltonians, and the supercharges as well [13]. We choose the explicit form of the superpotential as

$$A(\Phi^i) = \Phi^i \ln \Phi^i, \quad i = 1,2,3 \tag{4.1}$$

and consider the physical bosonic components of the superfields Φ^i as functions of the new variables x^i, namely,

$$\phi^i = (x^i)^2. \tag{4.2}$$

Making the following redefinition of the fermionic variables:

$$\xi^{ai} = \psi^{ai}\sqrt{2\frac{\partial^2 A}{(\partial\phi^i)^2}}, \quad \overline{\xi^i_a} = \overline{\psi^i_a}\sqrt{2\frac{\partial^2 A}{(\partial\phi^i)^2}}, \tag{4.3}$$

where no summation over the repeated indices is assumed, one obtains the canonical commutation relations between fermions

$$\{\xi^{ai}, \overline{\xi^j_b}\} = \delta^a_b \delta^{ij}. \tag{4.4}$$

Inserting expressions (4.1), (4.2), and (4.3) into Eq. (2.37), one obtains the three-dimensional superconformal $N=4$ quantum mechanics [21] with

$$H_{\text{quant}} = H^1 + H^2 + H^3, \tag{4.5}$$

i.e., as it could be concluded from the fact that the superpotential is diagonal with respect to the superfields considered, the total Hamiltonian is also a direct sum of three Hamiltonians, each of them containing the bosonic and fermionic operators of only one type. The explicit form of the Hamiltonians H^i, $(i=1,2,3)$ is

$$\begin{aligned} H^1 = &-\frac{1}{8}\frac{d^2}{(dx^1)^2} + \frac{1}{4}\Lambda_1^3(\sigma_3)_a^b \overline{\xi^1_b}\xi^{a1} + \frac{1}{8}(\Lambda_1^3)^2(x^1)^2 \\ &+ \frac{1}{(x^1)^2}\left(2(m^1)^2 + \frac{3}{32} - m^1(\overline{\xi^1_a}\xi^{a1} - 1) - \frac{1}{4}\overline{\xi^1_a}\xi^{a1}\right. \\ &\left.+\frac{1}{8}(\overline{\xi^1_a}\xi^{a1})(\overline{\xi^1_b}\xi^{b1})\right), \end{aligned} \tag{4.6}$$

$$\begin{aligned} H^2 = &-\frac{1}{8}\frac{d^2}{(dx^2)^2} + \frac{1}{4}\Lambda_2^1(\sigma_1)_a^b \overline{\xi^2_b}\xi^{a2} + \frac{1}{8}(\Lambda_2^1)^2(x^2)^2 \\ &+ \frac{1}{(x^2)^2}\left(2(N^2)^2 + \frac{3}{32} - \frac{1}{2}N^2(\xi^{a2}\xi^2_a + \overline{\xi^2_a}\overline{\xi^{a2}})\right. \\ &\left.-\frac{1}{4}\overline{\xi^2_a}\xi^{a2} + \frac{1}{8}(\overline{\xi^2_a}\xi^{a2})(\overline{\xi^2_b}\xi^{b2})\right), \end{aligned} \tag{4.7}$$

$$\begin{aligned} H^3 = &-\frac{1}{8}\frac{d^2}{(dx^3)^2} + \frac{1}{4}\Lambda_3^2(\sigma_2)_a^b \overline{\xi^3_b}\xi^{a3} + \frac{1}{8}(\Lambda_3^2)^2(x^3)^2 \\ &+ \frac{1}{(x^3)^2}\left(2(M^3)^2 + \frac{3}{32} - \frac{i}{2}M^3(\xi^{a3}\xi^3_a - \overline{\xi^3_a}\overline{\xi^{a3}})\right. \\ &\left.-\frac{1}{4}\overline{\xi^3_a}\xi^{a3} + \frac{1}{8}(\overline{\xi^3_a}\xi^{a3})(\overline{\xi^3_b}\xi^{b3})\right). \end{aligned} \tag{4.8}$$

The next step is to find the energy spectrum of the quantum Hamiltonian (4.5).

Since the bosonic and fermionic variables of each type are completely separated, the eigenfunctions of the Hamiltonian (4.5) is a direct product of the eigenfunctions of the Hamiltonians (4.6)–(4.8) and the total energy is just a sum of the energies corresponding to the Hamiltonians H^i.

Let us find the energy spectrum of the Hamiltonian H^1. Consider the general state in the ''reduced'' Fock space spanned by the fermionic creation and annihilation operators $\overline{\xi^1_a}$ and ξ^{a1} obeying the anticommutation relations (4.4) with $i=1$:

$$|\rho\rangle = X_1(x^1)|0\rangle + Y_1^a(x^1)\overline{\xi^1_a}|0\rangle + Z_1(x^1)\overline{\xi^1_a}\overline{\xi^{a1}}|0\rangle. \tag{4.9}$$

The operator H^1, acting on the state vector (4.9), gives the following four Shrödinger equations for the unknown functions $X_1(x^1)$, $Y_1^a(x^1)$, and $Z_1(x^1)$:

$$\begin{aligned} &\left[-\frac{1}{2}\frac{d^2}{(dx^1)^2} + \frac{1}{2}(\Lambda_1^3)^2(x^1)^2\right. \\ &\left.\quad + \frac{1}{(x^1)^2}\left(8(m^1)^2 + 4m^1 + \frac{3}{8}\right)\right]X_1(x^1) \\ &= 4E_I^1 X_1(x^1), \end{aligned} \tag{4.10}$$

$$\begin{aligned} &\left[-\frac{1}{2}\frac{d^2}{(dx^1)^2} + \Lambda_1^3 + \frac{1}{2}(\Lambda_1^3)^2(x^1)^2\right. \\ &\left.\quad + \frac{1}{(x^1)^2}\left(8(m^1)^2 - \frac{1}{8}\right)\right]Y_1^1(x^1) \\ &= 4E_{II}^1 Y_1^1(x^1), \end{aligned} \tag{4.11}$$

$$\begin{aligned} &\left[-\frac{1}{2}\frac{d^2}{(dx^1)^2} - \Lambda_1^3 + \frac{1}{2}(\Lambda_1^3)^2(x^1)^2\right. \\ &\left.\quad + \frac{1}{(x^1)^2}\left(8(m^1)^2 - \frac{1}{8}\right)\right]Y_1^2(x^1) \\ &= 4E_{III}^1 Y_1^2(x^1), \end{aligned} \tag{4.12}$$

$$\left[-\frac{1}{2}\frac{d^2}{(dx^1)^2}+\frac{1}{2}(\Lambda_1^3)^2(x^1)^2+\frac{1}{(x^1)^2}\left(8(m^1)^2-4m^1+\frac{3}{8}\right)\right]Z_1(x^1)=4E^1_{IV}Z_1(x^1). \tag{4.13}$$

The wave functions and the energy spectrum of the Hamiltonian of the type

$$\mathcal{H}=-\frac{1}{2}\frac{d}{dx^2}+\frac{1}{2}x^2+g\frac{1}{x^2}, \tag{4.14}$$

have been investigated in detail for the nonsupersymmetric theory [24,25] and in the framework of the $N=2$ supersymmetric quantum mechanics [25–28] as well. The most detailed and complete study has been done by Das and Pernice [25] where the eigenfunctions and energy spectrum of the Hamiltonian of the type (4.14) were found after appropriate regularization of the potential and superpotential, depending on whether one considers nonsupersymmetric or $N=2$ supersymmetric problem. However, as it can be seen from Eqs. (4.1) and (4.2) the superpotential in our $N=4$ case for the Hamiltonian with the $1/x^2$ term in the potential energy is regular in contrast with the case of $N=2$ supersymmetric quantum mechanics and, therefore, we use the results of [25] which are obtained after the regularization of the potential, but not of the superpotental.

For the problem considered one obtains (we take the value of the parameter Λ_1^3 without loss of generality to be equal to $+1$)

For $m^1<-\frac{1}{4}$,

$$\begin{aligned}4E^1_I&=2k^1_I-4m^1,\\4E^1_{II}&=2k^1_{II}-4m^1+2,\\4E^1_{III}&=2k^1_{III}-4m^1,\\4E^1_{IV}&=2k^1_{IV}-4m^1+2,\end{aligned} \tag{4.15}$$

where $k^A_M=0,1,2,\ldots,$ $(A=1,2,3)$ and $(M=I,II,III,IV)$. Each energy level E^A_M corresponds to a couple (even and odd) of wave functions and, therefore, is doubly degenerate. The minimal energy corresponds to the minima of E^1_I and E^1_{III} for $k^1_I=k^1_{III}=0$ and equals $-m^1$. Let us denote the corresponding states by $\pi^{1\pm}_I$ and $\pi^{1\pm}_{III}$.

For $-\frac{1}{4}<m^1<0$ one has

$$\begin{aligned}4E^1_I&=2k^1_I+4m^1+2,\\4E^1_{II}&=2k^1_{II}-4m^1+2,\\4E^1_{III}&=2k^1_{III}-4m^1,\\4E^1_{IV}&=2k^1_{IV}-4m^1+2.\end{aligned} \tag{4.16}$$

The minimal energy corresponds to the minimum of E^1_{III} for $k^1_{III}=0$ and equals $-m^1$. We denote the corresponding ground states by $\pi^{1\pm}_{III}$.

For $0<m^1<\frac{1}{4}$ one has

$$\begin{aligned}4E^1_I&=2k^1_I+4m^1+2,\\4E^1_{II}&=2k^1_{II}+4m^1+2,\\4E^1_{III}&=2k^1_{III}+4m^1,\\4E^1_{IV}&=2k^1_{IV}-4m^1+2.\end{aligned} \tag{4.17}$$

The minimal energy is m^1 for $k^1_{III}=0$, and the corresponding ground state is again $\pi^{1\pm}_{III}$.

Finally, for $m^1>\frac{1}{4}$:

$$\begin{aligned}4E^1_I&=2k^1_I+4m^1+2,\\4E^1_{II}&=2k^1_{II}+4m^1+2,\\4E^1_{III}&=2k^1_{III}+4m^1,\\4E^1_{IV}&=2k^1_{IV}+4m^1.\end{aligned} \tag{4.18}$$

The minimal energy is m^1 for $k^1_{III}=k^1_{IV}=0$, and the corresponding ground states are $\pi^{1\pm}_{III}$ and $\pi^{1\pm}_{IV}$.

The points $\pm\frac{1}{4}$ and 0 are the branching points. When m^1 gets these values, the corresponding energies and wave functions of the system in the regions of the parameter, divided by these points, just coincide.

If we also choose $\Lambda_2^1=\Lambda_3^2=1$, the energy spectra of the Hamiltonians H^2 and H^3 are absolutely the same as in Eqs. (4.15)–(4.18). The only difference is that the parameter m^1 should be replaced by N^2 or M^3, respectively. However, the eigenfunctions, corresponding to E^2_I and E^2_{IV}, are linear combinations of the states of zero and two fermionic sectors since the fermionic number operator $\overline{\xi^2_a}\xi^{a2}$ does not commute with the Hamiltonian H^2. The energies E^2_{II} and E^2_{III} are also linear combinations of both the states of one fermionic sector because the matrix $(\sigma_1)_a^b$ is not diagonal. An analogous situation takes place for the Hamiltonian H^3.

Now we are in a position to describe partial supersymmetry breaking following the lines of the previous section.

First, let us consider the one-dimensional case with m^1 equal to zero. As mentioned above, the zero value of m^1 is the branching point and, therefore, the energy spectra (4.16) and (4.17) as well as the wave functions in these regions completely coincide. Therefore, one has a couple of supersymmetric ground states $\pi^{1\pm}_{III}$ and all supersymmetries are exact.

As it has been mentioned in the previous section, in order to describe the halfbreaking of supersymmetry it is enough to consider only the spectrum of the Hamiltonian H^1. Inserting the corresponding eigenvalues of the operator H^1 for each range of the parameter m^1 into Eqs. (3.7), one obtains that half of supersymmetries are always broken.

Considering the spectra of the Hamiltonians H^1 and H^2, one can obtain the three-quarter breakdown of supersymmetry. Indeed, from Eqs. (3.9), (3.10), and (4.15)–(4.18), one can conclude that either T^1 or T^2 supersymmetries are exact depending on the range of the parameter m^1. The corresponding ground-state wave functions obviously are, for $m^1<-\frac{1}{4}$,

$$\pi_I^{1\pm}\times\pi_I^{2\pm}, \quad \pi_I^{1\pm}\times\pi_{III}^{2\pm}, \quad \pi_{III}^{1\pm}\times\pi_I^{2\pm}, \quad \pi_{III}^{1\pm}\times\pi_{III}^{2\pm}, \tag{4.19}$$

for $-\frac{1}{4}<m^1<0$

$$\pi_{III}^{1\pm}\times\pi_{III}^{2\pm}, \tag{4.20}$$

for $0<m^1<\frac{1}{4}$

$$\pi_{III}^{1\pm}\times\pi_{III}^{2\pm}, \tag{4.21}$$

for $m^1>\frac{1}{4}$

$$\pi_{III}^{1\pm}\times\pi_{III}^{2\pm}, \quad \pi_{III}^{1\pm}\times\pi_{IV}^{2\pm}, \quad \pi_{IV}^{1\pm}\times\pi_{III}^{2\pm}, \quad \pi_{IV}^{1\pm}\times\pi_{IV}^{2\pm}. \tag{4.22}$$

In order to study the possibility of the one-quarter breakdown of supersymmetry, one has to consider the three-dimensional case, i.e., the spectra and the wave functions of the Hamiltonians H^1, H^2, and H^3. Using Eqs. (3.12), (3.13)–(3.15), and (4.15)–(4.18) one can conclude that for the considered model the one-quarter supersymmetry breakdown is impossible since the energy of the ground state equals $3m^1$ rather than m^1, as is required for the annihilation of the ground state by the operators S^1, S^2, and T^1. This obviously does not mean that one-quarter supersymmetry breakdown is impossible, in principle; it means instead that this effect is impossible for the simple model we considered.

Indeed, let us consider the same three-dimensional problem, but restricting ourselves to non-negative values of coordinate x^1, i.e., $x^1\geqslant 0$.

The spectrum of $\mathcal{H}$ Eq. (4.14), when x belongs to the non-negative half-axis is slightly different [24][2] and it opens the possibility of constructing the ground state which is invariant under three unbroken supersymmetries. According to [24], we have

$$E_k^{(\pm\alpha)}=2k\pm\alpha+1, \tag{4.23}$$

where α is given by

$$\alpha=+\frac{1}{2}\sqrt{1+8g}, \tag{4.24}$$

and k is the nonnegative integer. If $\alpha\geqslant 1$, then the energies $E_k^{(-\alpha)}$ must be excluded from the spectrum since the corresponding wave functions are no longer normalizable. Otherwise one has to consider both sets of solutions. Applying these results to the problem under consideration, and setting again $\Lambda_1^3=\Lambda_2^1=\Lambda_3^2=1$, one obtains, for H^1,

$$\begin{aligned}\alpha_I^1&=|4m^1+1|,\\ \alpha_{II}^1&=|4m^1|,\\ \alpha_{III}^1&=|4m^1|,\\ \alpha_{IV}^1&=|4m^1-1|.\end{aligned} \tag{4.25}$$

And, therefore, the energy spectra have the form

$$\begin{aligned}4E_I^{1,(\pm)}&=2k_I^1\pm|4m^1+1|+1,\\ 4E_{II}^{1,(\pm)}&=2k_{II}^1\pm|4m^1|+2,\\ 4E_{III}^{1,(\pm)}&=2k_{III}^1\pm|4m^1|,\\ 4E_{IV}^{1,(\pm)}&=2k_{IV}^1\pm|4m^1-1|+1.\end{aligned} \tag{4.26}$$

Both the signs before the second terms have to be taken for E_I if $-\frac{1}{2}<m^1<0$; for E_{II} and for E_{III} if $-\frac{1}{4}<m^1<\frac{1}{4}$; for E_{IV} if $0<m^1<\frac{1}{2}$. Let us further restrict the value of the parameter so that it belongs to the open interval $-\frac{1}{4}<m^1<0$. Then due to Eqs. (3.13)–(3.15), (4.16), (4.17), and (4.26), the minimal energy of the system with $k_{III}^1=k_{III}^2=k_{III}^3=0$ is

$$E_{\min}=E_{III}^{1,-}+E_{III}^{2,\pm}+E_{III}^{3,\pm}=-m^1, \tag{4.27}$$

and according to Eq. (3.12) we have the supersymmetric ground states with three supersymmetries being unbroken.

In this section we have considered quite schematically the one-, two-, and three-dimensional $N=4$ supersymmetric versions of the quantum oscillator with the additional $1/x^2$ term in the potential energy. However, we believe that even this simple analysis gives a good illustration of all possible cases of the partial supersymmetry breakdown in the multidimensional $N=4$ SUSY QM. One should also stress the crucial meaning of the boundary conditions in the question of partial supersymmetry breakdown, as it has been shown for the case of one-quarter supersymmetry breakdown in the considered example.

V. DISCUSSION

In this paper, we have described the general formalism of the multidimensional $N=4$ supersymmetric quantum mechanics and studied various possibilities of partial supersymmetry breaking, illustrating them by the exactly solvable example.

However, several questions, which seem to be of particular importance, are left still open. Indeed, it would be interesting to investigate other possibilities of changing the bosonic end fermionic variables, namely, for the cases, when in contrast with Eqs. (4.1) and (4.2), the superpotential $A(\Phi^i)$ is not a direct sum of the terms, each containing only one superfield Φ^i and when the bosonic components of these superfields depend on several variables x^i. A detailed study

[2] In fact, as it has been recently shown by Das and Pernice [25], the energy spectrum, obtained in [24] is correct if one considers the problem only on the half-axis.

of this problem can lead to possible $N=4$ supersymmetrization and quantization of various pure bosonic integrable systems such as *n*-particle Calogero and Calogero-Moser models, which are related to the Reissner-Nordström (RN) black hole quantum mechanics and to $D=2$ Super Yang-Mills (SYM) theory [29]. This approach can also answer the question about the general class of potentials which lead to superconformal $N=4$ theories in higher dimensions.

Another topic, which is left uncovered in this paper, is the possible application of the constructed multidimensional $N=4$ SUSY QM to the problems of quantum cosmology. Possible embedding of pure bosonic effective Lagrangians, derived from the homogeneous cosmological models to $N=4$ SUSY QM can shed new light on the old problems of boundary conditions and spontaneous SUSY breaking in quantum cosmology which have recently been investigated in the framework of $N=2$ supersymmetric σ-model approach [30,31]. All these questions are now under intensive study and will be reported elsewhere.

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[1] E. Witten, Nucl. Phys. **B188**, 513 (1981).
[2] E. Witten, Nucl. Phys. **B202**, 253 (1982).
[3] H. P. Nilles, Phys. Rep. **110**, 1 (1984); H. E. Haber and G. L. Kane, *ibid.* **117**, 75 (1985).
[4] M. Shifman and A. Vainshtein, ''Instantons versus Supersymmery: Fifteen Years Later,'' TPI-MINN-99/08-T, UMN-TH-1743/99; hep-th/9902018.
[5] I. Antoniadis, H. Partouche, and T. R. Taylor, Phys. Lett. B **372**, 83 (1996).
[6] E. Ivanov and B. Zupnik, Yad. Fiz. **62**, 1110 (1999) [Phys. At. Nucl. **62**, 1043 (1999)].
[7] P. Salomonson and J. W. van Holten, Nucl. Phys. **B196**, 509 (1982).
[8] R. Abbott, Z. Phys. C **20**, 213 (1983); R. Abbott and W. Zakrzewski, *ibid.* **20**, 227 (1983).
[9] A. Khare and J. Maharana, Z. Phys. C **23**, 191 (1984).
[10] M. Claudson and M. Halpern, Nucl. Phys. **B250**, 689 (1985).
[11] V. P. Berezovoj and A. I. Pashnev, Class. Quantum Grav. **8**, 19 (1991).
[12] V. P. Berezovoj and A. I. Pashnev, ''Superfield Description of $N=2$ Extended One-Dimensional Supersymmetric Quantum Mechanics,'' Report KFTI 91-20, Kharkov (1991).
[13] E. A. Ivanov, S. O. Krivonos, and A. I. Pashnev, Class. Quantum Grav. **8**, 19 (1991).
[14] E. Witten and D. Olive, Phys. Lett. **78B**, 97 (1978).
[15] J. Bagger and J. Wess, Phys. Lett. **138B**, 105 (1984); J. Hughes and J. Polchinski, Nucl. Phys. **B278**, 147 (1986); J. Hughes, J. Liu, and J. Polchinski, Phys. Lett. B **180**, 37 (1986); M. Rocek and A. A. Tseytlin, Phys. Rev. D **59**, 106001 (1999); J. Bagger and A. Galperin, Phys. Lett. B **336**, 25 (1994); Phys. Rev. D **55**, 1091 (1997); Phys. Lett. B **412**, 296 (1997); S. Ferrara, L. Girardello, and M. Poratti, *ibid.* **376**, 275 (1996).
[16] J. P. Gauntlett, Nucl. Phys. **B411**, 443 (1994).
[17] J. Blum, Phys. Lett. B **333**, 92 (1994).
[18] R. Kallosh, A. Linde, T. Ortin, A. Peet, and A. Van Proeyen, Phys. Rev. D **46**, 5278 (1992).
[19] K. S. Stelle, ''BPS branes in Supergravity,'' CERN-TH/98-80, Imperial/TP/97-98/30, hep-th/9803116; P. K. Townsend, Nucl. Phys. B (Proc. Suppl.) **67**, 88 (1998); N. Ohta and P. K. Townsend, Phys. Lett. B **418**, 77 (1998); J. P. Gauntlett, R. C. Myers, and P. K. Townsend, Phys. Rev. D **59**, 025001 (1999).
[20] V. A. Tsokur and Yu. M. Zinovev, Yad. Fiz. **59**, 2277 (1996) [Phys. At. Nucl. **59**, 2192 (1996)].
[21] E. Ivanov, S. Krivonos, and V. Leviant, J. Phys. A **22**, 4201 (1989).
[22] P. Claus, M. Derix, R. Kallosh, J. Kumar, P. K. Townsend, and A. Van Proeyen, Phys. Rev. Lett. **81**, 4553 (1998); J. A. de Azca'rraga, J. M. Izquierdo, J. C. Perez-Bueno, and P. K. Townsend, Phys. Rev. D **59**, 084015 (1999); R. Kallosh, ''Black Holes and Quantum Mechanics,'' hep-th/9902007; Jian-Ge Zhou, ''Super 0-Brane and GS Superstring Action on $adS_2\times S^2$,'' hep-th/9906013.
[23] R. Casalbuoni, Nuovo Cimento A **33**, 389 (1976).
[24] L. Lathouwers, J. Math. Phys. **16**, 1393 (1975).
[25] A. Das and S. Pernice, ''Supersymmetry and Singular Potentials,'' hep-th/9905135.
[26] V. Akulov and A. Pashnev, Theor. Math. Phys. **56**, 862 (1983).
[27] A. Jevicki and J. Rodrigues, Phys. Lett. **146B**, 55 (1984).
[28] S. Fubini and E. Rabinovici, Nucl. Phys. **B245**, 17 (1984).
[29] G. W. Gibbons and P. K. Townsend, Phys. Lett. B **454**, 187 (1999).
[30] J. Bene and R. Graham, Phys. Rev. D **49**, 799 (1994).
[31] E. E. Donets, M. N. Tentyukov, and M. M. Tsulaia, Phys. Rev. D **59**, 023515 (1999).