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Holographic bound in Brans-Dicke cosmology

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We apply the holographic principle to Brans-Dicke cosmology. We analyze the holographic bound in both the Jordan and Einstein frames. The holographic bound is satisfied for both the $k=0$ and $k=-1$ universes, but it is violated for the $k=1$ matter-dominated universe.

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I. INTRODUCTION

In black hole theory, we know that the total entropy of matter inside a black hole cannot be greater than the Bekenstein-Hawking entropy, which is 1/4 of the area of the event horizon of the black hole measured in Planck units [1]. The extension of this statement to more general situations leads to the holographic principle [2]. The most radical version of the holographic principle motivated by the AdS conformal field theory (CFT) conjecture is that all information about a physical system in a spatial region is encoded in the boundary. The application of this idea to cosmology was first considered by Fischler and Susskind [3]. Because the universe does not have a boundary, how can we apply the holographic principle to cosmology? Fischler and Susskind answered this question by considering a space inside the particle horizon. They proposed that the matter entropy inside a spatial volume of the particle horizon would not exceed 1/4 of the area of the particle horizon measured in Planck units. They found that flat universes and open universes obeyed this version of the holographic principle. However, a closed universe violates this principle. This may imply that our universe is flat or open. On the other hand, this may imply that we need to revise the holographic principle somehow. Easther and Lowe use the generalized second law of thermodynamics to replace the holographic principle [4]. Bak and Rey [5] considered an apparent horizon instead of an event horizon to solve the problem. In cosmology, there is a natural choice of length scale, the Hubble distance, H^{-1}. H^{-1} coincides with the particle horizon and apparent horizon apart from an order 1 numerical factor for the flat universe, but it becomes much larger than the apparent horizon for a closed universe. So we know that the choice of H^{-1} as the horizon cannot solve the problem of the violation of the holographic principle in a closed universe. The holographic principle in cosmology is also discussed in [6]. Einstein's theory may not describe gravity at very high energy. The simplest generalization of Einstein's theory is Brans-Dicke theory. The recent interest in scalar-tensor theories of gravity arises from inflationary cosmology, supergravity, and string theory. There exists at least one scalar field, the dilaton field, in the low energy effective bosonic string theory. Scalar degrees of freedom arise also upon compactification of higher dimensions. In this paper, we apply the Fischler-Susskind proposal to Brans-Dicke cosmology in both the Jordan and Einstein frames.

II. BRANS-DICKE COSMOLOGY IN THE JORDAN FRAME

The Brans-Dicke Lagrangian in the Jordan frame is given by

$$\mathcal{L}_{BD}=\frac{\sqrt{-\gamma}}{16\pi}\left[\phi\tilde{R}-\omega\gamma^{\mu\nu}\frac{\partial_\mu\phi\partial_\nu\phi}{\phi}\right]-\mathcal{L}_m(\psi,\gamma_{\mu\nu}). \quad (1)$$

The above Lagrangian (1) is conformal invariant under the conformal transformations

$$g_{\mu\nu}=\Omega^2\gamma_{\mu\nu},\quad \Omega=\phi^\lambda \quad \left(\lambda\neq\frac{1}{2}\right),$$

$$\sigma=\phi^{1-2\lambda},\quad \bar{\omega}=\frac{\omega-6\lambda(\lambda-1)}{(2\lambda-1)^2}.$$

For the case $\lambda=1/2$, we make the following transformations:

$$g_{\mu\nu}=e^{\alpha\sigma}\gamma_{\mu\nu}, \quad (2)$$

$$\phi=\frac{8\pi}{\kappa^2}e^{\alpha\sigma}, \quad (3)$$

where $\kappa^2=8\pi G$, $\alpha=\beta\kappa$, and $\beta^2=2/(2\omega+3)$. Remember that the Jordan-Brans-Dicke Lagrangian is not invariant under the above transformations (2) and (3). The homogeneous and isotropic Friedmann-Robertson-Walker (FRW) space-time metric is

$$ds^2=-dt^2+a^2(t)\left[\frac{dr^2}{1-kr^2}+r^2d\Omega\right], \quad (4)$$

and the above metric can be written as

$$ds^2=-dt^2+a^2(t)[d\chi^2+\Sigma^2(d\theta^2+\sin^2\theta\, d\phi^2)], \quad (5)$$

where

$$\Sigma=\begin{cases}\chi, & k=0,\\ \sinh\chi, & k=-1,\\ \sin\chi, & k=1.\end{cases} \quad (6)$$

Based on the FRW metric and the perfect fluid $T_m^{\mu\nu}=(\rho+p)U^\mu U^\nu+pg^{\mu\nu}$ as the matter source, we can get the evolution equations of the universe from the action (1):

$$H^2+\frac{k}{a^2}+H\frac{\dot\phi}{\phi}-\frac{\omega}{6}\left(\frac{\dot\phi}{\phi}\right)^2=\frac{8\pi}{3\phi}\rho, \tag{7}$$

$$\ddot\phi+3H\dot\phi=4\pi\beta^2(\rho-3p), \tag{8}$$

$$\dot\rho+3H(\rho+p)=0. \tag{9}$$

If we are given a state equation for the matter $p=\gamma\rho$, then the solution to Eq. (9) is

$$\rho a^{3(\gamma+1)}=C_1. \tag{10}$$

Most of the cosmological solutions in this paper were given in [7]. For the case $k=0$, we can get the power-law solutions to Eqs. (7) and (8) with the help of Eq. (10):

$$a(t)=a_0t^p, \quad \phi(t)=\phi_0t^q, \tag{11}$$

where

$$p=\frac{2+2\omega(1-\gamma)}{4+3\omega(1-\gamma^2)}, \quad q=\frac{2(1-3\gamma)}{4+3\omega(1-\gamma^2)},$$

$$-1\leqslant\gamma<1-\frac{2}{3+\sqrt{6}/\beta}, \tag{12}$$

a_0 and ϕ_0 are integration constants, and $[q(q-1)+3pq]\phi_0=4\pi\beta^2(1-3\gamma)C_1a_0^{-3(\gamma+1)}$. The particle horizon is

$$r_H=\int_0^t\frac{d\tilde t}{a(\tilde t)}=\frac{4+3\omega(1-\gamma^2)}{a_0[2+\omega(1-\gamma)(1+3\gamma)]}t^{1-p}. \tag{13}$$

Therefore, the ratio between the entropy inside the particle horizon and the area of the horizon is

$$\frac{S}{GA/4}=\frac{4}{3G}\epsilon\frac{r_H}{a^2}=\frac{4\epsilon}{3G}\frac{4+3\omega(1-\gamma^2)}{a_0^3[2+\omega(1-\gamma)(1+3\gamma)]}t^{1-3p}, \tag{14}$$

where ϵ is the constant comoving entropy density, and $1-3p=-[2+3\omega(1-\gamma)^2]/[4+3\omega(1-\gamma^2)]$. The holographic bound is satisfied for γ in the range given by Eq. (12) if the above ratio is not greater than 1 initially.

For the case $k=\pm1$, we do not have a general solution for all values of γ, so we consider two special cases: the matter-dominated universe with $\gamma=0$ and the radiation-dominated universe with $\gamma=1/3$. It is convenient to use the cosmic time $d\eta=dt/a(t)$.

For $\gamma=1/3$, we can solve Eq. (8) to get

$$a^3\dot\phi=C_2, \tag{15}$$

where $C_2\neq0$ is an integration constant.

(*i*) $k=1$. The solutions are

$$\phi(\eta)=\phi_0\left[\frac{8\pi C_1\tan(\eta+\eta_0)/3+\sqrt{64\pi^2C_1^2/9+2C_2^2/3\beta^2}-\sqrt{2C_2^2/3\beta^2}}{8\pi C_1\tan(\eta+\eta_0)/3+\sqrt{64\pi^2C_1^2/9+2C_2^2/3\beta^2}+\sqrt{2C_2^2/3\beta^2}}\right]^{\sqrt{3/2}\beta}, \tag{16}$$

$$a^2(\eta)\phi(\eta)=\frac{4\pi C_1}{3}+\frac{1}{2}\sqrt{\frac{64\pi^2C_1^2}{9}+\frac{2C_2^2}{3\beta^2}}\sin[2(\eta+\eta_0)], \tag{17}$$

where η_0 is an integration constant. The entropy to area ratio is

$$\frac{S}{GA/4}=\frac{\epsilon(2\eta-\sin2\eta)\phi(\eta)}{G\sin^2\eta\{4\pi C_1/3+\sqrt{64\pi^2C_1^2/9+2C_2^2/3\beta^2}\sin[2(\eta+\eta_0)]\}}. \tag{18}$$

Note that $0\leqslant2(\eta+\eta_0)\leqslant\pi$, so we see that the holographic bound can be satisfied if it is satisfied initially.

(ii) $k=-1$ and $C_2^2<32\pi^2\beta^2C_1^2/3$. We have the solutions

$$a^2(\eta)\phi(\eta)=-\frac{4\pi C_1}{3}+\frac{1}{16}e^{2(\eta+\eta_0)}+\left(\frac{64\pi^2C_1^2}{9}-\frac{2C_2^2}{3\beta^2}\right)e^{-2(\eta+\eta_0)}, \tag{19}$$

$$\left(\frac{\phi}{\phi_0}\right)^{\sqrt{2}/\sqrt{3}\beta}=\frac{(-4\pi C_1/3-b)\tanh(\eta+\eta_0)-c+C_2/\sqrt{6}\beta}{(-4\pi C_1/3-b)\tanh(\eta+\eta_0)-c-C_2/\sqrt{6}\beta}, \tag{20}$$

where $b=1/16+64\pi^2C_1^2/9-2C_2^2/3\beta^2$ and $c=1/16-64\pi^2C_1^2/9+2C_2^2/3\beta^2$. The Brans-Dicke scalar field changes very slowly compared to the scale factor. Therefore the holographic bound

$$\frac{S}{GA/4}=\frac{\epsilon(\sinh 2\eta-2\eta)\phi}{G\sinh^2\eta[-4\pi C_1/3+e^{2(\eta+\eta_0)}/16+(64\pi^2C_1^2/9-2C_2^2/3\beta^2)e^{-2(\eta+\eta_0)}]}\leqslant 1 \tag{21}$$

will be satisfied if it is satisfied initially.

(iii) $k=0$. The solutions are

$$a^2(\eta)\phi(\eta)=\frac{8\pi C_1}{3}(\eta+\eta_0)^2-\frac{C_2^2}{16\pi\beta^2C_1}, \tag{22}$$

$$\phi(\eta)=\phi_0\left[\frac{\eta+\eta_0-\sqrt{6}C_2/16\pi\beta C_1}{\eta+\eta_0+\sqrt{6}C_2/16\pi\beta C_1}\right]^{\sqrt{3/2}\beta}. \tag{23}$$

The Brans-Dicke scalar field ϕ slowly increases up to ϕ_0 as the universe expands. The holographic bound

$$\frac{S}{GA/4}=\frac{4\epsilon}{3G}\frac{\eta\phi(\eta)}{8\pi C_1(\eta+\eta_0)^2/3-C_2^2/16\pi\beta^2C_1}\leqslant 1 \tag{24}$$

can be satisfied if it is satisfied initially.

For $\gamma=0$, the solutions are

$$a(\eta)=a_0e^{b\eta},\quad \phi=\phi_0e^{-b\eta}, \tag{25}$$

where $b^2=-2k/(2+\omega)$ and $4\pi\beta^2C_1=-a_0\phi_0b^2$.

(a) $k=-1$ and $-2<\omega<-3/2$. The above solutions (25) are exponential expansion in the cosmic time η or linear expansion in the coordinate time t. The entropy to area ratio is

$$\frac{S}{GA/4}=\frac{\epsilon(\sinh 2\eta-2\eta)}{Ga_0^2e^{2b\eta}\sinh^2\eta}. \tag{26}$$

So the holographic bound can be satisfied for $-2<\omega<-3/2$ if it is satisfied initially.

(b) $k=1$ and $\omega<-2$. The solutions (25) are linear in the coordinate time t. The entropy to area ratio is

$$\frac{S}{GA/4}=\frac{\epsilon(2\eta-\sin 2\eta)}{Ga_0^2e^{2b\eta}\sin^2\eta}. \tag{27}$$

It is obvious that the holographic bound can be violated when $\eta=n\pi$ for any integer n.

In fact, the current experimental constraint on ω is $\omega>500$ or $\beta^2<0.002$. The solutions (25) may not be physical. However, the low energy effective theory of the string theory can lead to $\omega=-1$; we may need to explore the possibility of negative ω. For positive ω, we need to solve the equations numerically. When $\omega\to\infty$ and at late times, the Brans-Dicke cosmological solutions become general relativistic solutions.

III. BRANS-DICKE COSMOLOGY IN THE EINSTEIN FRAME

The Brans-Dicke Lagrangian in the Einstein frame is obtained by the conformal transformations (2) and (3):

$$\mathcal{L}=\sqrt{-g}\left[\frac{1}{2\kappa^2}R-\frac{1}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma\right]-\mathcal{L}_m(\psi,e^{-\alpha\sigma}g_{\mu\nu}). \tag{28}$$

The perfect fluid becomes $T_m^{\mu\nu}=e^{-2a\sigma}[(\rho+p)U^\mu U^\nu+pg^{\mu\nu}]$. From the FRW metric in the Einstein frame, we can get the evolution equations of the universe from the action (28):

$$H^2+\frac{k}{a^2}=\frac{\kappa^2}{3}\left(\frac{1}{2}\dot{\sigma}^2+e^{-2\alpha\sigma}\rho\right), \tag{29}$$

$$\ddot{\sigma}+3H\dot{\sigma}=\frac{1}{2}\alpha e^{-2\alpha\sigma}(\rho-3p), \tag{30}$$

$$\dot{\rho}+3H(\rho+p)=\frac{3}{2}\alpha\dot{\sigma}(\rho+p). \tag{31}$$

With $p=\gamma\rho$, the solution to Eq. (31) is

$$\rho a^{3(\gamma+1)}e^{-3\alpha(\gamma+1)\sigma/2}=C_3, \tag{32}$$

where C_3 is a constant of integration. For the flat universe $k=0$, combining Eqs. (29), (30), and (31), we have

$$ae^{-\alpha(1-\gamma)\sigma/\beta^2(1-3\gamma)}=C_4, \tag{33}$$

where C_4 is an integration constant and the above equation is valid for $-1\leqslant\gamma<1-2/(3+\sqrt{6}/\beta)$ and $\gamma\neq 1/3$. To obtain the above solution, we assume that $\dot{\sigma}a^3\to 0$ and $\dot{a}a^2\to 0$ when $a\to 0$. From Eqs. (29), (32), and (33), we get

$$H^2=\frac{2\kappa^2(1-\gamma)^2C_3C_4^{+\beta^2(1-3\gamma)^2/2(1-\gamma)}}{6(1-\gamma)^2-\beta^2(1-3\gamma)^2}\times a^{[-6(1-\gamma^2)-\beta^2(1-3\gamma)^2]/2(1-\gamma)}. \tag{34}$$

In order to see the main result clearly, I omit constant coefficients in the following Eqs. (35) and (36). The particle horizon is

$$r_H=\int_0^a\frac{d\tilde{a}}{\tilde{a}^2H}\sim a^{[2(1-\gamma)(1+3\gamma)+\beta^2(1-3\gamma)^2]/4(1-\gamma)}. \tag{35}$$

The entropy to area ratio is

$$\frac{S}{GA/4} \sim a^{[-6(1-\gamma)^2-\beta^2(1-3\gamma)^2]/4(1-\gamma)}. \tag{36}$$

For $\gamma=1$, we find that

$$\left(e^{-2\alpha\sigma}\rho+\frac{1}{2}\dot{\sigma}^2\right)a^6=C_6,$$

where C_6 is an integration constant:

$$a^3=\sqrt{3\kappa^2C_6}t,$$

$$\frac{S}{GA/4}=\frac{2\epsilon}{G\sqrt{3\kappa^2C_6}}.$$

For $\gamma=1/3$, we have

$$a^3\dot{\sigma}=C_5, \tag{37}$$

where $C_5\neq 0$ is an integration constant.

(1) $k=0$. The entropy to area ratio is

$$\frac{S}{GA/4}=\frac{4\epsilon}{\sqrt{3}GC_3k}\frac{\sqrt{C_3a^2+C_5^2/2}-\sqrt{C_5^2/2}}{a^2}. \tag{38}$$

Therefore, from Eqs. (36) and (38), we see that the holographic principle is satisfied for $-1\leqslant\gamma<1-2/(3+\sqrt{6}/\beta)$ provided that it is satisfied initially.

(2) $k=-1$ and $\kappa^2C_3^2\geqslant 6C_5^2$. We have

$$e^{2\chi_H}=\frac{2\sqrt{a^4+\kappa^2C_3^2a^2/3+\kappa^2C_5^2/6}+2a^2+\kappa^2C_3/3}{2\sqrt{\kappa^2C_5^2/6}+\kappa^2C_3/3}. \tag{39}$$

The entropy to area ratio is

$$\frac{S}{GA/4}=\frac{\epsilon(\sinh 2\chi_H-2\chi_H)}{Ga^2\sinh^2\chi_H}. \tag{40}$$

As a increases, $4S/GA$ decreases. The holographic bound is satisfied if it is satisfied initially.

(3) $k=1$. We have

$$2\chi_H=\arcsin\frac{\kappa C_3}{\sqrt{\kappa^2C_3^2+6C_5^2}}+\arcsin\frac{6a^2-\kappa^2C_3}{\sqrt{\kappa^4C_3^2+6\kappa^2C_5^2}}. \tag{41}$$

The holographic bound

$$\frac{S}{GA/4}=\frac{\epsilon(2\chi_H-\sin 2\chi_H)}{Ga^2\sin^2\chi_H}\leqslant 1 \tag{42}$$

is satisfied if it is satisfied initially.

For $\gamma=0$ and $k^2=1$, we do not have any analytical solution. We need to solve the problem numerically.

IV. CONCLUSIONS

We analyze the holographic principle in Brans-Dicke theory. For a flat universe, we find that the holographic bound can be satisfied for any matter with $-1\leqslant\gamma<1-2/(3+\sqrt{6}/\beta)$. For a universe with $k^2=1$, we do not have general analytical solutions for all values of γ. In particular, we do not have an analytical solution for the matter-dominated $k^2=1$ universe. We know that in standard Friedmann cosmology the holographic principle is violated for a closed matter-dominated universe near the maximal expansion. To check the holographic bound for the $k=1$ matter-dominated Brans-Dicke cosmological model, we need to do a numerical calculation. However, the numerical results in [8] tell us that the expansion rate in Brans-Dicke models is slower than that in Friedmann models. At large times, the difference becomes negligible. Therefore we expect that the holographic bound is also violated for the $k=1$ matter-dominated universe in Brans-Dicke cosmology.

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