Calculation of density fluctuations in the inflationary epoch

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Starting from an initial state of thermal equilibrium, we derive an expression for the quantum fluctuation in the energy density during the inflationary epoch in terms of the mode functions for the inflaton field. The effect of this particular initial quantum state is not washed out in the final formula, contrary to what is usually believed. Numerically, however, the effect is completely negligible, validating the use of the two point function in the vacuum state. We also point out the requirement of conventional quantum field theory during inflation, that the quantum fluctuation in a wavelength must be evaluated, at the latest, when the wavelength crosses the Hubble length, in contrast with the usual practice in the literature.

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I. INTRODUCTION

The most attractive aspect of inflationary models of the early universe [1] is their potential to predict the present day density inhomogeneity from first principles [2]. In these models it is possible to calculate quantum fluctuation in the energy density on the homogeneous background in a region within the causal horizon (given by the Hubble length) during the inflationary epoch. This fluctuation provides the initial spectrum of density perturbation. As the region inflates into the observed universe or bigger, its propagation through different eras can be followed until the present time by the equations of linear perturbation theory of classical gravity [3–5].

In this work we consider some points in the calculation of quantum fluctuations during inflation. The basic ingredient is the expectation value of the product of two scalar field operators at a time when considerable inflation has already taken place. It is generally believed that as the inflation proceeds, the effects of all scales associated with a particular initial state tend to be wiped out, retaining only the extremely high energies associated with quantum fluctuations in the vacuum. So the expectation value is evaluated for the homogeneous background, which corresponds to the vacuum state of quantum field theory.

Here we investigate how far the density fluctuation becomes actually independent of the initial condition prevailing at the beginning of inflation. For this purpose, we start with a definite initial state, namely that of thermal equilibrium. The first attempt in this direction was by Guth and Pi [5]. We discuss it here in a more general framework [6,7]. The thermal propagator can be followed until the time when the quantum fluctuations are evaluated.

The existence of an initial thermal equilibrium distribution of particles, at least for the high wave numbers needed for the calculation of density fluctuation, appears quite probable. Even if the collision rates among the particles are too small to produce such a state, there could be another mechanism at work. As pointed out by Weinberg [8], the strong gravitational interaction at very early times would bring about thermal equilibrium, which, as we shall show, could be maintained at least until the beginning of inflation.

It must be pointed out here that the initial condition we discuss here concerns the computation of the scalar propagator in *quantum* field theory during the inflationary epoch. This condition has little to do with the usual initial conditions needed to have inflation at all, in particular, enough homogeneity of the *classical* scalar field on the scale of the Hubble radius at the begining of inflation [9]. Indeed, the volume of space in thermal equilibrium turns out to be only a small fraction of the Hubble volume at this time.

The other point we discuss is the time at which quantum fluctuation must be evaluated. It relates to the applicability of quantum field theory in curved space-time. As emphasized by DeWitt [10], conventional quantum field theory requires the mode functions to be oscillatory in time, allowing positive and negative frequencies to be identified. While on flat space-time such modes naturally arise for field theories describing physical particles, their existence is not guaranteed on space-times with non-zero curvature. The reason is that the curvature gives rise to a damping-like term in the equation of motion for the mode functions. In the inflationary period this makes a mode oscillatory or damped, according as the associated physical wavelength is smaller or bigger than the Hubble length. During this period physical wavelengths grow at a tremendous rate, while the Hubble length remains constant or approximately so. Thus even a wavelength lying initially deep inside the Hubble length would eventually go outside this length. So this time of exit marks the latest time at which we can evaluate the quantum fluctuation belonging to that particular wavelength.

In the literature, however, quantum fluctuations are actually evaluated at a time, when the modes have evolved well outside the Hubble length, so as to be frozen [11]. Of course, the complete problem of predicting the density fluctuation at the time of Hubble length reentry in a later radiation or matter dominated phase does involve the evolution of the fluctuation over a much longer period of time [4]. But the question at hand is where the quantum fluctuation can be evaluated reliably.

In Sec. II we review the derivation of the thermal scalar propagator in the early universe. In particular, we show the behavior of mode functions as the wavelengths grow and discuss the validity of the assumed initial thermal equilibrium state. In Sec. III we write the formula for density fluctuation in terms of the mode functions and discuss its dependence on the initial condition. As an example, we take the simple, original model of extended inflation [12] and comment on the earlier calculations of density fluctuation in this model in the context of the present work [13,14]. Finally our concluding remarks are contained in Sec. IV.

II. FINITE TEMPERATURE SCALAR PROPAGATOR

Consider a region of space in the early universe within the causal horizon. It can then be taken to be homogeneous and isotropic, admitting the (spatially flat) line element,

$$ds^2 = dt^2 - a^2(t) d\vec{x}^2, \tag{2.1}$$

where the scale factor $a(t)$ describes the expansion of the region of the universe. It constitutes the background space-time, which is perturbed by quantum fluctuations. The action for the scalar field in this space-time may be generally written as

$$S_\phi = \frac{1}{2}\int d^3x\, dt\, a^3(t)\left\{\dot{\phi}^2 - \frac{1}{a^2(t)}(\nabla\phi)^2 - \mu^2(t)\phi^2 - \lambda_1(t)\phi^3 - \lambda_2(t)\phi^4 + \cdots\right\}. \tag{2.2}$$

Here we have already shifted the scalar field by its homogeneous classical part, if any. The dot on ϕ indicates differentiation with respect to time and the other dots indicate interaction terms, if any, of the scalar field with other (gauge and matter) fields.

We assume the different species of particles to be in thermal equilibrium around some initial time t_0, which we conveniently take to be the time of transition of the radiation dominated phase to the inflationary phase. In particular, the scalar particles belonging to $\phi(x)$ are also assumed to be in thermal equilibrium. (This assumption will be examined at the end of this section.) The density matrix is then given by

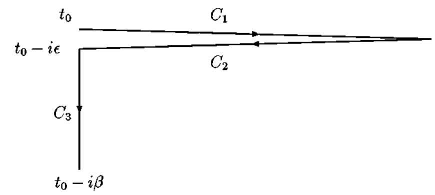

FIG. 1. Time path of real time thermal field theory.

$$\rho = e^{-\beta_0 \mathcal{H}(t_0)} / Tr e^{-\beta_0 \mathcal{H}(t_0)}, \tag{2.3}$$

where $1/\beta_0 = T(t_0)$ is the temperature at time t_0. The explicit time dependence of the Hamiltonian $\mathcal{H}(t)$ arises from that of the scale factor and the homogeneous classical field. Note that the density matrix is constant in the Heisenberg representation. Thus once the system is in the thermal equilibrium state, the thermal propagator continues to hold even when the system deviates from this state.

To describe the time evolution of the system, it is most convenient to use the real time formulation of thermal field theory [15]. In the context of the early universe, the time path C in the action integral consists of three segments as shown in Fig. 1 [6]. The points on it may be labeled by a complex parameter τ such that

$$\tau = \begin{cases} t & \text{on } C_1 \text{ and } C_2, \\ t_0 - it & \text{on } C_3. \end{cases} \tag{2.4}$$

The action in the path integral corresponding to the segments C_1 and C_2 is in Minkowski space, while it is Euclidean on C_3. It should be noted, however, that the scale factor is not continued to Euclidean space on C_3: since the Hamiltonian for the segment is $\mathcal{H}(t_0)$, the scale factor remains fixed at t_0.

We now review the derivation of the thermal propagator [6,7]. After a partial integration, the quadratic terms in S_ϕ becomes [16]

$$S_0 = -\frac{1}{2}\int_C d\tau \int d^3x\, \phi D \phi + \text{boundary terms}, \tag{2.5}$$

where

$$D = \begin{cases} a^3\left(\dfrac{d^2}{d\tau^2} + 3\dfrac{\dot{a}}{a}\dfrac{d}{d\tau} + \omega^2\right), & \omega^2(\tau) = -\dfrac{1}{a^2(\tau)}\nabla^2 + \mu^2(\tau), \quad \tau \in C_1, C_2 \\ a_0^3\left(\dfrac{d^2}{d\tau^2} + \omega_0^2\right), & \omega_0^2 = -\dfrac{1}{a_0^2}\nabla^2 + \mu^2(t_0), \quad \tau \in C_3. \end{cases} \tag{2.6}$$

We use the abbreviations, $a_0 = a(t_0)$, $\omega_0 = \omega(t_0)$, $T_0 = T(t_0)$. For the boundary terms to vanish, it is necessary that both ϕ and $d\phi/d\tau$ match at the joining of the segments C_1 and C_2, of C_2 and C_3 as well as at the free ends of C_3 and C_1.

The (time ordered) thermal propagator $Tr\, \rho T \phi(x)\phi(x')$ will be denoted by $G_\beta(x,\tau;x',\tau')$ or $\langle T\phi(x)\phi(x')\rangle$. It satisfies

$$DG_\beta(\vec{x},\tau;\vec{x}',\tau') = -i\delta^3(\vec{x}-\vec{x}')\delta(\tau-\tau'), \tag{2.7}$$

with boundary conditions following from the matching of ϕ

and $d\phi/d\tau$ mentioned above. For the spatial Fourier transform of the propagator, defined by

$$G_\beta(\vec{x},\tau,\tau')=\int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot\vec{x}}G_\beta(\vec{k},\tau,\tau'), \tag{2.8}$$

it reduces to

$$DG_\beta(\vec{k};\tau,\tau')=-i\delta(\tau-\tau'), \tag{2.9}$$

where $-\nabla^2$ appearing in the expressions for $\omega^2(\tau)$ and ω_0^2 is to be replaced now by k^2.

To construct the propagator we first find the mode functions. On the contour C_3 they satisfy

$$\left(\frac{d^2}{d\tau^2}+\omega_0^2\right)h^{\pm}(\tau)=0, \tag{2.10}$$

giving

$$h^{\pm}(\tau)=\frac{1}{\sqrt{2\omega_0 a_0^3}}e^{\mp i\omega_0\tau},\quad \tau=t_0-it\in C_3. \tag{2.11}$$

The normalization satisfies the Wronskian condition, $\dot{h}^+(\tau)h^-(\tau)-\dot{h}^-(\tau)h^+(\tau)=-i/a_0^3$. The mode functions on the real segments C_1 and C_2 are the solutions of

$$\left(\frac{d^2}{d\tau^2}+3\frac{\dot{a}}{a}\frac{d}{d\tau}+\omega^2(\tau)\right)g^{\pm}(\tau)=0,\quad \tau=t\in C_{1,2}, \tag{2.12}$$

with normalization fixed again by the Wronskian condition, $\dot{g}^+(\tau)g^-(\tau)-\dot{g}^-(\tau)g^+(\tau)=-i/a^3(\tau)$. To see the nature of these solutions, we put

$$g^{\pm}(\tau)=a^{-3/2}\overline{g}^{\pm}(\tau), \tag{2.13}$$

where $\overline{g}$ satisfies

$$\left(\frac{d^2}{d\tau^2}+\overline{\omega}^2(\tau)\right)\overline{g}^{\pm}(\tau)=0, \tag{2.14}$$

with

$$\overline{\omega}^2(t)=\frac{k^2}{a^2}+\mu^2-\frac{9}{4}\left(H^2+\frac{2}{3}\dot{H}\right),\quad H(t)=\frac{\dot{a}(t)}{a(t)}. \tag{2.15}$$

For a power law behavior of the scale factor, $a(t)\sim t^p$, it becomes

$$\overline{\omega}^2(t)=\frac{k^2}{a^2}+\mu^2-\frac{9}{4}\left(1-\frac{2}{3p}\right)H^2. \tag{2.16}$$

It is now simple to identify the modes, which belong to conventional quantum field theory. The magnitude of $\mu(t)$ is usually small compared to $H(t)$. Thus in the radiation dominated phase ($p=\frac{1}{2}$), $\overline{\omega}^2$ is positive for all values of k and it may be possible to define oscillatory modes belonging to positive and negative frequencies, at least in a quasi-static way. We show below that this is indeed the case around the time t_0, when we can solve Eq. (2.14) in the JWKB approximation to get [17]

$$\overline{g}^{\pm}(\tau)=\frac{1}{\sqrt{2\overline{\omega}(\tau)}}e^{\mp i\int_{t_0}^{\tau}d\tau'\overline{\omega}(\tau')},\quad \tau\simeq t_0. \tag{2.17}$$

We thus have a valid quantum field theory around the time t_0.

But in the inflationary phase ($p\gg 1$), a mode is oscillatory only if its physical wavelength $2\pi a(t)/k$ is small compared to the Hubble length $H^{-1}(t)$. As inflation progresses, the scale factor increases enormously, while $H(t)$ is approximately constant. Thus more and more modes go out of Hubble length and behave as damped waves, having no interpretation in quantum field theory.

The solutions $g^{\pm}(\tau)$ and $h^{\pm}(\tau)$ may now be joined to form the functions $f^{\pm}(\tau)$ on the entire contour C,

$$f^{\pm}(\tau)=\begin{cases} g^{\pm}(\tau), & \tau\in C_{1,2}\\ h^{\pm}(\tau), & \tau\in C_3. \end{cases} \tag{2.18}$$

By definition, $f^{\pm}(\tau)$ obey the continuity conditions relating the segments C_1 and C_2. Using Eqs. (2.11) and (2.17), we see that the conditions connecting C_2 and C_3 are also well satisfied if $H(t_0)$ is small compared to $k/a(t_0)$ [18]. A particular solution to Eq. (2.9) may now be written as

$$\begin{aligned}G_0(k:\tau,\tau')=&f^+(\tau)f^-(\tau')\theta_c(\tau-\tau')\\&+f^+(\tau')f^-(\tau)\theta_c(\tau'-\tau),\end{aligned} \tag{2.19}$$

where θ_c is a step function on the contour. This solution satisfies the continuity conditions at the junctions of segments C_1 and C_2 as well as of C_2 and C_3, because $f^{\pm}(\tau)$ does it. To satisfy the remaining (thermal) continuity condition at the ends of C_1 and C_3, we add to it the most general solution of the homogeneous equation,

$$G_\beta(k;\tau,\tau')=G_0(k,\tau,\tau')+\sum_{i,j=1}^{2}f^i(\tau)\Lambda^{ij}f^j(\tau'). \tag{2.20}$$

The superscript ($\pm$) on the mode functions are replaced temporarily by 1 and 2 to use matrix notation. The 2×2 constant coefficient matrix Λ is uniquely determined by the thermal conditions [19]. We get

$$\begin{aligned}G_\beta(k,\tau,\tau')=&f^+(\tau)f^-(\tau')\{\theta_c(\tau-\tau')+n(\omega_0)\}\\&+f^-(\tau)f^+(\tau')\{\theta_c(\tau'-\tau)+n(\omega_0)\},\end{aligned} \tag{2.21}$$

where $n(\omega_0)$ is the bosonic distribution function

$$n(\omega_0)=(e^{\beta_0\omega_0}-1)^{-1}. \tag{2.22}$$

For tree level calculations we need the Green's function only on the real axis C_1. Writing henceforth $\tau=t$, this is given by

$$\langle \phi(\vec{x},t)\phi(\vec{x}',t')\rangle = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}(1+n(\omega_0))g^+(t)g^-(t'), \quad t>t', \tag{2.23}$$

where the mode functions $g^{\pm}(t)$ are solutions of Eq. (2.12).

We now come back to the assumption of the initial thermal equilibrium state. Such an initial state can be ensured in an expanding universe if collisions among particles occur at a rate faster than the expansion rate of the universe. While this condition holds for species interacting through (relatively large) gauge coupling, it may not hold for particles of the inflaton field, which is a gauge singlet and has weak self-interaction. We discuss below the other mechanism, mentioned in the introduction, which could give rise to thermal equilibrium around the time t_0.

At the Planck time t_P, the strong gravitational interaction brings about thermal equilibrium for all species [8]. Let us quantize the system at this time in a cubic volume with sides of physical length, $L(t_P)$, small compared to the Hubble length, $L(t_P)<H^{-1}(t_P)\sim m_P^{-1}$, where m_P is the Planck mass. Then the longest wavelength will be well inside this length, i.e.

$$\frac{k}{a(t_P)}>\pi H(t_P), \tag{2.24}$$

even for the smallest wave number. Then Eq. (2.16) simplifies, to a good approximation, to that for a massless particle,

$$\bar{\omega}(t_P)=\frac{k}{a(t_P)}. \tag{2.25}$$

So the density distribution at the Planck time becomes

$$n(\omega(t_P))=\frac{1}{e^{k/a(t_P)T(t_P)}-1}. \tag{2.26}$$

The inequality (2.24), in turn, causes the wave numbers at the time t_0 to satisfy

$$\frac{k}{a(t_0)}>\frac{a(t_P)}{a(t_0)}\frac{H(t_P)}{H(t_0)}\pi H(t_0)=\frac{m_P}{T_0}\pi H(t_0). \tag{2.27}$$

In the last step we have used the radiation dominated solution for $a(t)$. The temperature T_0 is given by the grand unification scale, $T_0\sim 10^{15}$ GeV, so that $m_P/T_0\sim 10^5$. Thus the relation (2.24) at time t_P continues to hold throughout the radiation dominated phase; in fact, it becomes more and more accurate as t increases from t_P. Clearly the equilibrium distribution (2.26) established at time t_P is well maintained at least until the time t_0.

III. DENSITY FLUCTUATION FORMULA

The density inhomogeneity (at time t) is measured by the mean square fluctuation in the density function $\rho(\vec{x},t)$ [20,21],

$$\left(\frac{\delta\rho}{\rho}\right)^2=\left\langle\left(\frac{\rho(\vec{x},t)-\bar{\rho}(t)}{\bar{\rho}(t)}\right)^2\right\rangle_x, \tag{3.1}$$

where $\langle\cdots\rangle_x$ denotes averaging over space and $\bar{\rho}$ is the homogeneous background density, $\bar{\rho}=\langle\rho(\vec{x},t)\rangle_x$. In the inflationary scenario, this inhomogeneity in the early universe is supposed to arise from quantum fluctuation in the energy density on a homogeneous background. We may calculate the latter by evaluating an expression similar to Eq. (3.1), replacing $\rho(\vec{x},t)$ by the corresponding operator $\hat{\rho}(\vec{x},t)$ and the averaging by the expectation value in an appropriate quantum state.

There is, however, a technical problem with this quantum version, as it involves the product of $\hat{\rho}(\vec{x},t)$ with itself at the same space-time point, which is not defined in quantum field theory. The problem may be avoided by taking the smeared density function [5]

$$\rho_l(\vec{x},t)=N\int d^3y e^{-y^2/2l^2}\rho(\vec{x}+\vec{y},t), \tag{3.2}$$

where l is an arbitrary smearing length and N an irrelevant normalization factor. The classical fluctuation in ρ_l is then given by

$$\left(\frac{\delta\rho_l}{\rho_l}\right)_c^2=\left\langle\left(\frac{\rho_l(\vec{x},t)-\bar{\rho}_l(t)}{\bar{\rho}_l(t)}\right)^2\right\rangle_x, \tag{3.3}$$

where the subscript c stands for classical. The corresponding quantum fluctuation, denoted by the subscript q, is now well defined,

$$\left(\frac{\delta\rho_l}{\rho_l}\right)_q^2=\left\langle\left(\frac{\hat{\rho}_l(\vec{x},t)-\bar{\rho}(t)}{\bar{\rho}(t)}\right)^2\right\rangle, \tag{3.4}$$

where $\langle\cdots\rangle$ stands for the expectation value in the initial thermal state defined by Eq. (2.3).

To treat perturbation on different length scales, one writes

$$\rho(\vec{x},t)=\bar{\rho}(t)(1+\delta(\vec{x},t)), \tag{3.5}$$

and Fourier analyzes the so-called density contrast, $\delta(\vec{x},t)$,

$$\delta(\vec{x},t)=\frac{1}{\sqrt{V}}\sum_k \delta_k(t)e^{i\vec{k}\cdot\vec{x}}, \tag{3.6}$$

where V is a volume within the Hubble length. In the limit of large volume, Eq. (3.3) becomes

$$\left(\frac{\delta\rho_l}{\rho_l}\right)_c^2=\int\frac{d^3k}{(2\pi)^3}|\delta_k(t)|^2e^{-k^2l^2}. \tag{3.7}$$

The density inhomogeneity is conventionally expressed as

$$\left(\frac{\delta\rho}{\rho}\right)_k^2 \equiv \frac{k^3|\delta_k(t)|^2}{2\pi^2}. \tag{3.8}$$

The energy density operator $\hat{\rho}(\vec{x},t)$ is obtained from the time-time component of energy momentum tensor,

$$T_{\mu\nu}=\partial_\mu\Psi\partial_\nu\Psi-g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\Psi\partial_\beta\Psi-V(\Psi)\right), \tag{3.9}$$

for the full scalar field $\Psi(x)$. Here the potential function $V(\Psi)$ depends on the model considered. We shift $\Psi(x)$ by the homogeneous classical field $\psi(x)$,

$$\Psi(\vec{x},t)=\psi(t)+\phi(\vec{x},t), \tag{3.10}$$

such that for the quantum field, $\langle\phi(x)\rangle=0$. Then $\hat{\rho}(x)$ in quantum theory may be written as

$$\hat{\rho}(x)=\overline{\rho(t)}+\hat{U}(x), \tag{3.11}$$

where the two terms are

$$\overline{\rho(t)}=\frac{1}{2}\dot{\psi}^2+V(\psi), \tag{3.12}$$

and

$$\hat{U}(x)=r(t)\phi(x)+s(t)\dot{\phi}(x), \tag{3.13}$$

to first order in $\phi(x)$. The coefficients $r(t),s(t)$ in Eq. (3.13) depend on the classical field and other parameters in the potential $V(\Psi)$. Terms in $\hat{U}$, which are of higher order in ϕ would give loop contribution to the density fluctuation and are neglected. Note here also the expression for the homogeneous pressure,

$$\bar{p}=\frac{1}{2}\dot{\psi}^2-V(\psi). \tag{3.14}$$

The expectation value in Eq. (3.4) may now be evaluated to give

$$\begin{aligned}\left(\frac{\delta\rho_l}{\rho_l}\right)_q^2 &= \frac{1}{\bar{\rho}^2(t)}\int d^3x d^3y e^{-(x^2+y^2)/2l^2}\langle\hat{U}(x,t)\hat{U}(y,t)\rangle \\ &= \frac{1}{\bar{\rho}^2(t)}\int\frac{d^3k}{(2\pi)^3}(1+n(\omega_0)) \\ &\quad\times|r(t)g_k(t)+s(t)\dot{g}_k(t)|^2e^{-k^2l^2},\end{aligned} \tag{3.15}$$

where we used Eq. (2.23) for the two point function.

As a wavelength crosses the Hubble length at time t_h, $[k/a(t_h)=2\pi H(t_h)]$, we identify the associated quantum fluctuation with the classical density inhomogeneity. Comparing Eq. (3.7) with Eq. (3.15) we immediately get

$$|\delta_k(t_h)|^2=\frac{1}{\bar{\rho}^2}(1+n(\omega_0))|r(t_h)g_k(t_h)+s(t_h)\dot{g}_k(t_h)|^2. \tag{3.16}$$

It now evolves as linear perturbation in classical gravity [3]. It first oscillates with constant amplitude until $k/aH\sim 1$, and then its evolution as super-horizon sized perturbation is gauge dependent, until its re-entry within the horizon after inflation. But it turns out that the quantity, $\delta\rho/(\bar{\rho}+\bar{p})$, has a gauge invariant meaning: its magnitude on reentry has approximately the same value as it had at exit during inflation [4,20]. Assuming radiation dominance at re-entry, we thus get

$$\left(\frac{\delta\rho}{\rho}\right)_H=\frac{2\sqrt{2}}{3\pi}\sqrt{k^3(1+n(\omega_0))}\frac{|rg_k+s\dot{g}_k|_{t_h}}{(\bar{\rho}+\bar{p})_{t_h}}, \tag{3.17}$$

where the subscript H denotes horizon reentry.

It is simple to estimate $n(\omega_0)$ in the range of k/a_0, which is of interest. We write

$$\frac{k}{a_0}=\frac{k}{a(t_p)}\frac{a(t_p)}{a(t_e)}\frac{a(t_e)}{a(t_0)}. \tag{3.18}$$

From the time t_e when the inflation ends until the present time t_p, the universe expands adiabatically, so that $a(t_p)/a(t_e)\simeq T_0/T_p$. The other ratio $a(t_e)/a(t_0)\equiv Z$ gives the magnitude of inflation. We thus get

$$\frac{k}{a_0T_0}=\frac{2\pi}{\lambda(t_p)}\frac{Z}{T_p}\sim\frac{1}{\lambda_{Mpc}}\frac{Z}{10^{25}}, \tag{3.19}$$

where $T_p=2.7\ \mathrm{K}=11.8\ \mathrm{cm}^{-1}$ and λ_{Mpc} is $\lambda(t_p)$ expressed in Mpc. The wavelengths of interest stretch over the range $1<\lambda_{Mpc}<10^4$. In all models of inflation Z exceeds 10^{29} by many orders of magnitude. Thus k/a_0T_0 is large in these models and we may set $n(\omega_0)=0$ in the expression (3.17).

We thus see that although the initial thermal equilibrium state does produce a factor in the expression for the density inhomogeneity, its magnitude turns out to be unity, justifying the use of zero temperature propagator for its evaluation. Nevertheless it is important to know the initial state, as there are other quantities, such as the duration of inflation, which may depend sensitively on it.

In the discussion so far, we have been implicitly assuming that the initial thermal region of physical length $L(t_P)\sim m_P^{-1}$ at the Planck time grows to a size $L(t_p)$, which must be at least of order 10^{28}, the observed size of the universe. It is simple to check that this is indeed the case. The two lengths are related by

$$L(t_p)=L(t_P)\frac{a(t_p)}{a(t_P)}.$$

Evaluating the ratio of scale factors, we get

$$L(t_p)=\frac{1}{10}L(t_P)m_P Z\ \text{ cm}.$$

Thus $L(t_p)$ can easily exceed 10^{28} cm.

Finally we consider the example of the well-studied, original model of extended inflation [12] in the context of the ideas presented here. Attaining thermalization due to gravity in the Planck era is relevant for this model, as the collision rate of the scalar particles is known to be too small to produce it. Also the inflationary solution for the homogeneous classical field can be shown to join smoothly to its constant values during the two radiation dominated eras, before and after the inflation. We have already evaluated $(\delta\rho/\rho)_H$ in this model using the formula (3.17), which we now justify quite generally [14]. The difference in the time at which the mode functions are evaluated does show up in the numerical evaluation: Our estimate is about an order of magnitude bigger than that of others [13].

IV. CONCLUSION

In the present work we assume the inflationary epoch to begin in a state of thermal equilibrium and study its effect on the quantum fluctuation in the energy density calculated during this epoch. This initial quantum state including the scalar particles appears quite likely even if their self-interaction is too feeble to ensure it. We show that the thermal equilibrium established at very early times through the then strong gravitational interaction would be maintained until the beginning of inflation. By evaluating the scalar field propagator with thermal boundary conditions, we find a result for the density fluctuation, which differs from the one calculated with the vacuum propagator by the factor $\{1+(e^{k/a_0T_0}-1)^{-1}\}$. Clearly the factor does not go to unity as time passes but is a constant depending on the physical wave number and the temperature referred to the initial time t_0.

It turns out, however, that for wave numbers of interest in the present universe, this factor is unity in models where the amount of inflation exceeds by many orders of magnitude the minimal amount required to solve the problems of standard cosmology. Thus numerically the calculation of fluctuation in the vacuum state is justified.

We also point out that the conventional quantum field theory applies on curved space-time as long as the modes oscillate. This requires that we evaluate the quantum fluctuation, at the latest, when the corresponding wavelength crosses the Hubble length. Previous works [11], however, evaluate it as a rule for wavelengths well outside this length, where the modes freeze. As we have shown in a recent work [14], this difference in the calculation leads to an increase in the result by about an order of magnitude for the model of extended inflation.

Finally we note that during inflation as long as the modes are within the Hubble length, they retain a thermal equilibrium distribution. Thus although the initial state of thermal equilibrium comprising all modes is not maintained during inflation, the modes relevant for the calculation of density fluctuation are those still in an equilibrium distribution.

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[16] The quadratic action is generally improved by replacing $\mu(t)$ with the effective mass the scalar particle acquires at finite temperature through the self- and gauge-interactions. But for the inflaton field, which has no gauge interaction and is only weakly self-coupled, this modification is not significant.

[17] The approximation is valid for $\dot{\omega}\ll\omega^2$. In the present work where $\mu(t)$ is assumed small compared to $H(t)$, this condition is equivalent to $k/a(t)\gg H(t)$. In Ref. [7], it is ensured for all physical momenta in a different way, viz. by assuming a large thermal mass for the scalar particle.

[18] Let us mention here the problem associated with the density matrix (2.3) defined sharply at time t_0. This procedure of instant thermalization leads to additional short distance singu-

larities in the propagator, not present at zero temperature, making the theory non-renormalizable. Here we thermalize the system in a static background with constant scale factor prior to t_0 and then connect it smoothly to the actual scale factor. The resulting thermal propagator does not depend on the details of the interpolating scale factor, if the condition $k/a(t_0) \gg H(t_0)$ is satisfied. See Ref. [7] for more details.

[19] It is interesting to note that although we insisted on oscillatory modes with positive and negative frequencies to define quantum field theory, the propagator (2.20) is invariant under a reparametrization mixing the mode functions, $f^+(\tau) \rightarrow \alpha f^+(\tau) + \beta f^-(\tau)$, $f^-(\tau) \rightarrow \gamma f^+(\tau) + \delta f^-(\tau)$, the constant coefficients α, β, γ and δ satisfying $\alpha\delta - \beta\gamma = 1$. It is a property of the thermal propagator also on flat space-time.

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