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Pion electromagnetic form factor in the spacelike region and P phase $\delta_1^1(s)$ of $\pi\pi$ scattering from the value of the modulus of the form factor in the timelike region

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The problem of determining the pion electromagnetic form factor $F(q^2)$ in the spacelike region from the value of its modulus in the timelike region is solved by form factor analyticity. If $F(q^2)$ has no zeros in the complex plane then the form factor in the spacelike region is determined uniquely. If $F(q^2)$ has zeros in the complex plane q^2 it can be obtained in the spacelike region within narrow limits using experimental data from the timelike region. The form factor phase $\varphi(s)$ which coincides with the P-wave phase $\delta_1^1(s)$ of $\pi\pi$ scattering is calculated. The value of the pion radius has been improved.

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I. INTRODUCTION

The pion electromagnetic form factor $F(q^2)$ is calculated theoretically only in the spacelike region. For example, the form factor has been calculated with QCD sum rules [1] and in the lattice QCD [2].

On the other hand, the main part of experimental data on form factors are obtained in the timelike region via the reaction [3,4]

$$e^+e^-\to\pi^+\pi^-. \tag{1}$$

At small spacelike momentum transfers ($0<Q^2<0.253\ \text{GeV}^2$, $Q^2=-q^2$) the form factor has been measured by scattering of 300 GeV pions from atomic electrons [5]:

$$\pi^+e^-\to\pi^+e^-. \tag{2}$$

Colliding-beam measurements of $\sigma(e^+e^-\to\pi^+\pi^-)$ and measurements of $\sigma(\pi^+e^-\to\pi^+e^-)$ provide direct access to $F(q^2)$. At large spacelike momentum transfers form factor $F(q^2)$ was extracted from the reaction of the electroduction of pions from nucleons [6–9]

$$ep\to e\pi^+n,$$
$$en\to e\pi^-p. \tag{3}$$

But the presense of the nucleons and their structure complicates theoretical models and thus the determination of the pion form factor at large Q^2 is model dependent. Analyticity connects values of the pion form factor in spacelike and timelike regions.

As follows from microcausality the pion electromagnetic form factor $F(q^2)$ has the following analytical properties: (1) $F(q^2)$ is an analytical function of q^2 with a cut along positive q^2 from $q^2=4m_\pi^2$ to infinity; (2) on the real axis to the left of $q^2=4m_\pi^2$ the function $F(q^2)$ is real, and consenquently, takes complex conjugate values on the upper and lower edges of the cut; (3) for complex q^2 with $|q^2|\to\infty$ the function $F(q^2)$ grows no faster than a finite power (in QCD form factor decreases); (4) the function $F(q^2)$ is normalized by the condition $F(0)=1$.

It follows from unitarity that the form factor phase $\varphi(s)=\arctan[\operatorname{Im}F(s)/\operatorname{Re}F(s)]$ in the region $4m_\pi^2\leqslant s\leqslant s_{el}$ must coincide with the P phase of $\pi\pi$ scattering $\delta_1^1(s)$, $s_{el}\approx 0.8\ \text{GeV}^2$.

The purpose of this work is to calculate the form factor in the spacelike region from the known value of the modulus of the form factor in the timelike region measured in [3,4]. The phase of form factor $\varphi(s)$ which coincides with P phase $\delta_1^1(s)$ of $\pi\pi$ scattering $\delta_1^1(s)$ measured in [10–12] will be also calculated.

II. DETERMINATION OF THE FORM FACTOR $F(s)$ IN THE WHOLE COMPLEX PLANE FROM THE GIVEN VALUE OF ITS MODULUS IN THE TIMELIKE REGION IF THE FORM FACTOR HAS NO ZEROS IN THE COMPLEX s PLANE

We shall follow the method of the works [13]. If $F(s)$ has no zeros in complex s plane the function $\ln F(s)$ is an analytical function of s with the cut $[s_0,\infty]$, $s_0=4m_\pi^2$ and, consequently, we have the formula

$$\frac{1}{2\pi i}\int_C\frac{\ln F(s')ds'}{\sqrt{s'-s_0}(s'-s)s'}=\frac{\ln F(s)}{s\sqrt{s-s_0}}=\frac{1}{2\pi i}\int_{s_0}^{\infty}\frac{\ln|F(s')|^2ds'}{\sqrt{s'-s_0}s'(s'-s)}, \tag{4}$$

where the contour C contains the lower and upper edges of the cut and a large circle. It follows from Eq. (4) that when $F(s)$ has no zeros in the complex s plane the function $F(s)$ is uniquely determined

$$F(s)=F_0(s),$$

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$$F_0(s)=\exp\left[\frac{s\sqrt{s_0-s}}{2\pi}\int_{s_0}^{\infty}\frac{\ln|F(s')|^2 ds'}{\sqrt{s'-s_0}s'(s'-s)}\right]. \quad (5)$$

The result of the calculation of $F_0(s)$ is given in Tables I and II and in Fig. 1. We use for $|F(s')|s_0<s'<\infty$ the phenomenological formula for the form factor obtained in [14,15]. This formula satisfies the following requirements: (1) the form factor has correct analytical properties; (2) at large $Q^2=-s$ the form factor has the asymptotic behavior determined by QCD [16–19] (taking into account the preasymptotic power correction [20]); the phenomenological formula decsribes well the experiments [3,4] $\chi^2=123$ fitting over 120 experimental points in the interval $0.1296\ \text{GeV}^2\leqslant s\leqslant 4.951\ \text{GeV}^2$.

The phase of the form factor can be found using the following formula:

$$\frac{1}{s'-s-i\varepsilon}=\pi i\,\delta(s'-s)+P\frac{1}{s'-s}. \quad (6)$$

Substituting Eq. (6) into Eq. (5) we get

$$F_0(s)=|F(s)|e^{i\varphi_0(s)}, \quad s>s_0, \quad (7)$$

where the phase of the form factor $\varphi_0(s)$ is equal to

$$\varphi_0(s)=-\frac{s\sqrt{s-s_0}}{2\pi}P\int_{s_0}^{\infty}\frac{\ln|F(s')|^2 ds'}{\sqrt{s'-s_0}s'(s'-s)}, \quad s>s_0. \quad (8)$$

Let us write the integral (8) in the form more convenient for numerical calculation,

$$\varphi_0(s)=-\frac{s\sqrt{s-s_0}}{2\pi}\left\{\int_{s_0}^{\infty}\frac{\ln|F(s')/F(s)|^2 ds'}{\sqrt{s'-s_0}s'(s'-s)}-\frac{\pi}{s\sqrt{s_0}}\ln|F(s)|^2\right\}. \quad (9)$$

The result of the calculations of $\varphi_0(s)$ is given in Table III.

III. THE CONSIDERATION OF THE FORM FACTOR WITH ZEROS IN THE COMPLEX PLANE

Let us consider now the case when the form factor $F(s)$ has zeros in the complex plane at $s=s_k$, $k=1,2,\ldots,F(s_k)=F(s_k^*)=0$. We introduce instead of the form factor $F(s)$ the function $\tilde{F}(s)$ by formula

$$F(s)=\chi(s)\tilde{F}(s). \quad (10a)$$

The factor $\chi(s)$ is chosen so that

(1) $\chi(s_k)=0, \quad k=1,2,\ldots$

and consequently, the function $\tilde{F}(s)$ has no zeros in the complex s plane.

(2) The function $\chi(s)$ is an analytical function with a cut along positive s from $s=s_0$ to infinity.

(3) On the real axis to the left from $s=s_0$ the function $\chi(s)$ is real.

(4) The modulus of the function $\chi(s)$ is equal to unity on the cut.

The function

$$\chi(s)=\prod_k \chi_k(s), \quad (10b)$$

$$\chi_k(s)=\frac{\sqrt{s_0-s}-\sqrt{s_0-s_k}}{\sqrt{s_0-s}+\sqrt{s_0-s_k}}\,\frac{\sqrt{s_0-s}-\sqrt{s_0-s_k^*}}{\sqrt{s_0-s}+\sqrt{s_0-s_k^*}} \quad (10c)$$

satisfies these requirements.

The function $\chi(s)$ is unique. In order to prove this we conformally map the plane with the cut s inside the unit circle W by the formula

$$W=\frac{1-\sqrt{s_0-s}}{1+\sqrt{s_0-s}}, \quad W_k=\frac{1-\sqrt{s_0-s_k}}{1+\sqrt{s_0-s_k}}. \quad (10d)$$

We obtain

$$\chi_k(W)=\frac{W-W_k}{1-W_k^*W}\,\frac{W-W_k^*}{1-W_kW}. \quad (10e)$$

The function $(W-W_k)/(1-W_k^*W)$ is named by the unimodular Blashcke factor [21], which is known to be unique. The validity of formula (10a) for the form factor with infinite number of complex zeros has been considered in the second paper in Ref. [13]. If the form factor vanished somewhere like $(s-s_k)^m$, a Blashcke factor of χ_k^m would be necessary.

The function $\tilde{F}(s)$ has no zeros in the complex plane, on the cut its modulus equals to $|F(s')|$, and it is normalized by the condition

$$\tilde{F}(0)=\chi^{-1}(0)\geqslant 1 \quad (11)$$

and, consequently, we may apply the Cauchy theorem to the function $[\ln\tilde{F}(s)]/s\sqrt{s-s_0}$,

$$\frac{1}{2\pi i}\int\frac{\ln\tilde{F}(s')ds'}{\sqrt{s'-s_0}(s'-s)s'}=\frac{\ln\tilde{F}(s)}{s\sqrt{s-s_0}}+\frac{\ln\tilde{F}(0)}{i(-s)\sqrt{s_0}}=\frac{1}{2\pi i}\int_{s_0}^{\infty}\frac{\ln|F(s')|^2 ds'}{\sqrt{s'-s_0}s'(s'-s)}. \quad (12)$$

Therefore

$$F(s)=\psi(s)F_0(s),$$

$$\psi(s)=\chi(s)/[\chi(0)]^{\sqrt{1-s/s_0}}, \quad (13)$$

TABLE I. The result of the calculations of the pion form factor $F(-Q^2)$ in the spacelike region $0 \leqslant Q^2 \leqslant 0.253$ GeV^2. (1) $F_0(-Q^2)$ is free from the complex zeros form factor. (2) The form factor $F_{min}(-Q^2)$ has complex zeros so that $F(-Q^2)$ is minimal. (3) The form factor $F_{max}(-Q^2)$ has complex zeros so that $F(-Q^2)$ is maximal. (4) The form factor $F_{max.impr}(-Q^2)$ has complex zeros so $F(-Q^2)$ is maximal and the phase of the form factor $\varphi(s)$ coincides with p-phase $\delta_1^1(s)$ of the $\pi\pi$ scattering (Ref. [11]). (5) $F_{exp}(-Q^2)$ is the experimental value of the form factor from Ref. [5]. $F_{exp}(Q^2)$ is measured by scattering 300 GeV pions from the electrons of a liquid hydrogen target.

n	Q^2	$F^2(-Q^2)_{Exp}$	$F_0^2(-Q^2)$	$F_{min}^2(-Q^2)$	$F_{max}^2(-Q^2)$	$F_{max,impr.}^2(-Q^2)$
0	0	1	1	1	1	1
1	0.015	0.944 ± 0.007	0.943	0.943	0.966	0.944
2	0.017	0.921 ± 0.006	0.935	0.935	0.961	0.936
3	0.019	0.933 ± 0.006	0.928	0.928	0.957	0.929
4	0.021	0.926 ± 0.006	0.921	0.921	0.952	0.922
5	0.023	0.914 ± 0.007	0.914	0.914	0.948	0.915
6	0.025	0.905 ± 0.007	0.907	0.907	0.943	0.908
7	0.027	0.898 ± 0.008	0.900	0.900	0.938	0.901
8	0.029	0.884 ± 0.008	0.894	0.894	0.934	0.895
9	0.031	0.884 ± 0.009	0.887	0.887	0.929	0.888
10	0.033	0.890 ± 0.009	0.881	0.881	0.925	0.882
11	0.035	0.866 ± 0.010	0.871	0.871	0.917	0.872
12	0.037	0.876 ± 0.011	0.868	0.868	0.916	0.869
13	0.039	0.857 ± 0.011	0.861	0.861	0.911	0.862
14	0.042	0.849 ± 0.009	0.852	0.852	0.905	0.854
15	0.046	0.837 ± 0.009	0.840	0.840	0.896	0.842
16	0.050	0.830 ± 0.010	0.828	0.828	0.887	0.830
17	0.054	0.801 ± 0.011	0.816	0.816	0.878	0.818
18	0.058	0.800 ± 0.012	0.805	0.870	0.870	0.807
19	0.062	0.809 ± 0.012	0.793	0.793	0.861	0.795
20	0.066	0.785 ± 0.014	0.782	0.782	0.853	0.784
21	0.070	0.785 ± 0.015	0.772	0.772	0.844	0.775
22	0.074	0.777 ± 0.016	0.761	0.761	0.836	0.764
23	0.078	0.769 ± 0.017	0.751	0.751	0.828	0.754
24	0.083	0.757 ± 0.010	0.738	0.738	0.818	0.741
25	0.089	0.715 ± 0.016	0.724	0.724	0.806	0.727
26	0.095	0.724 ± 0.018	0.710	0.710	0.795	0.714
27	0.101	0.680 ± 0.017	0.696	0.696	0.783	0.700
28	0.107	0.696 ± 0.019	0.683	0.683	0.772	0.687
29	0.013	0.688 ± 0.020	0.670	0.670	0.761	0.674
30	0.119	0.676 ± 0.021	0.657	0.657	0.750	0.661
31	0.125	0.665 ± 0.023	0.645	0.645	0.740	0.650
32	0.131	0.651 ± 0.024	0.633	0.633	0.729	0.638
33	0.137	0.646 ± 0.027	0.621	0.621	0.719	0.626
34	0.144	0.616 ± 0.023	0.608	0.608	0.708	0.613
35	0.153	0.654 ± 0.023	0.592	0.592	0.693	0.597
36	0.163	0.563 ± 0.024	0.575	0.575	0.677	0.581
37	0.173	0.534 ± 0.038	0.558	0.558	0.662	0.564
38	0.183	0.586 ± 0.034	0.542	0.542	0.648	0.548
39	0.193	0.544 ± 0.036	0.527	0.527	0.634	0.533
40	0.203	0.529 ± 0.040	0.513	0.513	0.620	0.520
41	0.213	0.616 ± 0.048	0.499	0.499	0.607	0.506
42	0.223	0.487 ± 0.049	0.486	0.485	0.594	0.493
43	0.233	0.417 ± 0.058	0.473	0.473	0.581	0.480
44	0.243	0.593 ± 0.074	0.460	0.460	0.569	0.468
45	0.253	0.336 ± 0.073	0.449	0.448	0.558	0.457

TABLE II. The results of the calculations of the pion form factor $F(-Q^2)$ in the spacelike region $0.18 \leqslant Q^2 \leqslant 9.77\ \mathrm{GeV}^2$. (1) $F_0(-Q^2)$ is free from the complex zeros form factor. (2) The form factor $F_{min}(-Q^2)$ has complex zeros so that $F(-Q^2)$ is minimal. (3) The form factor $F_{max}(-Q^2)$ has complex zeros so that $F(-Q^2)$ is maximal. (4) The form factor $F_{max.impr.}(-Q^2)$ has complex zeros, so $F(-Q^2)$ is maximal and the phase of $\varphi(s)$ coincides with p-phase $\delta_1^1(s)$ of the $\pi\pi$ scattering (Ref. [11]). (5) $F_{exp}(-Q^2)$ is the experimental value of the form factor from Refs. [6–9]. $F(-Q^2)_{exp}$ has been obtained from the electroproduction of pions from nucleons $ep \to e\pi^+ n$ by extrapolation.

Q^2/GeV^2	$(F(-Q^2))_{Exp}$	$F_0(-Q^2)$	$F_{min}(-Q^2)$	$F_{max}(-Q^2)$	$F_{max.impr.}(-Q^2)$
			Ref. [6]		
0.18	0.850±0.044	0.740	0.740	0.807	0.744
0.29	0.634±0.029	0.639	0.639	0.719	0.646
0.40	0.570±0.016	0.562	0.562	0.649	0.571
0.79	0.384±0.014	0.391	0.391	0.483	0.407
1.19	0.238±0.017	0.295	0.295	0.383	0.315
			Ref. [7]		
0.62	0.445±0.016	0.452	0.452	0.543	0.465
1.07	0.309±0.019	0.319	0.319	0.409	0.338
1.20	0.269±0.012	0.293	0.293	0.381	0.313
1.31	0.242±0.015	0.274	0.274	0.361	0.295
1.20	0.262±0.014	0.293	0.293	0.381	0.313
2.01	0.154±0.014	0.191	0.191	0.270	0.216
			Ref. [8]		
1.22	0.290±0.030	0.290	0.289	0.378	0.310
1.20	0.294±0.019	0.293	0.293	0.381	0.313
1.71	0.238±0.020	0.221	0.229	0.303	0.245
3.30	0.102±0.023	0.118	0.117	0.184	0.146
1.99	0.179±0.021	0.193	0.193	0.272	0.218
3.99	0.004±0.678	0.096	0.095	0.157	0.124
			Ref. [9]		
1.18	0.256±0.026	0.297	0.297	0.385	0.317
1.94	0.193±0.025	0.198	0.197	0.277	0.223
3.33	0.086±0.033	0.117	0.116	0.183	0.145
6.30	0.059±0.030	0.056	0.055	0.105	0.083
9.77	0.070±0.019	0.032	0.030	0.068	0.056

where $F_0(s)$ is determined by Eq. (5). Essentially, the formulas (5), (10a), (13) allow us to fill the requirements of the analyticity (1)–(4) for the form factor having any modulus $|F(s')|$ on the cut. Five parameters are enough to describe experimental data of [3,4] where the modulus of the form factor is measured in 120 points ($\chi^2_{d.o.f.}=123/120$). Clearly the modulus of the form factor is a smooth function.

It is clear from Eq. (10a) that the knowledge of the modulus of the form factor on the cut determines the value of the form factor in the entire complex plane up to within the factor $\psi(s)$.

It is easy to obtain from Eq. (13) the asymptotic formula for the form factor $F(s)$ for $s \to -\infty$. Since $\chi(-s) \to 1$ as $s \to \infty$, we have

$$|F(-s)| \to |F(s)| \exp(-[a+\ln \chi(0)]\sqrt{s/s_0}), \tag{14}$$

where

$$a = \frac{\sqrt{s_0}}{2\pi} \int_{s_0}^{\infty} \frac{\ln |F(s)|^2 ds}{\sqrt{s-s_0}\, s}. \tag{15}$$

From QCD there follows the asymptotic formula at $Q^2 = -q^2 \to \infty$ for the pion form factor [16–19],

$$F(-Q^2) \to \frac{16\pi\alpha_s(Q^2)}{Q^2} f_\pi^2, \tag{16}$$

where $f_\pi = 93$ MeV. Therefore

$$a + \ln \chi(0) = 0. \tag{17}$$

If the form factor $F(s)$ has no zeros in the complex plane then $a=0$ and the form factor $F(s)$ can be determined uniquely. If the value a is small then the uncertainty of the determination of the form factor $F(s)$ due to the function $\psi(s)$ will be small. The value a was calculated in the work [15] from the analysis of the experiments [3,4] and was found to be small:

$$a = 0.069 \pm 0.003. \tag{18}$$

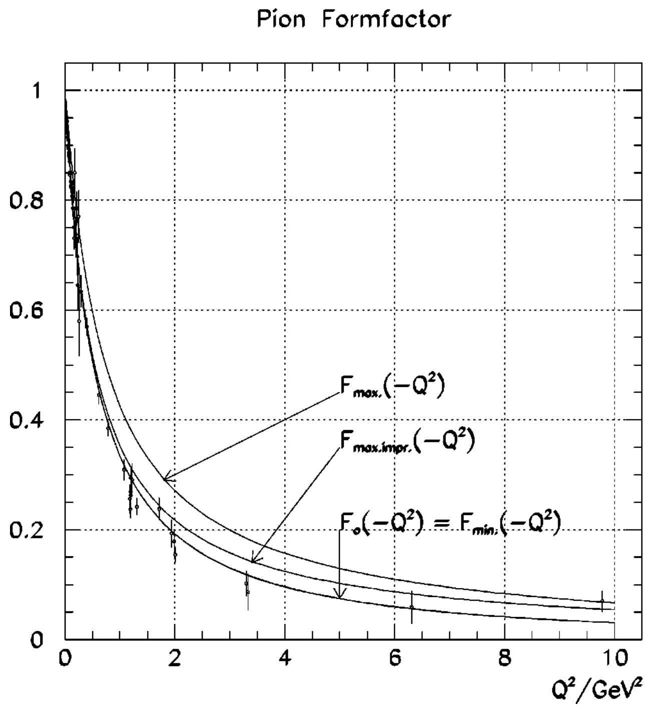

FIG. 1. (1) $F_0(-Q^2)$ is free from the complex zeros form factor. (2) The form factor $F_{min}(-Q^2)$ has complex zeros so that $F(-Q^2)$ is minimal. (3) The form factor $F_{max}(-Q^2)$ has complex zeros so that $F(-Q^2)$ is maximal. (4) The form factor $F_{max.impr.}(-Q^2)$ has complex zeros so $F(-Q^2)$ is maximal and the phase of the form factor $\varphi(s)$ coincides with p-phase $\delta_1^1(s)$ of the $\pi\pi$ scattering (Ref. [11]). (5) The experimental value of the form factor from Refs. [5–9].

IV. BOUNDS ON THE FORM FACTOR $F(s)$ IN THE SPACELIKE REGION

The form factor $F(s), s<0$ is determined up to the factor $\psi(s)$. To determine upper and lower bounds on the value of the form factor $F(s)$, for $s<0$ we look for the maximum and the minimum of the factor $\psi(s)$, by fixing the value of s and changing the distribution of the zeros s_k so that condition (17) is satisfied. Let us write the relation

$$w_k=(\sqrt{s_0}-\sqrt{s_0-s_k})/(\sqrt{s_0}+\sqrt{s_0-s_k}). \tag{19}$$

Write w_k in the form

$$w_k=r_k e^{i\varphi_k}, \tag{20}$$

where $r_k<1$.

Let us write the factor $\chi_k(s)$ in Eq. (10a) in the form

$$\chi_k(s)=\frac{(w-w_k)(w-w_k^*)}{(1-ww_k)(1-ww_k^*)}=\frac{w^2-2wr_k\cos\varphi_k+r_k^2}{1-2wr_k\cos\varphi_k+w^2r_k^2}, \tag{21}$$

where

$$w=(\sqrt{s_0}-\sqrt{s_0-s})/(\sqrt{s_0}+\sqrt{s_0-s}). \tag{22}$$

It is obvious that the maximum $\chi_k(s)$ is achieved if $\cos\varphi_k=1$ and the minimum $\chi_k(s)$ is achieved if $\cos\varphi_k=-1$ and

$$\max\chi_k(s)=\frac{(w-r_k)^2}{(1-wr_k)^2}, \tag{23}$$

$$\min\chi_k(s)=\frac{(w+r_k)^2}{(1+wr_k)^2}. \tag{24}$$

The value $\chi(0)$ is equal

$$\chi(0)=\prod_k r_k^2. \tag{25}$$

It follows from Eq. (17) that

$$\chi(0)=e^{-a}. \tag{26}$$

And it follows from inequality that

$$\frac{|w|+r_1}{1+|w|r_1}\,\frac{|w|+r_2}{1+|w|r_2}<\frac{|w|+r_1r_2}{1+|w|r_1r_2} \tag{27}$$

and from Eqs. (25), (26) that the maximum $\psi(s)$ is achieved if $F(s)$ has one double real zero:

$$w_{max}=e^{-a/2},\quad s_{max}=\frac{4e^{-a/2}}{(1+e^{-a/2})^2}s_0=0.9997s_0.$$

Inequalities $r_1>|w|,\ r_2>|w|,\ r_1,r_2>|w|,$

$$\frac{r_1-|w|}{1-|w|r}\,\frac{r_2-|w|}{1-|w|r}>\frac{r_1r_2-|w|}{1-|w|r_1r_2}, \tag{28}$$

and Eqs. (25), (26) prove that minimum $\psi(s)$ is achieved if $F(s)$ has one double real zero:

$$w_{min}=-e^{-a/2},$$

$$s_{min}=-\frac{4e^{-a/2}}{(1-e^{-a/2})^2}s_0=-3360.31s_0.$$ [1]

V. THE CONTRIBUTION OF COMPLEX ZEROS INTO THE PHASE OF THE PION FORM FACTOR $\varphi(s)$

The contribution of complex zeros in the phase of the form factor $\varphi(s)$ is defined by the formula

$$\delta\varphi(s)=\frac{1}{2i}\ln\psi(s),\quad s>s_0. \tag{29}$$

The contribution of pair complex conjugate zeros in the phase of the function χ_k can be obtained from Eq. (21) putting $w=(1-0)e^{i\theta}$

$$\chi_k(s)=e^{2i(\theta+\theta_k)}, \tag{30}$$

[1]Strictly speaking, minimum and maximum $\psi(s)$ are achieved at simple real zero, but practically it does not change all the results.

TABLE III. The results of the calculations of the phase $\varphi(s)$ of the pion form factor. (1) $\varphi_0(s)$ is phase of the pion form factor free from complex zeros. (2) $\delta\varphi^+(s)$ is the maximal contribution of complex zeros in the phase of the pion form factor. (3) $\delta\varphi^-(s)$ is the minimal contribution of complex zeros in the phase of the pion form factor. (4) δ_1^1(s) is the P phase of the $\pi\pi$ scattering (Ref. [11]).

$\sqrt{s}$/GeV	$\varphi_0(s)$/deg.	$\delta\varphi^{(+)}(s)$/deg.	$\delta\varphi^{(-)}(s)$/deg.	$\delta_1^1(s)$/deg.
0.51	9.86	351.35	−0.0020	9.3±0.7
0.53	11.31	351.15	−0.0023	10.4±0.6
0.55	12.95	350.94	−0.0026	13.1±0.8
0.57	14.83	350.72	−0.0029	13.5±0.7
0.59	17.02	350.49	−0.0033	17.6±0.8
0.61	19.64	350.26	−0.0037	19.4±0.8
0.63	22.84	350.02	−0.0041	20.9±0.8
0.65	26.70	349.78	−0.0045	25.5±0.7
0.67	31.71	349.53	−0.0050	32.1±0.7
0.69	38.31	349.28	−0.0055	37.5±0.5
0.71	47.07	349.03	−0.0060	46.1±0.9
0.73	58.65	348.52	−0.0071	73.0±2.3
0.75	73.28	348.52	−0.0071	73.0±2.3
0.79	106.12	348	−0.0084	113.3±1.9
0.81	117.09	347.74	−0.0091	118.1±1.1

where

$$\theta=-i\ln\frac{1+i\sqrt{s/s_0-1}}{1-i\sqrt{s/s_0-1}}=2\arcsin\sqrt{1-s_0/s}, \tag{31}$$

$$\theta_k=\frac{1}{2i}\ln\frac{1-2r_ke^{-i\theta}\cos\varphi_k+r_k^2e^{-2i\theta}}{1-2r_ke^{i\theta}\cos\varphi_k+r_k^2e^{2i\theta}}. \tag{32}$$

Maximum and minimum of the function θ_k by φ_k are achieved if cos $\varphi_k=\pm1$:

$$\theta_k^{\pm}=\frac{1}{i}\ln\frac{1\mp r_ke^{-i\theta}}{1\mp r_ke^{i\theta}}=\pm2\arcsin\frac{r_k\sin\theta}{\sqrt{1\mp2r_k\cos\theta+r_k^2}}. \tag{33}$$

The contribution of the pair complex conjugate zeros satisfying the condition (26) has the form

$$\delta\varphi^{(\pm)}(s)=2(\theta+\theta_2^{(\pm)}+\delta_0), \tag{34}$$

where θ_2 follows from Eq. (33) by changing $r_k\to r=e^{-a/2}$ and $\delta_0=-(a/2)\sqrt{s/s_0-1}$.

In Table II we give the values of φ_0, $\delta\varphi^{(+)}$, $\delta\varphi^{(-)}$ and the values of the P-phase $\pi\pi$-scattering $\delta_1^1(s)$ from [11]. It is seen from Table III that there is a very good lower bound on the phase $\varphi(s)$. Upper bound on the phase $\varphi(s)$ is practically absent.

VI. IMPROVED DETERMINATION OF THE UPPER BOUND OF THE FORM FACTOR IN THE SPACELIKE REGION

It can be seen from Table III that $\varphi_0(s)$ within the limits of experimental errors coincides with $\delta_1^1(s)$. This means that the value $\delta\varphi^+(s)$ has the order of the experimental error in $\delta_1^1(s)$. Thus we change $\theta_k^{(+)}$ on θ_k and $\cos\varphi_k=1$ on $\cos\varphi_k=-0.96$. If $-1\leqslant\cos\varphi_k\leqslant-0.96$ the phase $\varphi(s)$ coincides with $\delta_1^1(s)$ to within the experimental errors. Improved upper bound of the form factor in the spacelike region is obtained from Eq. (21) if $(\cos\varphi_k)_{max.impr.}=-0.96$.

The results of the calculation of $F_0, F_{min}, F_{max}, F_{max.impr.}$ and the experimental data from Refs. [5–9] are shown in Tables I and II and Fig. 1. The curves $F_0(s)$ and $F_{min}(s)$ merge together.

VII. IMPROVED CALCULATION OF THE PION RADIUS

The formulas for the bounds on the pion radius were obtained in [13,21–23],

$$(r_\pi^2)_{max}=\frac{3}{2m_\pi^2}\left[b+\frac{1}{2}(sha-a)\right], \tag{35}$$

$$(r_\pi^2)_{min}=\frac{3}{2m_\pi^2}\left[b-\frac{1}{2}(sha+a)\right], \tag{36}$$

where

$$b=\frac{s_0^{3/2}}{2\pi}\int_{s_0}^{\infty}\frac{\ln|F(s)|^2ds}{s^2\sqrt{s-s_0}}. \tag{37}$$

The value b was calculated in [15],

$$b=0.1544\pm0.0016. \tag{38}$$

Formulas (35), (36) were obtained from the derivative of the form factor $F(s)$ with respect to s at $s=0$,

$$F'(0)=\left(b-\sum_k\frac{1-w_k}{4w_k}+\frac{1}{2}\ln\left|\prod_k w_k\right|\right)/s_0, \tag{39}$$

and, by definition,

$$r_\pi^2=6F'(0). \tag{40}$$

The maximum value of $F'(0)$ is reached when the function $F(s)$ has one negative zero $(s_1)_{max}$, so that $(w_1)_{max}=-e^{-a}$ and then $(s_1)_{max}=-839.8s_0$, $(r_\pi^2)_{max}=(0.463\pm0.005)$ fm^2. The minimum value of $F'(0)$ is reached when the function $F(s)$ has one positive zero $(s_1)_{min}$, so that $(w_1)_{min}=e^{-a}$ and then $(s_1)_{min}=0.9988s_0$, $(r_\pi^2)_{min}=0.256$ fm^2. This zero gives in the phase $\varphi(s)$ the additional term ~180° what is inconsistent with the experimental data on $\delta_1^1(s)$ [11]. The minimum of $F'(0)$ which is consistent with the experimental data of $\delta_1^1(s)$ is reached when form factor $F(s)$ has two complex conjugate zeros

$(w_1)_{min.impr}=r_1e^{i\varphi_1}$, $(w_1^*)_{min.impr.}=r_1e^{-i\varphi_1}$, and $(r_1)_{min.impr.}=e^{-a/2}$, $(\cos\varphi_1)_{min.impr.}=-0.96$, $(s_1)_{min.impr.}=(46.23+11.33i)s_0$.

We have obtained

$$F'_{min.impr.}(0)=b-\frac{a}{2}+0.96sh\frac{a}{2}=0.1530\pm0.0016 \tag{41}$$

and

$$(r_\pi^2)_{min.impr.}=(0.4623\pm0.0048)\ \ \mathrm{fm}^2. \tag{42}$$

Taking into account the closeness of $(r_\pi^2)_{max}$ and $(r_\pi^2)_{min.impr.}$ we have obtained

$$r_\pi^2=(0.463\pm0.005)\ \ \mathrm{fm}^2. \tag{43}$$

This value of r_π^2 is slightly larger than those obtained in [3,5],

$$r_\pi^2=(0.422\pm0.003\pm0.013)\ \ \mathrm{fm}^2\ \ [3]$$
$$=(0.439\pm0.008)\ \ \mathrm{fm}^2\ \ [5]. \tag{44}$$

This disagreement is due to the fact that the authors of [3,5] used models which give the underestimated value of r_π^2 [15].

In a recent paper [24] Buck and Lebed have solved the same problem and obtained results opposite to the results of the present paper. Buck and Lebed claimed that the existing world sample of the timelike data for the form factor gives only loose bounds on the form factor in the spacelike region in contrast with the results of this paper. Despite the fact that 61 fitting parameters were used in [24], the fit of the data in the timelike region is not satisfactory, $\chi^2/\mathrm{DOF}=3.2$ for 145 data points. In this aspect it is worth mentioning the papers in Refs. [14,15], where the pion form factor model was obtained, which has correct analytical properties, the asymptotics of QCD and describes well the experiments [3,4]. The model of [14,15] has only 5 parameters and gives $\chi^2/\mathrm{DOF}\approx1$ for 120 data points in the timelike region $0.13\ \mathrm{GeV}^2\leqslant s\leqslant4.95\ \mathrm{GeV}^2$ [3,4].

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