Solitons in a gauged Landau-Lifshitz model

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We study the gauged Landau-Lifshitz model with the aim of obtaining self-dual solitonic configurations. It is shown that with the introduction of a suitably chosen triplet of background scalar fields, this model admits topological solitons.

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Topologically nontrivial localized structures such as vortices are possible candidates of anyonic objects in quasiplanar condensed matter physics. In the case of charged matter fields coupled to Chern-Simons (CS) terms, the vortex is electrically charged and behaves as an anyon [1]. In this context, self-dual CS models are all the more relevant since the equations of motion for the gauge fields in CS models are of the first order form to begin with, unlike the ones involving Maxwell terms, and hence these equations can be used directly as a part of self-duality equations [2].

The first nonrelativistic Abelian CS model to admit self-dual solitonic configurations was constructed by Jackiw and Pi [3]. This is a planar gauged Schrödinger model with a quartic potential term. It was generalized subsequently to models involving the Maxwell term [2,4] also, which makes the gauge field acquire a physical propagating massive mode. The CS system still admits a self-dual formulation, a sign of its robustness, provided one includes additional real scalar fields of mass equal to the mass of the propagating gauge mode. Alternatively, as shown by Barashenkov *et al.* [5], self-dual solitons can be obtained also by introducing a suitable background charge density. This latter model can accommodate the case of repulsive gases as well, with an asymptotically nonvanishing matter field.

Here it is pertinent to note that the Schrödinger model has to be gauged, along with dynamical terms such as the CS term, in order to obtain self-dual solutions. This is in contrast to the case of the Landau-Lifshitz (LL) model, or the nonrelativistic $O(3)$ nonlinear sigma model (NLSM) involving a triplet of spin fields [6,7], which admits self-dual solitonic configurations even at the ungauged level, just as its relativistic counterpart [8]. However, this model has certain coordinate singularities which can be got rid of by rewriting it in terms of the CP^1 variables [7]. This LL model describes a Heisenberg ferromagnetic system in the long wavelength limit. In the CP^1 formulation this model becomes a nonrelativistic U(1) gauge theory without any dynamical terms such as CS or Maxwell terms. The model also has a *global* SU(2) invariance. The question naturally arises whether the gauged CP^1 model obtained by gauging the global SU(2) group and adding a corresponding CS term also admits solitonic configurations.

It has been shown by Nardelli, Cho, and Kimm [9] that the relativistic gauged CP^1 model admits a new kind of solitonic configuration which cannot always be characterized by the second homotopy group (π_2) of the configuration space, unlike the case with the pure CP^1 model. A partially [U(1)] gauged LL model with both the CS and Maxwell terms has been considered by Tchrakian and Tomaras [10] who showed that their model admits topologically stable self-dual vortices. It is therefore desirable to analyze the case where the entire SU(2) group is gauged. The purpose of this paper is to study this model and look for any solitonic configuration.

To this end, we consider the model given by

$$\mathcal{L}=\frac{i}{2}[(D_0Z)^\dagger Z-Z^\dagger(D_0Z)]-|D_iZ|^2-a_0(Z^\dagger Z-1)$$
$$+\theta\epsilon^{\mu\nu\lambda}\left(A^a_\mu\partial_\nu A^a_\lambda+\frac{g}{3}\epsilon^{abc}A^a_\mu A^b_\nu A^c_\lambda\right)+\frac{g}{2}A^a_0\phi^a. \quad (1)$$

Here we have introduced a triplet of real scalar (background) fields ϕ^a, for reasons that will become clear later, and $Z=\binom{z_1}{z_2}$ is a complex doublet satisfying $Z^\dagger Z=1$. θ represents the CS parameter. The covariant derivatives are given by

$$D_0=\partial_0-igA^a_0T^a,$$
$$D_i=\partial_i-ia_i-igA^a_iT^a \quad (2)$$

with $T^a=\sigma^a/2$. Here a_i and A^a_i represent the U(1) and SU(2) gauge fields, respectively. Note that since we are considering a nonrelativistic model, we have the freedom to introduce different temporal and spatial ''covariant'' derivatives in Eq. (2) without violating any principles. Note here that whereas the Jackiw-Pi model [3] involved a single component Schrodinger field with a potential term, the present model involves a doublet of scalar fields subject to the condition $Z^\dagger Z=1$.

The Legendre transformed Hamiltonian density which is given by

$$\mathcal{H}=|D_iZ|^2+a_0(Z^\dagger Z-1)+A^a_0\left(\frac{g}{2}(M^a-\phi^a)-2\theta B^a\right), \quad (3)$$

where $B^a=(\partial_1A^a_2-\partial_2A^a_1+g\epsilon^{abc}A^b_1A^c_2)$ is the non-Abelian SU(2) magnetic field, and

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$$M^a = Z^\dagger \sigma_a Z \tag{4}$$

is a unit vector obtained by using the Hopf map.

Clearly, a_0 and A_0^a are the Lagrange multipliers enforcing the constraints

$$G_1 = (Z^\dagger Z - 1) \approx 0,$$

$$G_2^a = \frac{g}{2}(M^a - \phi^a) - 2\theta B^a \approx 0. \tag{5}$$

As this nonrelativistic model is first order in time derivative, the symplectic structure can readily be obtained by using the Faddeev-Jackiw [12] method to yield the following brackets:

$$\{z_\alpha(x), z_\beta^*(y)\} = i\,\delta_{\alpha\beta}\delta(x-y),$$

$$\{A_i^a(x), A_j^b(y)\} = \frac{\epsilon_{ij}}{2\theta}\delta^{ab}\delta(x-y). \tag{6}$$

G_1 and G_a^2 are the first class Gauss constraints of this model, and it can be shown that G_1 and G_a^2 generate the appropriate U(1) and SU(2) transformations, respectively: i.e.,

$$\delta Z(x) = \int d^2y f(y)\{Z(x), G_1(y)\} = i f(x) Z(x),$$

$$\delta Z(x) = \int d^2y f^a(y)\{Z(x), G_2^a(y)\} = \frac{ig}{2} f^a(\sigma^a Z),$$

$$\delta A_\mu^a(x) = \int d^2y f(y)\{A_\mu^a(x), G_1(y)\} = 0,$$

$$\delta A_i^a(x) = \int d^2y f^b(y)\{A_i^a(x), G_2^b(y)\} = \partial_i f^a(x) + g\epsilon^{abc} A_i^b f^c(x). \tag{7}$$

The momentum variable π_i conjugate to a_i vanishes:

$$\pi_i = \frac{\delta \mathcal{L}}{\delta \dot{a}_i} = 0. \tag{8}$$

Preservation of this primary constraint in time yields the secondary constraint

$$a_i \approx -iZ^\dagger \partial_i Z - \frac{g}{2} A_i^a M^a. \tag{9}$$

Clearly, Eqs. (8) and (9) form a pair of second class constraints, and therefore are ''strongly'' implemented by the Dirac brackets

$$\{a_i, \pi_j\} = 0. \tag{10}$$

With this, a_i ceases to be an independent degree of freedom. It may be mentioned here that the same thing happens to the U(1) gauge field for the case of the gauged CP^1 model coupled to the CS term in its relativistic avatar which was studied in Refs. [9,10].

One can show that G_2^a satisfy an algebra isomorphic to the SU(2) algebra

$$\{G_2^a(x), G_2^b(y)\} \approx 2\epsilon^{abc} G_2^c(x)\delta(x-y). \tag{11}$$

Following the group theoretical arguments, as in Ref. [10], one can show that the following relations between the Gauss constraints are expected:

$$gG_1 = M^a G_2^a. \tag{12}$$

This can be verified using $M^a M^a = (Z^\dagger Z)^2 \approx 1$. This shows that G_1 is not an independent constraint, and, this model therefore, has only SU(2) gauge invariance. This is again similar to the case of the relativistic version of the gauged CP^1 model coupled to the Hopf term [10].

We are now equipped to address the question of solitonic configurations [8] in this model. To investigate the existence of solitons, consider the energy functional obtained from the Hamiltonian density (3) [on the constraint surface (5)], which is given by

$$E = \int d^2x (D_i Z)^\dagger (D_i Z) \tag{13}$$

which can be rewritten as

$$E = \int d^2x \left|(D_1 \pm iD_2)Z\right|^2 \pm 2\pi N, \tag{14}$$

where

$$N = \frac{1}{2\pi i}\int d^2x\,\epsilon_{ij}(D_i Z)^\dagger (D_j Z). \tag{15}$$

In order for topological solitons to exist, the corresponding static configurations must satisfy the following self-dual or anti-self-dual saturation condition:

$$(D_1 \pm iD_2)Z = 0 \tag{16}$$

and N should be given by some number of topological origin. This provides the lower Bogomol'nyi bound for the energy functional (14). Since the energy is minimized by this self-dual static solution (16), it must also correspond to the static solution of the Euler-Lagrange equation. Note that N(15) is SU(2) gauge invariant, and so we can evaluate N in any gauge of our choice. At this stage we rewrite N as

$$N = \frac{1}{4\pi}\int d^2x\,\epsilon_{ij}\left[-2i\,\partial_i Z^\dagger \partial_j Z + gA_i^a \times\left(\partial_j M^a + \frac{g}{2}\epsilon^{abc} A_j^b M^c\right)\right]. \tag{17}$$

By making use of the local SU(2) gauge invariance of the model, we can go to a configuration where $Z(x) = \text{const}$. Correspondingly, M^a's are also constant, so that quantities such as $\partial_j Z$ and $\partial_j M^a$ vanish, and N (17) reduces to

$$N = \frac{g^2}{4\pi}\int d^2x \epsilon^{abc} M^a A_1^b A_2^c. \quad (18)$$

On the other hand, the invariance under SU(2) (global) transformation of the model (1) implies the existence of the following triplet of conserved SU(2) charges

$$Q^a = 2\theta\int d^2x(\partial_1 A_2^a - \partial_2 A_1^a) \quad (19)$$

as follows from Noether's theorem. In terms of the corresponding charge density $J^{0a} = 2\theta(\epsilon_{ij}\partial_i A_j^a)$, N becomes

$$N = \frac{g}{4\pi} M^a \int d^2x\left(B^a - \frac{J^{0a}}{2\theta}\right). \quad (20)$$

Making use of the Gauss constraint G_2 (5), one can write

$$N = \frac{g^2}{16\pi\theta}\int d^2x(1 - M^a\phi^a) - \frac{g}{8\pi\theta} M^a Q^a. \quad (21)$$

Now if the background field ϕ^a satisfies the condition

$$M^a\phi^a = 1 \quad (22)$$

then using Eqs. (21) and (5) one gets

$$N = -\frac{g}{8\pi\theta} M^a Q^a, \quad (23)$$

$$M^a B^a = 0. \quad (24)$$

Note that all $(M^a\phi^a)$, $(M^a B^a)$, and $(M^a Q^a)$ are SU(2) scalars.

At this point we make a particular gauge choice[1] $Z = \binom{0}{1}$. Correspondingly, $M^a = -\delta^{a3}$, and it follows from Eq. (24) that

$$B^3 = \partial_1 A_2^3 - \partial_2 A_1^3 + g\epsilon^{\alpha\beta} A_1^\alpha A_2^\beta = 0 \quad (25)$$

$(\alpha,\beta = 1,2)$. With this Q^3 (19) can be reexpressed as

$$Q^3 = -2\theta g\int d^2x(A_1^1 A_2^2 - A_2^1 A_1^2). \quad (26)$$

(Just to remind the reader, the subscripts and superscripts of A stand for spatial and group indices, respectively.) Note that in this gauge, one has only a surviving U(1) symmetry. This corresponds to an SO(2) rotation around the M^3 axis. Q^3 is the corresponding conserved (Noether) charge. Making use of Eqs. (2) and (9), one of the saturation conditions (16), [i.e., $(D_1 + iD_2)Z = 0$] in the gauge $Z = \binom{0}{1}$, can be shown to yield

$$A_1^1 = -A_2^2,$$

$$A_2^1 = A_1^2. \quad (27)$$

Using Eqs. (26) and (27) one finds

$$Q^3 = 2\theta g\int d^2x[(A_1^1)^2 + (A_2^1)^2]$$
$$= 2\theta g\int d^2x[(A_2^2)^2 + (A_1^2)^2] > 0 \quad (28)$$

and consequently [using Eq. (23)],

$$N = \frac{g}{8\pi\theta} Q^3 > 0 \quad (29)$$

and the minimum value of energy is

$$E_{(\min)} = 2\pi N = \frac{g}{4\theta} Q^3 = \frac{g^2}{2}\int d^2x[(A_1^1)^2 + (A_2^1)^2] > 0. \quad (30)$$

One can easily check that the other saturation condition $[(D_1 - iD_2)Z = 0]$ yields

$$A_1^1 = A_2^2,$$
$$A_2^1 = -A_1^2, \quad (31)$$

and Q^3 (26) becomes

$$Q^3 = -2\theta g\int d^2x[(A_1^1)^2 + (A_2^1)^2] < 0. \quad (32)$$

With this, N is given as

$$N = \frac{g}{8\pi\theta} Q^3 < 0. \quad (33)$$

But now, $E_{\min}$ is given by

$$E_{\min} = -2\pi N = -\frac{g}{4\theta} Q^3 = \frac{g^2}{2}\int d^2x[(A_1^1)^2 + (A_2^1)^2] > 0. \quad (34)$$

Thus, either of the saturation conditions (16) yields the same (positive definite) value for $E_{\min}$ as desired. The only difference is that the two conditions in Eq. (16) correspond to the positive and negative values for the number N.

Thus we must have either of the saturation conditions corresponding to the configuration $Z = \binom{0}{1}$. Also note that for $E_{(\min)}$ to be finite, A^1 and A^2 must vanish asymptotically. Thus asymptotically, B^3 (25) becomes a U(1) magnetic field with A_i^3 as the Abelian gauge field. The points at infinity can be identified, so that the two-dimensional plane gets effectively compactified to S^2. From Eq. (19), it then follows that Q^3 represents a topological index, which is nothing but the first Chern class. Clearly, $E_{\min}$ [Eqs. (30) and (34)] is given by a topological index. Thus these field configurations satisfying either of the saturation conditions (27) or (31) correspond to topological solitons, with positive or negative ''winding numbers'' (N). The same holds for all other con-

[1]Although the presence of the CS term allows for only those gauge transformations which tend to a constant at infinity, E (13) and N (15) are invariant under arbitrary gauge transformations.

figurations obtained by gauge transformations. For example, for any other configuration with $Z(x)=\text{const}$, the E_{min} will be given by an appropriate Q^a obtained by making a suitable SO(3) rotation corresponding to the SU(2) transformation required to get $Z=\text{const}$ configuration from $Z=\binom{0}{1}$. But the value of E_{min} will remain the same. It is important to note that we had to choose a convenient gauge here to identify E_{min} with a topological index (up to a constant). It is rather nontrivial to make this identification in an arbitrary gauge.

Note further, that if in adition, $\phi^a=M^a$, then $B^a=0$ (5) and A_i^a are pure gauges. In this case we can go to a gauge where $A_i^a=0$ and $D_i|_{A_i=0}=\mathcal{D}_i$ (the covariant derivative operator of the pure CP^1 model). Here also N corresponds to the topological index (Chern class) and we have a topological soliton.

Finally, note that for $\phi^a=0$, the first term in N (21) diverges, and there does not exist any solitonic configuration. It was thus necessary to introduce a triplet of background scalar fields ϕ^a in Eq. (1), satisfying Eq. (22) in order to obtain solitonic configurations. This is somewhat similar to the model [5] where a uniform background charge density was introduced. The difference is that here we have a triplet of background scalar fields instead of a dynamical one. However, such additional fields or charge density are not required for the case of the relativistic model [9] and its reduced phase space version [11]. On the other hand, in the model [4], it was required to introduce a dynamical scalar field of appropriate mass.

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