Supersymmetric three-cycles and (super)symmetry breaking

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(Received 17 September 1999; published 22 December 1999)

We describe physical phenomena associated with a class of transitions that occur in the study of supersymmetric three-cycles in Calabi-Yau (CY) threefolds. The transitions in question occur at real codimension one in the complex structure moduli space of the Calabi-Yau manifold. In type-IIB string theory, these transitions can be used to describe the evolution of a Bogomol'nyi-Prasad-Sommerfeld (BPS) state as one moves through a locus of marginal stability: at the transition point the BPS particle becomes degenerate with a supersymmetric two-particle state, and after the transition the lowest energy state carrying the same charges is a nonsupersymmetric two-particle state. In the IIA theory, wrapping the cycles in question with D6 branes leads to a simple realization of the Fayet model: for some values of the CY modulus gauge symmetry is spontaneously broken, while for other values supersymmetry is spontaneously broken.

PACS number(s): 11.25.Mj

I. INTRODUCTION

In the study of string compactifications on manifolds of reduced holonomy, odd-dimensional supersymmetric cycles play an important part (see, for instance [1–10], and references therein). In type-IIB string theory, a supersymmetric three-cycle can be wrapped by a D3 brane to yield a Bogomol'nyi-Prasad-Sommerfield (BPS) state whose properties are amenable to exact study; in the type-IIA theory or in *M* theory, Euclidean membranes can wrap the three-cycle and contribute to ''holomorphic'' terms in the low-energy effective action of the spacetime theory (that is, terms that are integrated over only a subset of the fermionic superspace coordinates).

Of particular interest, partially due to their role in mirror symmetry [5,7], have been special Lagrangian submanifolds in Calabi-Yau (CY) threefolds. In an interesting recent paper by Joyce [10], various transitions which these cycles undergo as one moves in the complex structure or Kähler moduli space of the underlying CY manifold were described. In this paper, we study some of the physics associated with the simplest such transitions discussed in Secs. 6 and 7 of [10]. These transitions are reviewed in Sec. II. The physical picture which one obtains by wrapping D3 branes on the relevant cycles in IIB string theory is described in Sec. III, while the physics of wrapped D6 branes in type-IIA string theory occupies Sec. IV. Our discussion is purely local (in both the moduli space and the Calabi-Yau manifold), as was the analysis performed in [10]; we close with some speculations about more global aspects in Sec. V.

At all points in this paper, we will be concerned with *rigid* special Lagrangian three cycles. Since the moduli space of a special Lagrangian three cycle *N* (including Wilson lines of a wrapped D brane) is a complex Kähler manifold of dimension $b_1(N)$ [11,6], this means we have to focus on so-called ''rational homology three spheres'' with $H_1(N,\mathbb{Z})$ at most a discrete group. We will further assume that $H_1(N,\mathbb{Z})$ is trivial.

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II. SPLITTING SUPERSYMMETRIC CYCLES

A. Definitions

Let *M* be a Calabi-Yau threefold equipped with a choice of complex structure and Kähler structure. Let ω be the Kähler form on *M*, and let Ω be the holomorphic three form, normalized to satisfy

$$\frac{\omega^3}{3!} = \frac{i}{8}\Omega \wedge \bar{\Omega}. \tag{2.1}$$

This also allows us to define two real, closed three forms on *M*, $\mathrm{Re}(\Omega)$ and $\mathrm{Im}(\Omega)$.

Let *N* be an oriented real three-dimensional submanifold of *M*. We call *N* a special Lagrangian with phase $e^{i\theta}$ if

(a) $\omega|_N = 0$,
(b) $[\sin(\theta)\mathrm{Re}(\Omega) - \cos(\theta)\mathrm{Im}(\Omega)]|_N = 0$.
(a) and (b) together imply that

$$\int_N [\cos(\theta)\mathrm{Re}(\Omega) + \sin(\theta)\mathrm{Im}(\Omega)] = \mathrm{vol}(N), \tag{2.2}$$

where $\mathrm{vol}(N)$ is the volume of *N*.

Physically, the relevance of θ for us will be the following. Let *N* and N' be three-cycles which are special Lagrangian with different phases θ and θ'. Compactifying, say, IIB string theory on *M*, we can obtain BPS states which preserve half of the $\mathcal{N}=2$ spacetime supersymmetry by wrapping three branes on *N* or N'. In the notation of [1], the surviving supersymmetries in the presence of a D3 brane on *N*, for example, are generated by

$$\epsilon_\delta = e^{i\delta}\epsilon_+ + e^{-i\delta}\epsilon_-,$$

0556-2821/99/61(2)/026001(4)/$15.00

with $\delta = -\theta/2 - \pi/4$. For generic $\theta \neq \theta'$, however, N and N' preserve different $\mathcal{N}=1$ supersymmetries and the state with both wrapped three branes would break all of the supersymmetry.

B. Transitions

The following supersymmetric three-cycle transitions are conjectured by Joyce to occur in compact Calabi-Yau threefolds M. Choose two homology classes $\chi^{\pm} \in H_3(M,\mathbb{Z})$ which are linearly independent in $H_3(M,\mathbb{R})$. For any $\Phi \in H^3(M,\mathbb{C})$, define

$$\Phi \cdot \chi^{\pm} = \int_{\chi^{\pm}} \Phi. \tag{2.3}$$

Thus $\Phi \cdot \chi^{\pm}$ are complex numbers. Following Joyce, define a subset $W(\chi^+,\chi^-)$ in $H^3(M,\mathbb{C})$ by

$$W(\chi^+,\chi^-) = \{\Phi \in H^3(M,\mathbb{C}) : (\Phi \cdot \chi^+)(\overline{\Phi} \cdot \chi^-) \in (0,\infty)\}. \tag{2.4}$$

So $W(\chi^+,\chi^-)$ is a real hypersurface in $H^3(M,\mathbb{C})$.

Fix some small, positive angle ϵ. For $\Phi \in H^3(M,\mathbb{C})$ write

$$(\Phi \cdot \chi^+)(\overline{\Phi} \cdot \chi^-) = \mathrm{Re}^{i\theta},$$

where $R \geqslant 0$ and $\theta \in (-\pi,\pi]$. Then we say Φ lies in $W(\chi^+,\chi^-)$ if $R>0$ and $\theta=0$. We say that Φ lies on the positive side of $W(\chi^+,\chi^-)$ if $R>0$ and $0<\theta<\epsilon$. We say that Φ lies on the negative side of $W(\chi^+,\chi^-)$ if $R>0$ and $-\epsilon<\theta<0$. Then, Joyce argues that the following kinds of transitions should occur. We are given a Calabi-Yau M with compact, nonsingular three cycles $N^{\pm}$ in homology classes $[N^{\pm}] = \chi^{\pm}$. $N^{\pm}$ are taken to be special Lagrangians with phases $\theta^{\pm}$. We assume $N^{\pm}$ intersect at one point $p \in M$, with $N^+ \cap N^-$ a positive intersection. As we deform the complex structure of M, the holomorphic three form moves around in $H^3(M,\mathbb{C})$ and therefore the phases $\theta^{\pm}$ of $N^{\pm}$ change.

When $[\Omega]$ is on the positive side of $W(\chi^+,\chi^-)$ there exists a special Lagrangian threefold N which is diffeomorphic to the connected sum $N^+ \# N^-$, with $[N]=[N^+]+[N^-]$ in $H_3(M,\mathbb{Z})$. N can be taken to be special Lagrangian with phase $\theta=0$ (this fixes the phase of Ω for us). As we deform $[\Omega]$ through $W(\chi^+,\chi^-)$, N converges to the singular union $N^+ \cup N^-$. When $[\Omega]$ is in $W(\chi^+,\chi^-)$, the phases $\theta^{\pm}$ align with $\theta=0$. On the negative side of $W(\chi^+,\chi^-)$, N ceases to exist as a special Lagrangian submanifold of M (while $\theta^{\pm}$ again become distinct).

For completeness and to establish some notation we will find useful, we briefly mention some motivation for the existence of these transitions [10]. In Joyce's model of the transition, there exists a manifold D with boundary $S \subset N$, which is a special Lagrangian with phase i. If we call its volume A, this means that $iA = \int_D \Omega$. S defines a two-chain in N; since we are assuming that $H_1(N,\mathbb{Z})$ is trivial, by Poincaré duality, S must be trivial in homology. Because S is real codimension one in N, it actually splits N into two parts:

$$N = C^+ \cup C^-, \quad \partial C^+ = -S, \quad \partial C^- = S.$$

So $C^{\pm} \pm D$ define three-chains and in fact it turns out that

$$[C^{\pm} \pm D] = \chi^{\pm} = [N^{\pm}]. \tag{2.5}$$

We see that we can determine the volume of D just from knowledge of $\chi^{\pm}$:

$$A = \frac{1}{i}\int_D \Omega = \int_D \mathrm{Im}(\Omega) = \int_{\chi^+} \mathrm{Im}(\Omega) \tag{2.6}$$

using $\mathrm{Re}(\Omega)|_D = 0$ and $\mathrm{Im}(\Omega)|_N = 0$. But when $[\Omega]$ goes through $W(\chi^+,\chi^-)$, we see from Eq. (2.6) and from the definition of $W(\chi^+,\chi^-)$ that A becomes negative; at least in the local model in $\mathbb{C}^3$, this means that N does not exist.

III. FORMERLY BPS STATES IN IIB STRING THEORY

Now, consider type-IIB string theory compactified on M. When the complex structure is such that $[\Omega]$ is on the positive side of $W(\chi^+,\chi^-)$, one can obtain a BPS hypermultiplet by wrapping a D3 brane on N. One can also obtain BPS hypermultiplets by wrapping D3 branes on N^+ or N^-.

Because

$$[N] = [N^+] + [N^-],$$

one can make a state carrying the same charges as the BPS brane wrapping N by considering the two-particle state with D3 branes wrapping both N^+ and N^-. How does the energy of the two states compare?

Recall that the disc D with boundary on N splits N into two components, $C^{\pm}$. Define

$$B^{\pm} = \int_{C^{\pm}} \Omega. \tag{3.1}$$

Then if we let V denote the volume of N and $V^{\pm}$ denote the volumes of $N^{\pm}$, we recall

$$V = B^+ + B^-, \tag{3.2}$$

$$V^{\pm} e^{i\theta^{\pm}} = B^{\pm} \pm iA, \tag{3.3}$$

where A is the volume of D. Since on this side of the transition A is positive, θ^+ is small and positive while θ^- is small and negative. In fact, the reality of the volumes $V^{\pm}$ lets us solve for $\theta^{\pm}$ in terms of $B^{\pm}$ yielding

$$\theta^{\pm} = \pm \frac{A}{B^{\pm}}. \tag{3.4}$$

The energy of the single-particle state obtained by wrapping a D3 brane on N is $T_{D3} \times V$ where T_{D3} is the D3 brane tension. The energy of the (nonsupersymmetric) state obtained by wrapping D3 branes on both $N^{\pm}$ can be approximated by $T_{D3} \times (V^+ + V^-)$. Expanding Eq. (3.3) for small $\theta^{\pm}$, we find

$$V^{+}+V^{-}=V+A(\theta^{+}-\theta^{-})=V+A^{2}\left(\frac{1}{B^{+}}+\frac{1}{B^{-}}\right). \tag{3.5}$$

So since $A>0$ and $\pm\theta^{\pm}>0$ on this side of the transition, we see that the single wrapped brane on N is energetically preferred.

Therefore, when the complex structure is on the positive side of $W(\chi^{+},\chi^{-})$, the BPS state indeed has lower energy than the nonsupersymmetric two-particle state carrying the same charges, by roughly $T_{D3}\times A(\theta^{+}-\theta^{-})$.

Now as one moves in the complex structure moduli space of M through a point where $[\Omega]$ lies in $W(\chi^{+},\chi^{-})$, A and $\theta^{\pm}$ vanish. Therefore, Eq. (3.5) shows that that mass of the two-particle state becomes equal to that of the single-particle state: we are passing through a locus of marginal stability. On this locus, the two-particle state consisting of branes wrapping $N^{\pm}$ is supersymmetric, since $N^{\pm}$ are special Lagrangians with the same phase.

Finally, we move through to the region where $[\Omega]$ lies on the negative side of $W(\chi^{+},\chi^{-})$. Here, $\pm\theta^{\pm}<0$. Since N ceases to exist as a supersymmetric cycle, the two-particle state with D3 branes wrapping $N^{\pm}$ is the lowest energy state carrying its charges.[1] Note that the two-particle state is nonsupersymmetric, since $N^{\pm}$ are special Lagrangians with different phases. Here, we are making the conservative assumption that there is no stable, nonsupersymmetric bound state of these two particles—such a bound state would be reflected in the existence of a (nonsupersymmetric) cycle in the homology class $[N^{+}]+[N^{-}]$ with lower volume than $V^{+}+V^{-}$. This is tantamount to assuming that the force between the two particles is repulsive for slightly negative A. This is reasonable since for A positive there is an attractive force and a (supersymmetric) bound state, and as A decreases to zero the magnitude of the force and the binding energy decrease until they vanish when $A=0$.

This phenomenon is an interesting variant on the examples of [12]. There, a stable nonsupersymmetric state passes through a locus of marginal stability and becomes unstable to decay to a pair of BPS particles (which together break all of the supersymmetries). In the present example, a BPS particle becomes, as we move in complex structure moduli space, unstable to decay to a pair of BPS particles. Moving slightly further in moduli space, we see that the two BPS particles together break all of the supersymmetries.

IV. D6 BRANES AND THE FAYET MODEL

Now, consider type-IIA string theory on the Calabi-Yau M in which the phenomena of Sec. II are taking place. Instead of studying particles in the resulting $\mathcal{N}=2$ supersymmetric theory, we wrap the three-cycle N with a space-filling D6-brane (i.e., $3+1$ of the dimensions of the D6 brane fill the noncompact space). This yields an $\mathcal{N}=1$ supersymmetric theory in the noncompact dimensions. For simplicity (since all our considerations are local), we can assume M is noncompact so we do not have to worry about canceling the D6 Ramond-Ramond charge. Alternatively, we could imagine the model discussed below arising as part of a larger system of branes and/or orientifolds on M.

First, let us discuss the physics when $[\Omega]$ is on the positive side of $W(\chi^{+},\chi^{-})$. Since $b_{1}(N)=0$, N has no moduli in M. Therefore, there are no moduli in the effective $(3+1)$-dimensional field theory on the wrapped D6 brane. The U(1) gauge field on the brane survives reduction on N, so the $(3+1)$-dimensional low-energy effective theory has a U(1) gauge symmetry. Finally, because N is a supersymmetric cycle with $H_{1}(N,\mathbb{Z})$ trivial, there is a unique supersymmetric ground state in the gauge theory (as opposed to a discrete set of ground states parametrized by Wilson lines around N).

What about the physics when $[\Omega]$ is on the negative side of $W(\chi^{+},\chi^{-})$? The D6 which was wrapping N has now split into two D6 branes, wrapping N^{+} and N^{-}. The U(1) gauge field on each survives, yielding a $U(1)^{2}$ gauge theory. Because N^{+} and N^{-} are supersymmetric cycles with different phases, the theory has no supersymmetric ground state. We do expect a stable nonsupersymmetric ground state, as long as $[\Omega]$ is close enough to $W(\chi^{+},\chi^{-})$.

What is the physics associated with the phase transition when $[\Omega]$ lies in $W(\chi^{+},\chi^{-})$? At this point, the two D6 branes wrapping N^{+} and N^{-} preserve the same supersymmetry, and intersect at a point in M. Because the light states are localized at the intersection, the global geometry of the intersecting cycles does not matter and we can model the physics by a pair of flat special Lagrangian three planes intersecting at a point. This kind of system was discussed in [13], and using their results it is easy to see that the resulting light strings give rise to precisely one chiral multiplet with charges $(+,-)$ under the $U(1)^{2}$ gauge group of the two wrapped D branes. Therefore, one linear combination of the U(1)'s [the normal ''center of mass'' U(1)] remains free of charged matter, while the other [the ''relative'' U(1)] gains a single charged chiral multiplet Φ. The relative U(1) is therefore anomalous [13]; demonstrates that the anomaly is canceled by inflow from the bulk.

Ignoring the center of mass U(1) [which we identify with the surviving U(1) on the positive side of W], the physics of this model is precisely reproduced by the Fayet model, the simplest model of spontaneous (super)symmetry breaking [14]. This is a U(1) gauge theory with a single charged chiral multiplet Φ (containing a complex scalar ϕ). There is no superpotential, but including a Fayet-Iliopoulos term rD in the spacetime Lagrangian, the potential energy is

$$V(\phi)=\frac{1}{g^{2}}(|\phi|^{2}-r)^{2}, \tag{4.1}$$

where g is the gauge coupling.

The phase structure of the model is quite simple: For $r>0$, there is a unique supersymmetric minimum, and the U(1) gauge symmetry is Higgsed. For $r<0$, there is a unique

[1]In a global model, even if there do exist other supersymmetric cycles in the same class, there will be some region in moduli space close to the transition where the energy cost for moving to them in the Calabi-Yau will be larger than the energy gained.

nonsupersymmetric minimum at $\phi=0$, so the U(1) symmetry is unbroken. Precisely when $r=0$, there is a U(1) gauge theory with a massless charged chiral field and a supersymmetric ground state.

Thus, we are led to identify the regions of positive, vanishing, and negative r with the positive side of $W(\chi^{+},\chi^{-})$, the locus where $[\Omega]$ is in W, and the negative side of W. The single real modulus which varies in the transition experienced by the supersymmetric three-cycle N can be identified with the Fayet-Iliopoulos parameter r. This identification is consistent with the conjecture in [9] that in worldvolume gauge theories of A-type D branes on Calabi-Yau spaces, complex structure moduli only enter as D terms.[2]

V. DISCUSSION

Exploration of the phenomena involving supersymmetric cycles in a Calabi-Yau manifold M under variation of the moduli of M has just started. It should be clear that as such phenomena are understood, they will have interesting implications for the physics of D branes on Calabi-Yau spaces (for a nice discussion of various aspects of this, see [9]).

One of the most enticing possibilities is that as more such phenomena are uncovered, we will find new ways to ''geometrize'' the study of supersymmetry breaking models in string theory. This would provide a complementary approach to attempts to write down interesting nonsupersymmetric string models informed by anti–de Sitter-conformal field theory considerations [15] or insights about tachyon condensation and nonsupersymmetric branes [16].

As a small step in this direction, it would be nice to find ways of going over small potential hills between different supersymmetric vacua of string theory. The transitions studied here, when put in the more global context of a manifold M with (possibly) several supersymmetric cycles in each homology class, might provide a way of doing this. For instance in Sec. IV, as one moves $[\Omega]$ into the negative side of $W(\chi^{+},\chi^{-})$, it is clear that one is increasing the scale of supersymmetry breaking (at least in the region close to the transition). Suppose that after one moves through the negative side of W in complex structure moduli space, eventually N^{+} and N^{-} approach each other and intersect again and the phenomenon of Sec. II occurs in reverse, with a new supersymmetric cycle N' in the same homology class as $[N^{+}]+[N^{-}]$ popping into existence. In such a case, one would have a nonsupersymmetric ground state for some range of parameters on the negative side of W, and then eventually reach a supersymmetric ground state again (with the D6 brane wrapping N').

Similarly, on the negative side of W there could exist ''elsewhere'' in M a supersymmetric cycle $\tilde{N}$ in the same class as $[N^{+}]+[N^{-}]$. Although the cost in energy to move from wrapping N to wrapping $\tilde{N}$ is nonzero and hence on the negative side of W the phenomena of Secs. III and IV occur, eventually it may become advantageous for the D6 branes to shift over to wrapping $\tilde{N}$. This would again be a situation where supersymmetry is broken, and then restored, as one dials the complex structure modulus of the Calabi-Yau space.

ACKNOWLEDGMENTS

We are grateful to J. Harvey, G. Moore, and E. Silverstein for discussions. The research of S.K. is supported by the A. P. Sloan Foundation and a DOE OJI program. The research of J.M. is supported by the Department of Defense NDSEG program.

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