Stochastic dynamics of correlations in quantum field theory: From the Schwinger-Dyson to Boltzmann-Langevin equation

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The aim of this paper is twofold: to probe the statistical mechanical properties of interacting quantum fields, and to provide a field theoretical justification for a stochastic source term in the Boltzmann equation. We start with the formulation of quantum field theory in terms of the set of Schwinger-Dyson equations for the correlation functions, which we describe by a closed-time-path master ($n=\infty$PI) effective action. When the hierarchy is simply truncated to a certain order, one obtains the usual closed system of correlation functions up to that order, and from the nPI effective action, a set of time-reversal invariant equations of motion. (This is the Dyson equation, the quantum field theoretical parallel of the collisionless Boltzmann equation.) But when the effect of the higher order correlation functions is included through a causal factorization condition (such as the molecular chaos assumption in Boltzmann's theory) called slaving, the dynamics of the lower order correlations shows dissipative features, as familiar in the usual (dissipative yet noiseless) Boltzmann equation, the field-theoretical version of which being the dissipative Dyson equations. We show that a fluctuation-dissipation relation should exist for such effectively open systems, and use this fact to show that a stochastic term, which explicitly introduces quantum fluctuations in the lower order correlation functions, necessarily accompanies the dissipative term. This leads to a stochastic Dyson equation, which is the quantum field theoretic parallel of the classical Boltzmann-Langevin equation, encompassing both the dissipative and stochastic dynamics of correlation functions.

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I. INTRODUCTION

The main result of this paper is a derivation of the stochastic Dyson equation in the form of the Boltzmann-Langevin equation as the correct description of the kinetic stochastic limit of quantum field theory. We begin with the set of Schwinger-Dyson equations in pure quantum field theory in parallel to the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy in kinetic theory, reexamine briefly how dissipation appears in the Dyson equations similar to that of the Boltzmann equation in field theory, analyze the effect of higher order correlation functions and their quantum fluctuations on the lower order ones, and by invoking a basic relation in stochastic processes, reason out the necessity for a stochastic term in the dissipative Dyson equation.

The significance of such an inquiry is twofold: to probe the statistical mechanical properties of interacting quantum fields, and to provide a field theoretical justification for a stochastic source term in the Boltzmann equation, first for quantum kinetic theory and then for quantum field theory. The former, sometimes known as ''correlation dynamics'' [1,2] has been investigated mainly for classical or quantum mechanical, but not field-theoretical, systems (see, however, [3,4]) and the latter primarily studied for a classical gas [5,6]. Extending previous studies to quantum fields is essential in the establishment of a quantum field theory of nonequilibrium processes. Previous work on this subject [7–14] showed how the Boltzmann equation can be derived from first-principles in quantum field theory, and dealt with its dissipative or response properties. In this paper we want to focus on the fluctuation or noise aspects in the derivation of a stochastic Dyson or Langevin-Boltzmann equation from quantum field theory [15–17].

The physical motivation for us to claim that the Boltzmann equation needs a noise term stems from the fluctuation-dissipation theorem. This relation is usually understood in the context of open systems where one defines a system of interest at the outset and denotes what it interacts with and whose details we do not particularly care as the environment, the coarse-graining of which leads to noise, which generates dissipative dynamics in the open system. This is captured in the stochastic equations such as the Langevin equation. In kinetic theory, the full BBGKY hierarchy gives complete information of the closed system (of molecules, say). It is upon (1) the ''truncation'' of the hierarchy and (2) the imposition of causal factorization conditions—the combination of these two procedures we call ''slaving'' (the molecular chaos assumption being a familiar example)—that the equation for the low order correlation functions (such as the Boltzmann equation for the one particle distribution function) acquires dissipative behavior. The key conceptual observation in this paper is that while the low order correlation functions constitute the ''system'' of interest, which obeys dissipative dynamics, there is always the equivalent of an ''environment'' acting on the system from the slaved higher order correlation functions, their fluctuations being the source of noise in the kinetic equations. The combination of a truncated system (of low order correlation functions) acted on by a slaved hierarchy (of higher order correlation functions) is an example of what we call an ''effectively open system''—the key process which renders a closed system effectively open being in this case ''slaving.'' (Note that truncation would only yield two disjoint partitions—each one being a smaller closed system—of the original closed system, i.e., the complete hierarchy.) This stochastic generalization of the Boltzmann equation gives rise to a Boltzmann-Langevin equation and its field theoretical parallel is the stochastic Dyson equation. Let us begin a brief exposition of these ideas and procedures with the fluctuation-dissipation relation.

A. Fluctuations and dissipation

It has long been known in statistical physics that the equilibrium state is far from being static, quite the opposite, it is the fluctuations around equilibrium which underlie and give meaning to such phenomena as Brownian motion [18] and transport processes, and determine the responses (such as heat capacity and susceptibility function) of the system in equilibrium. The condition that equilibrium constantly reproduces itself in the course of all these activities means that the equilibrium state is closely related both to the structure of the fluctuations and to the dynamical processes by which equilibrium sustains itself; these simple but deep relations are embodied in the so-called fluctuation-dissipation theorems: If a fluctuating system is to persist in the neighborhood of a given equilibrium state, then the overall dissipative processes in the system (due mainly to its interaction with the environment) are determined. Vice versa, if the dissipative processes are known, then we may describe the properties of equilibrium fluctuations without detailed knowledge of the system's microscopic structure. This is the aspect of the fluctuation-dissipation relations which guided Einstein in his pioneering analysis of the corpuscular structure of matter [19], Nyquist in his stochastic theory of electric resistivity [20], and Landau and Lifshitz to the theory of hydrodynamical fluctuations [21].

These ideas apply to systems described by an infinite number of degrees of freedom as well as only a few macroscopic variables, such as the long wavelength modes in hydrodynamics or a single particle distribution function as in kinetic theory. In the latter case, the dynamics is described by a dissipative Boltzmann equation, which depicts under general conditions the approach to equilibrium and, by virtue of the fluctuation-dissipation relation, one expects the existence of nontrivial fluctuations in equilibrium. The stochastic properties of the Boltzmann equation has been discussed by Zwanzig, Kac and Logan, and others [5,22].

For field theory, in the kinetic theory regime, where there is a clear separation of microscopic and macroscopic scales the field may be described in terms of quasiparticles, whose distribution function obeys a Boltzmann equation [10,3]. Formally, the one particle distribution function is introduced as a partial Fourier transform of a suitable Green function of the field. The same arguments which lead to a fluctuating Boltzmann equation in classical and quantum mechanics lead us to expect fluctuations in quantum kinetic field theory as well.

The end result of our investigation is a highly nonlinear, explicitly stochastic Dyson equation for the Green functions. By going to the kinetic theory limit, we derive a stochastic Boltzmann equation, and the resulting noise may be compared with that required by the fluctuation-dissipation relation. Here we see clearly the contrast between the predictions of field theory with and without statistical physics considerations.

Before we describe the technical procedures, we wish to point out that statistical connotations of fluctuations in quantum field theory is not an entirely foreign concept or construct.

B. Fluctuations in composite operators

There are a variety of problems in nonequilibrium field theory which are most naturally described in terms of the evolution of composite operators, the familiar lowest order ones being the particle number and the energy momentum densities and their fluctuations. The usual approach to these problems assumes that these operators have small fluctuations around their expectation values, in which case they can be expressed in terms of the Green's functions of the theory. However, when fluctuations are large, typically when correlations among several particles are important, this approximation breaks down.

A familiar example is critical phenomena: by choosing a suitable order parameter to describe the different phases, one can obtain a wealth of information on the phase diagram of a system. But to study the dynamics of a phase transition, especially in the regime where fluctuations get large, the single order parameter must be replaced by a locally defined field obeying a stochastic equation of motion, for example, a time-dependent Ginzburg-Landau equation with noise (which though often put in by hand, should in theory be derived from fluctuation-dissipation considerations—*if* one can identify the closed system and show the origin of noise). It is important for our discussion to observe that by going, say, from the time-independent Landau-Ginzburg equation to the Langevin equation one has introduced a new field, since the solution to the latter can no longer be understood as the ''expectation value'' of the order parameter, nor can it be identified as the actual field (which will usually be a q-number).

The same phenomenon occurs more generally in effective field theories, where the light fields are randomized by the back reaction from the heavy fields [23], and in semiclassical theories, where the classical field (for example, the gravitational field in the early Universe) is subject to random driving forces from activities in the quantum field, such as particle creation [24]. The object of our present concerns is yet another example: in the stochastic Boltzmann equation [5,22] the stochastic distribution function is neither the expected value of the number of particles in a given phase space cell, nor the actual number (whose dynamics is given by the full, not the truncated, hierarchy).

The influence of noise on the classical dynamics of a quantum system is discussed at length by Gell-Mann and Hartle [25]; the conversion of quantum fluctuations to classical noise is discussed by a number of authors [26–30]. This scheme was also used by us for the study of decoherence of correlation histories and correlation noise in interacting field theories [31–33].

C. Self-contained dynamics for the propagators

In terms of the technical procedures, our goal is to obtain a self-contained dynamics for the propagators. What this means is that the propagators or correlation functions should carry in them the effect of their interaction with the higher correlation functions as embodied in the BBGKY or Dyson-Schwinger hierarchy. [A similar consideration in purely field theoretical rather than statistical mechanical terms is the incorporation of radiative corrections which manifest as loop effects in perturbative renormalization theory. In a later section we will make clear the relation between loop order and correlation order in terms of the n particle irreducible (PI) effective action.] Since there is *ab initio* an infinite tower of higher order correlation functions which interact with the propagators of a given order, it is hopeless to accomplish this goal, not unlike what Boltzmann confronted with the full molecular dynamics in terms of the distribution and correlation functions. The key step which makes this possible is slaving—the imposition of molecular chaos assumption for Boltzmann. The necessary consequence is the appearance of dissipative behavior in the dynamics of the correlation functions, and, in view of the dissipation-fluctuation relation explained above, the necessary existence of noise as well.

To capture these new aspects of the problem, and following the precedents from Langevin and Boltzmann-Langevin equations, we shall seek a description of the field in terms of a new object, namely, a stochastic correlation function $\mathbf{G}^{ab}$ whose fluctuations reproduce the quantum fluctuations in the binary products of field operators $\phi^a\phi^b$, and whose noise average gives the usual two point functions $G^{ab}=\langle\phi^a\phi^b\rangle$ [we use closed-time-path (CTP) techniques and notation, described more fully in the Appendix [34]].

It ought to be clear that, since the composite operator $\phi^a\phi^b$ is a q-number, a substitute depiction in terms of a classical stochastic kernel cannot be complete. It is suitable for certain types of problems where the mean value behaves classically while quantum fluctuations can be mimiced by statistical distributions (some examples are mentioned above). While the advantage for such a description is evident, i.e., greater simplicity of the c-number formalism, its justification ultimately rests on how much relevant quantum features it can retain.

Consider a theory of a scalar field ϕ^a (we use a condensed notation where the index a denotes both a space-time point and one or the other branch of the time path—see Appendix). The CTP action is $S=S[\phi^1]-S^*[\phi^2]$. Introduce the generating functional

$$Z[K_{ab}]=e^{iW[K_{ab}]}=\int D\phi^a e^{i\{S+(1/2)K_{ab}\phi^a\phi^b\}}, \quad (1.1)$$

then

$$G^{ab}=\langle\phi^a\phi^b\rangle=2\left.\frac{\delta W}{\delta K_{ab}}\right|_{K=0} \quad (1.2)$$

but also

$$\left.\frac{\delta^2 W}{\delta K_{ab}\delta K_{cd}}\right|_{K=0}=\frac{i}{4}\{\langle\phi^a\phi^b\phi^c\phi^d\rangle-\langle\phi^a\phi^b\rangle\langle\phi^c\phi^d\rangle\}. \quad (1.3)$$

This suggests viewing the stochastic kernel $\mathbf{G}^{ab}$ as a Gaussian process defined (formally) by the relationships

$$\langle\mathbf{G}^{ab}\rangle=\langle\phi^a\phi^b\rangle;\quad \langle\mathbf{G}^{ab}\mathbf{G}^{cd}\rangle=\langle\phi^a\phi^b\phi^c\phi^d\rangle. \quad (1.4)$$

Or else, calling

$$\mathbf{G}^{ab}=G^{ab}+\Delta^{ab}, \quad (1.5)$$

$$\langle\Delta^{ab}\rangle=0,\quad \langle\Delta^{ab}\Delta^{cd}\rangle=-4i\left.\frac{\delta^2 W}{\delta K_{ab}\delta K_{cd}}\right|_{K=0}. \quad (1.6)$$

To turn the intuitive ansatz Eqs. (1.4) and (1.6) into a rigorous formalism we must deal with the obvious fact that we are manipulating complex expressions; in particular, it is not clear the Δs define a stochastic process at all. However, for our present purposes it will prove enough to deal with the propagators as if they were real quantities. The reason is that we are primarily concerned with the large occupation numbers or semiclassical limit, where the propagators do become real. We will see that this prescription will be sufficient to extract unambiguous results from the formal manipulations below.

Before getting involved in formal manipulations, notwithstanding, let us dwell on one qualitative aspect of the ansatz above, namely, that the correlations may be described by a Gaussian kernel.

Our assumption of correlations as a Gaussian process may be compared with the usual modeling of the action of a large system in equilibrium over some small system as Gaussian white noise. It would not be easy to find an environment whose driving action were exactly white; indeed, if we believe that no physical system can be excited to arbitrarily large frequencies, we may say it is impossible to find such an environment. But in actual applications this approximation is well within the accuracy of the phenomenological models (such as the time-dependent Landau-Ginzburg equation containing a white noise as stochastic source), which have proven to be quite widely useful.

In setting forth to construct the present theory we do have a concrete set of applications in mind, namely, those where the kinetic limit is relevant. The assumptions which enter into the construction of this theory should therefore be consistent with such limits. This means that we shall obtain, from the short distance behavior of the correlations, a distribution function F, and then we want to study the correlation between the value of F in some region of space with its value in some other region which is far away in terms of the correlation length of the field itself. In other words, an expectation value such as $\langle F(x)F(y)F(z)\rangle$, where the separation between x, y and z is large with respect to the Compton length of the field, and spacelike, may be written in terms of averages of quantities such as $\langle \phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)\phi(z_1)\phi(z_2)\rangle$, where the points $x_i(y_i,z_i)$ stay close to $x(y,z)$ and remain far away from the other pairs.

It is a general consequence of the clustering properties of correlations [4] that the irreducible part of this expectation value will be much weaker than its reducible parts. Retaining only the latter amounts to treating F as a Gaussian variable, and then the ansatz Eq. (1.4) is optimal in that it guarantees that the 2-point stochastic averages exactly match the corresponding quantities as computed from quantum field theory. These considerations give a physical basis for the Gaussian ansatz.

D. Stochastic Boltzmann equation

We can define the process Δ^{ab} also in terms of a stochastic equation of motion. Consider the Legendre transform of W, the so-called 2 particle irreducible (2PI) effective action (EA)

$$\Gamma[\mathbf{G}^{ab}]=W[K^*_{ab}]-\frac{1}{2}K^*_{ab}\mathbf{G}^{ab},\quad K^*_{ab}=-2\frac{\delta\Gamma}{\delta\mathbf{G}^{ab}}. \tag{1.7}$$

We have the identities

$$\frac{\phi\Gamma}{\delta G^{ab}}=0;\quad \frac{\delta^2 W}{\delta K_{ab}\delta K_{cd}}=\frac{-1}{4}\left[\frac{\delta^2\Gamma}{\delta\mathbf{G}^{ab}\delta\mathbf{G}^{cd}}\right]^{-1} \tag{1.8}$$

the first of which is just the truncated Schwinger-Dyson equation for the propagators; we therefore propose the following equations of motion for $\mathbf{G}^{ab}$:

$$\frac{\delta\Gamma}{\delta\mathbf{G}^{ab}}=\frac{-1}{2}\kappa_{ab}, \tag{1.9}$$

where κ_{ab} is a stochastic nonlocal Gaussian source defined by

$$\langle\kappa_{ab}\rangle=0,\quad \langle\kappa_{ab}\kappa_{cd}\rangle=4i\left[\frac{\delta^2\Gamma}{\delta G^{ab}\delta G^{cd}}\right]^{\dagger}. \tag{1.10}$$

If we linearize Eq. (1.9) around G, then the correlation Eq. (1.10) for κ implies Eq. (1.6) for Δ. Consistent with our recipe of handling G as if it were real we should treat κ also as if it were a real source.

It is well known that the noiseless Eq. (1.9) can be used as a basis for the derivation of transport equations in the near equilibrium limit. Indeed, for a $\lambda\phi^4$ type theory, the resulting equation is simply the Boltzmann equation for a distribution function f defined from the Wigner transform of G^{ab} (details are given below). We shall show in this paper that the full stochastic equation (1.9) leads, in the same limit, to a Boltzmann-Langevin equation, thus providing the microscopic basis for this equation in a manifestly relativistic quantum field theory.

Let us first examine some consequences of Eq. (1.10). For a free field theory, we can compute the 2PI EA explicitly (derivation in Sec. V)

$$\Gamma[\mathbf{G}^{ab}]=\frac{-i}{2}\ln[\mathrm{Det}\,\mathbf{G}]-\frac{1}{2}c_{ab}(-\Box+m^2)\mathbf{G}^{ab}(x,x), \tag{1.11}$$

where c_{ab} is the CTP metric tensor (see the Appendix). We immediately find

$$\frac{\delta^2\Gamma}{\delta G^{ab}\delta G^{cd}}=\frac{i}{2}(G^{-1})_{ac}(G^{-1})_{db}. \tag{1.12}$$

Therefore

$$\langle\Delta^{ab}\Delta^{cd}\rangle=i\left[\frac{\delta^2\Gamma}{\delta G^{ab}\delta G^{cd}}\right]^{-1}=G^{ac}G^{db}+G^{da}G^{bc}, \tag{1.13}$$

an eminently sensible result. Observe that the stochastic source does not vanish in this case, rather

$$\langle\kappa_{ab}\kappa_{cd}\rangle=G^{-1}_{ac}G^{-1}_{db}+G^{-1}_{da}G^{-1}_{bc}. \tag{1.14}$$

However

$$(G^{-1})_{ac}\sim -ic_{ac}(-\Box+m^2) \tag{1.15}$$

does vanish on mass-shell. Therefore, when we take the kinetic theory limit, we shall find that for a free theory, there are no on-shell fluctuations of the distribution function. For an interacting theory this is no longer the case.

The physical reason for this different behavior is that the evolution of the distribution function for an interacting theory is dissipative, and therefore basic statistical mechanics considerations call for the presence of fluctuations [35]. Indeed it is this kind of consideration which led us to think about a Boltzmann-Langevin equation in the first place. This is fine if one takes a statistical mechanical viewpoint, but one is used to the idea that quantum field theories are unitary and complete with no information loss, so how could one see dissipation or noise?

In field theory there is a particular derivation of the self consistent dynamics for Green functions which resolves this puzzle, namely when the Dyson equations are derived from the variation of a nonlocal action functional, the two-particle irreducible effective action (2PI-EA). This was originally introduced [36] as a convenient way to perform nonperturbative resummation of several Feynman graphs. When cast in the Schwinger-Keldysh ''closed time path'' (CTP) formulation [34], it guarantees real and causal evolution equations for the Green functions of the theory. It is conceptually clear if one begins with a ''master'' effective action (MEA) [32] where all Green functions of the theory appear as arguments, and then systematically eliminate all higher-than-two point functions to arrive at the 2PIEA.

As mentioned earlier, to us the correct approach is to view the two point functions as an effectively open system [37,18,38], separated from yet interacting with the hierarchy of higher correlation functions obeying the set of Schwinger-

Dyson equations. The averaged effect of its interaction with an environment of slaved higher irreducible correlations brings about dissipation and the attending fluctuations give rise to the correlation noise [32]. This is the conceptual basis of our program.

E. Organization of this paper

In this paper, we shall concentrate on the issue of what kind of fluctuations may be convincingly derived from the 2PI-CTP-EA for Green functions, and how they compare to the fluctuation-dissipation noise in the kinetic theory limit. Given the complexity of the subject, we shall adopt a line of development which favors at least in the beginning ease of understanding over completeness. That is, instead of starting from the master effective action of n point functions and work our way down in a systematic way, we shall begin with the Boltzmann equation for one-particle distributions and work our way up.

In the next section, we present briefly the fluctuation-dissipation theorem in a nonrelativistic context, and use it to derive the fluctuating Boltzmann equation. The discussion, kept at the classical level, simply reviews well established results in the theory of the Boltzmann equation. Section III reviews the basic tenets of nonequilibrium quantum field theory as it concerns the dynamics of correlations, and the retrieval of the Boltzmann equation therefrom. We refrain from using functional methods, so as to keep the discussion as intuitive as possible.

Section IV discusses how the functional derivation of the Schwinger-Dyson hierarchy suggests that these equations ought to be enlarged to include stochastic terms. By going through the kinetic limit we use these results to establish a comparison with the purely classical results of Sec. II.

Our investigation into the physical origin of noise and dissipation in the dynamics of correlation functions shows that in the final analysis this is an effective dynamics, obtained from averaging out the higher correlations. This point is made most explicit in the approach whereby the 2PI EA for the correlations is obtained through truncation of the master effective action, this being the formal functional whose variations generate the full Schwinger-Dyson hierarchy. In Sec. V, we briefly discuss the definition and construction of the master effective action, the relationship of truncation to common approximation schemes, and present explicitly the calculation leading to the dynamics of the two point functions at three loops accuracy [32,10].

In the last section we give a brief discussion of the meaning of our results and possible implications on renormalization group theory.

II. STOCHASTIC BOLTZMANN EQUATION FROM FDT

As a primer, we wish to introduce the fluctuation-dissipation theorem (FDT) or relation (FDR) in a rudimentary yet complete form, and use it to give a simple derivation of the stochastic Boltzmann equation. In this way its physical content can stand out clear before we get formal.

There are many different versions [39]: It could be taken to mean the formulas relating dissipative coefficients to time integrals of correlation functions (sometimes called the Landau-Lifshitz FDT) or the relations between the susceptibility and the space integral of the correlation function. In this paper, the fluctuation-dissipation theorem addresses the relation between the dissipative coefficients of the effectively open system and the autocorrelation of random forces acting on the system, as illustrated below.

A. Fluctuation-dissipation theorem (FDT)

The simplest setting [40] for the FDT is a homogeneous system described by variables x^i. The thermodynamics is encoded in the form of the entropy $S(x^i)$. The thermodynamic fluxes are the derivatives $\dot{x}^i$, and the thermodynamic forces are the components of the gradient of the entropy

$$F_i = -\frac{\partial S}{\partial x^i}. \tag{2.1}$$

The dynamics is given by

$$\dot{x}^i = -\gamma^{ij} F_j + j^i. \tag{2.2}$$

The first term describes the mean regression of the system towards a local entropy maximum, γ^{ij} being the dissipative coefficient or function, and the second term describes the random microscopic fluctuations induced by its interaction with an environment. Near equilibrium, we also have the phenomenological relations for linear response

$$F_i = c_{ij} x^j, \tag{2.3}$$

where c_{ij} is a nonsingular matrix.

In a classical theory, the equal time statistics of fluctuations is determined by Einstein's law

$$\langle x^i(t) F_j(t) \rangle = \delta^i_j. \tag{2.4}$$

Take a derivative to find

$$0 = c_{jk}\{\langle(-\gamma^{il}F_l + j^i)x^k\rangle + \langle x^i(-\gamma^{kl}F_l + j^k)\rangle\}. \tag{2.5}$$

If the noise is Gaussian,

$$\langle x^i(t) j^k(t)\rangle = \int dt' \frac{\delta x^i(t)}{\delta j^l(t')} \langle j^l(t') j^k(t)\rangle,$$

and white

$$\langle j^l(t') j^k(t)\rangle = \nu^{lk}\delta(t'-t), \tag{2.6}$$

then

$$\langle x^i(t) j^k(t)\rangle = \frac{1}{2}\nu^{ik}. \tag{2.7}$$

From Eqs. (2.5) and (2.4) we find the noise-noise autocorrelation function ν^{ik} is related to the symmetrized dissipative function γ^{ik} by

$$\nu^{ik} = [\gamma^{ik} + \gamma^{ki}], \tag{2.8}$$

which is the FDT in a simple classical formulation.[1]

In the case of a one-dimensional system, the above argument can be simplified even further because there is only one variable x, and γ, c, ν are simply constants. In equilibrium, we have $\langle x^2\rangle = c^{-1}$. On the other hand, the late time solution of the equations of motion reads

$$x(t)=\int^t du e^{-\gamma c(t-u)} j(u),$$

which implies $\langle x^2\rangle = \nu/2\gamma c$. Thus $\nu = 2\gamma$, in agreement with Eq. (2.8).

B. Boltzmann equation for a classical relativistic gas

We shall apply the theory above to a dilute gas of relativistic classical particles [44]. The system is described by its one particle distribution function $f(X,k)$, where X is a position variable, and k is a momentum variable. Momentum is assumed to lie on a mass shell $k^2+M^2=0$ [we use the Misner-Thorne-Wheeler (MTW) convention, with signature $-+++$ for the background metric [45]] and have positive energy $k^0>0$. In other words, given a spatial element $d\Sigma^\mu = n^\mu d\Sigma$ and a momentum space element d^4k, the number of particles with momentum k lying within that phase space volume element is

$$dn = -4\pi f(X,k)\,\theta(k^0)\,\delta(k^2+M^2)k^\mu n_\mu d\Sigma \frac{d^4k}{(2\pi)^4}. \tag{2.9}$$

The dynamics of the distribution function is given by the Boltzmann equation, which we give in a notation adapted to our later needs, and for the time being without the sought-after stochastic terms

$$k^\mu \frac{\partial}{\partial X^\mu} f(k) = I_{col}(X,k), \tag{2.10}$$

$$I_{col}=\frac{\lambda^2}{4}(2\pi)^3\int\left[\prod_{i=1}^{3}\frac{d^4p_i}{(2\pi)^4}\,\theta(p_i^0)\,\delta(p_i^2+M^2)\right] \times[(2\pi)^4\delta(p_1+p_2-p_3-k)]\mathbf{I}, \tag{2.11}$$

$$\mathbf{I}=\{[1+f(p_3)][1+f(k)]f(p_1)f(p_2) -[1+f(p_1)][1+f(p_2)]f(p_3)f(k)\}. \tag{2.12}$$

The entropy flux is given by

$$S^\mu(X)=4\pi\int\frac{d^4p}{(2\pi)^4}\,\theta(p^0)\,\delta(p^2+M^2)p^\mu\{[1+f(p)] \times \ln[1+f(p)]-f(p)\ln f(p)\}, \tag{2.13}$$

while the entropy itself S is (minus) the integral of the flux over a Cauchy surface. Now consider a small deviation from the equilibrium distribution

$$f=f_{eq}+\delta f, \tag{2.14}$$

$$f_{eq}=\frac{1}{e^{\beta p^0}-1}, \tag{2.15}$$

corresponding to the same particle and energy fluxes

$$\int\frac{d^4p}{(2\pi)^4}\,\theta(p^0)\,\delta(p^2+M^2)p^\mu\,\delta f(p)=0, \tag{2.16}$$

$$\int\frac{d^4p}{(2\pi)^4}\,\theta(p^0)\,\delta(p^2+M^2)p^\mu p^0\,\delta f(p)=0. \tag{2.17}$$

Then the variation in entropy becomes

$$\delta S=-2\pi\int d^3X\int\frac{d^4p}{(2\pi)^4}\,\theta(p^0) \times\delta(p^2+M^2)p^0\frac{1}{[1+f_{eq}(p)]f_{eq}(p)}(\delta f)^2. \tag{2.18}$$

In the classical theory, the distribution function is concentrated on the positive frequency mass shell. Therefore, it is convenient to label momenta just by its spatial components $\vec{p}$, the temporal component being necessarily $\omega_p = \sqrt{M^2+\vec{p}^2}>0$. In the same way, it is simplest to regard the distribution function as a function of the three momentum $\vec{p}$ alone, according to the rule

$$f^{(3)}(X,\vec{p})=f[X,(\omega_p,\vec{p})], \tag{2.19}$$

where f represents the distribution function as a function on four dimensional momentum space, and $f^{(3)}$ its restriction to three dimensional mass shell. With this understood, we shall henceforth drop the superscript, using the same symbol f for both functions, since only the distribution function on mass shell enters into our discussion. The variation of the entropy now reads

$$\delta S=-\frac{1}{2}\int d^3X\int\frac{d^3p}{(2\pi)^3}\frac{1}{[1+f_{eq}(p)]f_{eq}(p)}(\delta f)^2. \tag{2.20}$$

From Einstein's formula, we conclude that, in equilibrium, the distribution function is subject to Gaussian fluctuations, with equal time mean square value

$$\langle\delta f(t,\vec{X},\vec{p})\,\delta f(t,\vec{Y},\vec{q})\rangle=(2\pi)^3\delta(\vec{X}-\vec{Y})\,\delta(\vec{p}-\vec{q}) \times[1+f_{eq}(p)]f_{eq}(p). \tag{2.21}$$

[1]To be concrete, this is the FDT of the second kind in the classification of Ref. [41]. The FDT of the first kind is further discussed in Ref. [42]. Also observe that we are only concerned with small deviations from equilibrium; FDT's valid arbitrarily far from equilibrium are discussed in Ref. [43].

One of the goals of this paper is to rederive this result as the kinetic theory limit of the general fluctuation formula given for the propagators in the Introduction, Eq. (1.6). For the time being, we only observe that this fluctuation formula is quite independent of the processes which sustain equilibrium; in particular, it holds equally for a free and an interacting gas, since it contains no coupling constants.

In the interacting case, however, a stochastic source is necessary to sustain these fluctuations. Following the discussion of the FDR above, we compute these sources by writing the dissipative part of the equations of motion in terms of the thermodynamic forces

$$F(X,\vec{p})=\frac{1}{[1+f_{eq}(p)]f_{eq}(p)}\frac{\delta f(X,\vec{p})}{(2\pi)^3}. \tag{2.22}$$

To obtain an equation of motion for $f(X,\vec{p})$ multiply both sides of the Boltzmann equation, Eq. (2.10), by $\theta(k^0)\delta(k^2+M^2)$ and integrate over k^0 to get

$$\frac{\partial f}{\partial t}+\frac{\vec{k}}{\omega_k}\vec{\nabla} f=\frac{1}{\omega_k}I_{col}. \tag{2.23}$$

Upon variation we get

$$\frac{\partial(\delta f)}{\partial t}+\frac{\vec{k}}{\omega_k}\vec{\nabla}(\delta f)=\frac{1}{\omega_k}\delta I_{col}. \tag{2.24}$$

When we write δI_{col} in terms of the thermodynamic forces, we find local terms proportional to $F(k)$ as well as nonlocal terms where F is evaluated elsewhere. We shall keep only the former, as it is usually done in deriving the ''collision time approximation'' to the Boltzmann equation [46] (also related to the Krook-Bhatnager-Gross kinetic equation); thus we write

$$\delta I_{col}(k)\sim-\omega_k\nu^2(X,\vec{k})F(X,\vec{k}), \tag{2.25}$$

where

$$\nu^2(X,\vec{k})=\frac{\lambda^2}{4\omega_k}(2\pi)^6\int\left[\prod_{i=1}^{3}\frac{d^4p_i}{(2\pi)^4}\theta(p_i^0)\delta(p_i^2-M^2)\right]\times[(2\pi)^4\delta(p_1+p_2-p_3-k)]I_+, \tag{2.26}$$

$k^0=\omega_k$, and

$$I_+=[1+f_{eq}(p_1)][1+f_{eq}(p_2)]f_{eq}(p_3)f_{eq}(k). \tag{2.27}$$

Among other things, the linearized form of the Boltzmann equation provides a quick estimate of the relevant relaxation time. Let us assume the high temperature limit, where $f\sim T/M$, and the integrals in Eq. (2.26) are restricted to the range $p\leqslant M$. Then simple dimensional analysis yields the estimate $\tau\sim M/\lambda^2T^2$ for the relaxation time appropriate to long wavelength modes.

C. Fluctuations in the Boltzmann equation

Observance of the FDT demands that a stochastic source j be present in the Boltzmann equation, Eq. (2.10) [and its linearized form, Eq. (2.24)], which should assume the Langevin form:

$$\frac{\partial f}{\partial t}+\frac{\vec{k}}{\omega_k}\nabla f=\frac{1}{\omega_k}I_{col}+j(X,\vec{k}). \tag{2.28}$$

Then

$$\langle j(X,\vec{p})j(Y,\vec{q})\rangle=-\left\{\frac{1}{\omega_p}\frac{\partial I_{col}(X,\vec{p})}{\delta F(Y,\vec{q})}+\frac{1}{\omega_q}\frac{\partial I_{col}(Y,\vec{q})}{\delta F(X,\vec{p})}\right\}. \tag{2.29}$$

From Eqs. (2.25), (2.26) and (2.27) we find the noise autocorrelation

$$\langle j(X,\vec{k})j(Y,\vec{p})\rangle=2\delta^{(4)}(X-Y)\delta(\vec{k}-\vec{p})\nu^2(X,\vec{k}), \tag{2.30}$$

where ν^2 is given in Eq. (2.26). Equation (2.30) and (2.26) are the solution to our problem, that is, they describe the fluctuations in the Boltzmann equation, required by consistency with the FDT. Observe that, unlike Eq. (2.21), the mean square value of the stochastic force vanishes for a free gas.

In this discussion, of course, we accepted the Boltzmann equation as given without tracing its origin. We now want to see how the noises in Eq. (2.30) originate from a deeper level, that related to the higher correlation functions, which we call the correlation noises.

III. KINETIC FIELD THEORY, FROM DYSON TO BOLTZMANN

Our goal in this section is to show how the Boltzmann equation arises as a description of the dynamics of quasiparticles in the kinetic limit of field theory. To this end, we shall adopt the view that the main element in the description of a nonequilibrium quantum field is its Green functions, whose dynamics is given by the Dyson equations. This connects with the results of our earlier paper on dissipation in Boltzmann equations [10]. The task is to find the noise or fluctuation terms. The need to upgrade the Boltzmann equation to a Langevin form will lead to a similar generalization of Dyson's equations, whose physical origin will be the subject of the remaining of the paper.

The discussion of propagators is simplest for a free field theory, and so, following our choice of physical clarity over formal rigor in the exposition, we shall first discuss nonequilibrium free fields. The general case follows.

A. Free fields and propagators

Let us focus on the nonequilibrium dynamics of a real scalar quantum (Heisenberg) field $\Phi(x)$, obeying the Klein-Gordon equation

$$(\Box-m^2)\Phi(x)=0 \tag{3.1}$$

and the canonical equal time commutation relations

$$[\dot{\Phi}(\vec{x},t),\Phi(\vec{y},t)]=-i\hbar\,\delta(\vec{x}-\vec{y}) \tag{3.2}$$

(from here on, we take $\hbar=1$).

We shall assume throughout that the expectation value of the field vanishes. Thus the simplest nontrivial description of the dynamics will be in terms of the two-point or Green functions, namely the expectation values of various products of two field operators. Of particular relevance is the Jordan propagator

$$G(x,x')=\langle[\Phi(x),\Phi(x')]\rangle, \tag{3.3}$$

which for a free field is independent of the state of the field. From the Jordan propagator we derive the causal propagators, advanced and retarded,

$$G_{adv}(x,x')=-iG(x,x')\,\theta(t'-t),$$
$$G_{ret}(x,x')=iG(x,x')\,\theta(t-t'). \tag{3.4}$$

These propagators describe the evolution of small perturbations (they are fundamental solutions to the Klein-Gordon equation) but contain no information about the state. For that purpose we require other propagators, such as the positive and negative frequency ones

$$G_{+}(x,x')=\langle\Phi(x)\Phi(x')\rangle,\quad G_{-}(x,x')=\langle\Phi(x')\Phi(x)\rangle. \tag{3.5}$$

Observe that $G=G_{+}-G_{-}$. The symmetric combination gives the Hadamard propagator

$$G_1=G_{+}+G_{-}=\langle\{\Phi(x),\Phi(x')\}\rangle. \tag{3.6}$$

Note that while the Jordan, advanced and retarded propagators emphasize the dynamics, and the negative, positive frequency and Hadamard propagators emphasize the statistical aspects, two other propagators contain both kinds of information. They are the Feynman and Dyson propagators

$$G_F(x,x')=\langle T[\Phi(x)\Phi(x')]\rangle$$
$$=\frac{1}{2}[G_1(x,x')+G(x,x')\mathrm{sgn}(t-t')], \tag{3.7}$$

$$G_D(x,x')=\langle\tilde{T}[\Phi(x)\Phi(x')]\rangle$$
$$=\frac{1}{2}[G_1(x,x')-G(x,x')\mathrm{sgn}(t-t')], \tag{3.8}$$

where T stands for time-ordered product

$$T[\Phi(x)\Phi(x')]=\Phi(x)\Phi(x')\,\theta(t-t')+\Phi(x')\Phi(x)\,\theta(t'-t) \tag{3.9}$$

and $\tilde{T}$ for antitemporal ordering

$$\tilde{T}[\Phi(x)\Phi(x')]=\Phi(x')\Phi(x)\,\theta(t-t')+\Phi(x)\Phi(x')\,\theta(t'-t). \tag{3.10}$$

B. Equilibrium structure of propagators

In this subsection, we shall review several important properties of the equilibrium propagators which follow from the Kubo-Martin-Schwinger (KMS) condition [Eq. (3.12) below] [47], and general invariance properties.

In equilibrium, all propagators must be time-translation invariant, and may be Fourier transformed

$$G(x,x')=\int\frac{d^4k}{(2\pi)^4}e^{ik(x-x')}G(k). \tag{3.11}$$

In particular, because the Jordan propagator is antisymmetric, we must have $G(\omega,\vec{k})=-G(-\omega,\vec{k})$. Also, since $G(x,x')=G(x',x)^*=-G(x,x')^*,G(k)=G(k)^*$.

The positive and negative frequency propagators are further related by the KMS condition

$$G_{+}[(t,\vec{x}),(t',\vec{x}')]=G_{-}[(t+i\beta,\vec{x}),(t',\vec{x}')], \tag{3.12}$$

where β is the inverse temperature. With $G_{+}-G_{-}=G$, we get

$$G_{+}(k)=\frac{G(k)}{1-e^{-\beta k^0}}=\mathrm{sgn}(k^0)\left[\theta(k^0)+\frac{1}{e^{\beta|k^0|}-1}\right]G(k), \tag{3.13}$$

$$G_{-}(k)=\frac{G(k)}{e^{\beta k^0}-1}=\mathrm{sgn}(k^0)\left[\theta(-k^0)+\frac{1}{e^{\beta|k^0|}-1}\right]G(k). \tag{3.14}$$

Adding these two equations, we find

$$G_1(k)=2\ \mathrm{sgn}(k^0)\left[\frac{1}{2}+\frac{1}{e^{\beta|k^0|}-1}\right]G(k). \tag{3.15}$$

We may consider this formula as the quantum generalization of the FDT, as we shall see below. Let us stress that Eqs. (3.12)–(3.15) hold for interacting as well as free fields.

Of course, since G is an odd homogeneous solutions to the Klein-Gordon equation we must have

$$G(k)=\delta(k^2+m^2)\mathrm{sgn}(k^0)g(k), \tag{3.16}$$

which leads to

$$G_{ret}(x,x')=\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-x')}}{-(k^0-i\epsilon)^2+\omega_k^2}\left[\frac{g(\omega_k,\vec{k})}{2\pi}\right] \tag{3.17}$$

and to

$$G_F(x,x')=\int\frac{d^4k}{(2\pi)^4}e^{ik(x-x')}\times\left[\frac{(-i)}{-k^{02}+\omega_k^2-i\epsilon}+\frac{2\pi\delta(k^{02}-\omega_k^2)}{e^{\beta|k^0|}-1}\right]\times\left[\frac{g(\omega_k,\vec{k})}{2\pi}\right], \tag{3.18}$$

with similar formulas for G_{adv} and G_D, respectively. It is remarkable that all propagators may be split into a vacuum and a thermal contribution, with the thermal part being the same for all propagators except G, G_{ret} and G_{adv}, where it vanishes. Also, we have expressed all propagators in terms of g; in the language of the Lehmann decomposition, this is just the density of states [48].

We shall finish this subsection by expanding our remark on Eq. (3.15) being the fluctuation dissipation theorem [49]. Suppose we try to explain the quantum and statistical fluctuations of the field by adding an external source $-j(x)$ to the right hand side of the Klein-Gordon equation (3.1). The resulting field would be

$$\Phi(x)=\int d^4x' G_{ret}(x,x')j(x').$$

If the process is stationary

$$\langle j(x)j(x')\rangle=\int \frac{d^4k}{(2\pi)^4}e^{ik(x-x')}\nu(k), \tag{3.19}$$

we get

$$\nu(k)=\frac{G_1(k)}{2|G_{ret}(k)|^2}.$$

From Eqs. (3.15) and (3.17)

$$\nu(k)=\left[1+\frac{2}{e^{\beta|k^0|}-1}\right]|\mathrm{Im}\,G_{ret}^{-1}(k)|, \tag{3.20}$$

which is a generalized form of the FDT, including both quantum and thermal fluctuations.

So far, we have intentionally left everything expressed in terms of the density of states $g(k)$. For a free field, we can compute this explicitly

$$g(k)=2\pi, \tag{3.21}$$

with which we can fill in the remaining results.

C. Interacting fields and the Dyson equation

Let us now consider a weakly interacting field, obeying the Heisenberg equation

$$(\Box-m^2)\Phi(x)-\frac{\lambda}{6}\Phi^3(x)=0 \tag{3.22}$$

and the same equal time canonical commutation relations, Eq. (3.2). As before, we shall assume that the expectation value of the field vanishes identically, and seek to describe the dynamics in terms of the propagators introduced earlier.

In the usual approach to field theory, where one focuses on computing the *S*-matrix elements, rather than the causal evolution of fields, the leading role is played by the Feynman propagator, which is directly related to the *S* matrix through the Lehmann-Symanzik-Zimmermann (LSZ) reduction formulas, and has a simple perturbative expansion [48,50,51]. We may obtain a dynamical equation for the Feynman propagator by noting that, from Eq. (3.9),

$$(-\Box+m^2)T[\Phi(x)\Phi(x')]=T[(-\Box+m^2)\Phi(x)\Phi(x')]-i\delta(x-x').$$

Therefore

$$(-\Box+m^2)G_F(x,x')=-i\delta(x-x')-\frac{\lambda}{6}\langle T[\Phi^3(x)\Phi(x')]\rangle \tag{3.23}$$

[cf. Eq. (3.18)]. This is the Dyson equation for the propagator, relating the evolution of the Feynman propagator to higher order (in this case, four point) correlation functions. As different from an IN-OUT matrix element of the *S*-matrix, in this case we have an IN-IN expectation value taken with respect to a nontrivial state defined at some initial time.

Equation (3.23) does not yet define a self-contained dynamics for the propagators. To achieve this goal, we must further ''slave'' the higher correlation funtion $\langle T[\Phi^3(x)\Phi(x')]\rangle$, meaning that we must adopt some scheme that will allow us to express this correlation as a functional of the propagators themselves. These schemes may be generally understood as imposing specific boundary conditions on the Schwinger-Dyson equations for the higher correlations [10] which is similar to the role of the molecular chaos assumption in Boltzmann's theory. In our case, we shall substitute $\langle T[\Phi^3(x)\Phi(x')]\rangle$ by its perturbative expansion. Because of causality, the perturbative expansion of the self energy term cannot be expressed in terms of the IN-IN Feynman propagator alone. We should rather have

$$\langle T[\Phi^3(x)\Phi(x')]\rangle\sim 3G_F(x,x)G_F(x,x') -i\lambda\int d^4y\{G_F^3(x,y)G_F(y,x') -G_-^3(x,y)G_+(y,x')\}$$

[since we use full propagators in the internal lines, two-particle reducible (2PR) graphs must not be included]. Thus to obtain a self-consistent dynamics, we must enlarge the set to include other propagators as well. Of course, we are assuming that the initial state is such that Wick's theorem holds (for example, that it is Gaussian)—this issue is discussed in detail in [10]. We are also leaving aside issues of renormalization [52].

We overcome this difficulty by adopting as fundamental object the closed-time-path ordered propagator $G_P(x^a,y^b)$. This object is equivalent to four ordinary propagators: if we write $G_P(x^a,y^b)=G^{ab}(x,y)$, then $G^{11}(x,y)=G_F(x,y)$, $G^{12}(x,y)=G_-(x,y)$, $G^{21}(x,y)=G_+(x,y)$ and $G^{22}(x,y)=G_D(x,y)$.

We can obtain closed dynamical equations for these four propagators. Actually there is a slight redundancy, but this set has the advantage of being very simple to handle (see Chou *et al.* in [34] for details). The equations read

$$\left[-\Box+m^2+\frac{\lambda}{2}G_F(x,x)\right]G^{ab}(x,x')-\frac{i\lambda^2}{6}c_{cd}\times\int d^4y\Sigma^{ac}(x,y)G^{db}(y,x')=-ic^{ab}\delta(x-x'), \quad (3.24)$$

$$\Sigma^{ac}(x,y)=[G^{ac}(x,y)]^3. \quad (3.25)$$

The matrix c ($c_{11}=c^{11}=1, c_{22}=c^{22}=-1$; all others zero) keeps track of the sign inversions associated with the reverse temporal ordering of the second branch. This form of the Dyson equation is relevant to our discussion.

D. The kinetic theory limit

In equilibrium the propagators are time-translation invariant. Out of equilibrium this is no longer true. In the kinetic theory regime, however, the propagators depend mostly on the difference variable $u=x-x'$, with the corresponding Fourier transform depending weakly on the center of mass variable $X=(1/2)(x+x')$. As such, the propagators take the form

$$G^{ab}(x,x')=\int\frac{d^4k}{(2\pi)^4}e^{ik(x-x')}G^{ab}(X,k). \quad (3.26)$$

The Σ kernel has a similar expression

$$\Sigma^{ab}(x,x')=\int\frac{d^4k}{(2\pi)^4}e^{ik(x-x')}\Sigma^{ab}(X,k),$$

$$\Sigma^{ab}(X,k)=\int\prod_{i=1}^{3}\left\{\frac{d^4p_i}{(2\pi)^4}G^{ab}(X,p_i)\right\}\times\left[(2\pi)^4\delta\left(\sum p_i-k\right)\right]. \quad (3.27)$$

The weak dependence on X allows for the approximations (details in [10])

$$G^{ab}(x,x)=\int\frac{d^4k}{(2\pi)^4}G^{ab}(x,k)\sim\int\frac{d^4k}{(2\pi)^4}G^{ab}(X,k),$$

$$\int d^4y\Sigma^{ac}(x,y)G^{db}(y,x')\sim\int\frac{d^4k}{(2\pi)^4}e^{ik(x-x')}\Sigma^{ac}(X,k)G^{db}(X,k)$$

and the equations of motion become

$$\left[k^2-ik^\mu\frac{\partial}{\partial X^\mu}-\frac{1}{4}\Box_X+M^2(X)\right]G^{ab}(X,k)-\frac{i\lambda^2}{6}c_{cd}\Sigma^{ac}(X,k)G^{db}(X,k)=-ic^{ab}, \quad (3.28)$$

$$M^2(X)=m^2+\frac{\lambda}{2}\int\frac{d^4k}{(2\pi)^4}G^{ab}(X,k). \quad (3.29)$$

Alternatively, we may think of the propagators as functions of x', leading to an equation of the form [cf. Eq. (3.24)]

$$\left[-\Box'+m^2+\frac{\lambda}{2}G_F(x',x')\right]G^{ab}(x,x')-\frac{i\lambda^2}{6}c_{cd}\int d^4yG^{ac}(x,y)\Sigma^{db}(y,x')=-ic^{ab}\delta(x-x'). \quad (3.30)$$

In the kinetic limit, this yields

$$\left[k^2+ik^\mu\frac{\partial}{\partial X^\mu}-\frac{1}{4}\Box_X+M^2(X)\right]G^{ab}(X,k)-\frac{i\lambda^2}{6}c_{cd}G^{ac}(X,k)\Sigma^{db}(X,k)=-ic^{ab}. \quad (3.31)$$

Taking the average and the difference of Eqs. (3.28) and (3.31) we get

$$\left[k^2-\frac{1}{4}\Box_X+M^2(X)\right]G^{ab}(X,k)-\frac{i\lambda^2}{12}c_{cd}\{\Sigma^{ac}(X,k)G^{db}(X,k)+G^{ac}(X,k)\Sigma^{db}(X,k)\}=-ic^{ab}, \quad (3.32)$$

$$k^\mu\frac{\partial}{\partial X^\mu}G^{ab}(X,k)+\frac{\lambda^2}{12}c_{cd}\{\Sigma^{ac}(X,k)G^{db}(X,k)-G^{ac}(X,k)\Sigma^{db}(X,k)\}=0. \quad (3.33)$$

We recognize the first equation as a mass shell condition on the nonequilibrium propagator. The second equation is the kinetic equation proper, describing relaxation towards equilibrium.

To investigate further this equation, we observe that since both terms are already of second order in λ (see [10]), it is enough to solve the mass shell condition to zeroth order. That is, we assume that the renormalized mass M^2 is actually position independent, and write

$$G^{ab}(X,k)=G_0^{ab}(M^2,k)+G_{stat}^{ab}(X,k), \quad (3.34)$$

where the $G_0^{ab}(M^2,k)$ are vacuum propagators for a free field with mass M^2, and G_{stat} is the nonvacuum part

$$G_{stat}^{ab}(X,k)=2\pi\delta(k^2+M^2)f(X,k), \quad (3.35)$$

which we assume is the same for all propagators involved, as in the free field case. $f(X,k)$ has the physical interpretation of a one particle distribution function for quasiparticles built out of the field excitations. Substituting Eqs. (3.34) and (3.35) into (3.33), and assuming, for example, that $k^0>0$ (f

must be even in k, because of the symmetries of the propagators) immediately shows that the dynamics of f is given by the Boltzmann equations (2.10), (2.11) and (2.12).

We shall not discuss further the region of validity of the hypothesis underlying the kinetic limit, except to observe that this issue is far from trivial. On general grounds, one expects that propagators will depend strongly on the difference variable on scales $\tau_C \sim M^{-1}$. For smooth initial conditions, the scale for dependence on the average variable is set by the relaxation time $\tau \sim M/\lambda^2 T^2$. A nontrivial kinetic limit exists if $\tau \gg \tau_C$. Already this simple estimate shows that one would expect trouble in theories with strictly massless particles, such as gauge or Goldstone bosons [12]. If particle masses are not specially protected, then at large temperature the physical mass $M \sim \sqrt{\lambda} T$, and $\tau_C/\tau \sim \lambda$ will in general be suitably small.

E. Stochastic Dyson equations

Our derivation of kinetic theory from the perturbative Dyson equations leads to a dissipative equation similar to the (noiseless) Boltzmann equation. But we know that in addition to the usual collision integral an explicitly stochastic term ought to be in place. Our earlier considerations of the fluctuation-dissipation relations attest that this stochastic term has autocorrelation Eqs. (2.30) and (2.26). Since no manipulation of the deterministic Dyson equations (from the truncated Schwinger-Dyson set) will yield a stochastic term like this, we posit that when quantum field theory is viewed in the statistical mechanical context, they need be supplemented with a noise term. Suppose we add a stochastic driving term F^{ab} to them (we shall justify this in the next section) as follows:

$$\left[-\Box + m^2 + \frac{\lambda}{2} G_F(x,x)\right] G^{ab}(x,x') - \frac{i\lambda^2}{6} c_{cd} \int d^4y \Sigma^{ac}(x,y) G^{db}(y,x') = -ic^{ab}\delta(x-x') - iF^{ab}(x,x'), \tag{3.36}$$

$$\left[-\Box' + m^2 + \frac{\lambda}{2} G_F(x',x')\right] G^{ab}(x,x') - \frac{i\lambda^2}{6} c_{cd} \int d^4y G^{ac}(x,y) \Sigma^{db}(y,x') = -ic^{ab}\delta(x-x') - i\tilde{F}^{ab}(x,x'). \tag{3.37}$$

In the kinetic limit, the random forces become

$$F^{ab}(x,x') = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} F^{ab}(X,k) \tag{3.38}$$

(and similarly for $\tilde{F}$). Leaving aside the random fluctuations of the mass shell, we find the new kinetic equation

$$k^\mu \frac{\partial}{\partial X^\mu} G^{ab}(X,k) + \frac{\lambda^2}{12} c_{cd}\{\Sigma^{ac}(X,k) G^{db}(X,k) - G^{ac}(X,k)\Sigma^{db}(X,k)\} = H^{ab}(X,k), \tag{3.39}$$

where

$$H^{ab} \equiv \frac{1}{2}[F - \tilde{F}]^{ab}(X,k). \tag{3.40}$$

Our problem now is to justify changing the truncated Dyson equation to Eq. (3.36), and to expound the physical meaning of this new stochastic equation. To do this we need to use functional methods, to which we now turn.

IV. CORRELATION NOISE AND STOCHASTIC BOLTZMANN EQUATION

Our goal in this section is to show how noise terms such as those introduced above from phenomenological considerations may actually be systematically identified from an appropriate effective action. In this section we shall limit ourselves to finding a suitable recipe to identify the noise terms, and to compare the results to the phenomenological discussion above. The physical foundations of the recipe shall be discussed in the following section.

A. Fluctuations in the propagators

We shall now adapt the foregoing discussion to the study of fluctuations in the dynamics of the two point functions (which arises from the truncated Schwinger-Dyson hierarchy). The first step is to notice that their dynamics can be obtained from the variation of the 2PI action functional [we derive this formula in Sec. V, Eq. (5.51)]

$$\Gamma[G^{ab}] = \frac{-i}{2}\ln[\operatorname{Det} G] - \frac{1}{2} c_{ab} \int d^4x(-\Box + m^2) G^{ab}(x,x) - \frac{\lambda}{8} c_{abcd} \int d^4x G^{ab}(x,x) G^{ab}(x,x) + \frac{i\lambda^2}{48} c_{abcd} c_{efgh} \int d^4x d^4x' G^{ae}(x,x') \times G^{bf}(x,x') G^{cg}(x,x') G^{dh}(x,x'). \tag{4.1}$$

The resulting equations of motion

$$\frac{-i}{2} G^{-1}_{ab} - \frac{1}{2}\left[c_{ab}(-\Box + m^2) + \frac{\lambda}{2} c_{abcd} G^{cd}(x,x)\right]\delta(x,x') + \frac{i\lambda^2}{12} c_{ac} c_{bd} [G^{cd}(x,x')]^3 = 0 \tag{4.2}$$

are seen to be equivalent to the truncated Dyson equations, Eq. (3.24).

As we discussed in the Introduction, we shall incorporate quantum fluctuations in the evolution of the Green function G^{ab} by explicitly adding a stochastic source $(-1/2)\kappa_{ab}$ to the right hand side of Eq. (4.2), and reinterpreting it as an equation for a stochastic correlation function $\mathbf{G}^{ab}$. Let us write $\mathbf{G}^{ab}=G^{ab}+\Delta^{ab}$, and expand the 2PI CTP EA to second order [the first order term vanishes by virtue of Eq. (4.2)],

$$\delta\Gamma=\delta_2\Gamma, \tag{4.3}$$

$$\delta_2\Gamma[\Delta^{ab}]=\frac{i}{4}G^{-1}_{ab}\Delta^{bc}G^{-1}_{cd}\Delta^{da} - \frac{\lambda}{8}c_{abcd}\int d^4x\Delta^{ab}(x,x)\Delta^{ab}(x,x) + \frac{i\lambda^2}{8}c_{abcd}c_{efgh}\int d^4xd^4x'G^{ac}(x,x') \times G^{bf}(x,x')\Delta^{cg}(x,x')\Delta^{dh}(x,x'). \tag{4.4}$$

From now on, we shall assume that the background tadpole vanishes, and identify the mass with its renormalized value. We now have, as discussed in the Introduction,

$$\langle\Delta^{ab}\Delta^{cd}\rangle=i\left[\frac{\delta^2\Gamma}{\delta G^{ab}\delta G^{cd}}\right]^{-1}. \tag{4.5}$$

To sustain these fluctuations, the noise autocorrelation must be

$$\langle\kappa_{ab}\kappa_{cd}\rangle=(4i)\left[\frac{\delta^2}{\delta\Delta^{ab}\delta\Delta^{cd}}\,\delta_2\Gamma\bigg|_{\Delta=0}\right]^{\dagger}.$$

That is

$$\langle\kappa_{ab}(x,x')\kappa_{cd}(y,y')\rangle=N_{abcd}(x,x',y,y')+N^{int}_{abcd}(x,x',y,y'),$$

$$N_{abcd}(x,x',y,y')=G^{-1}_{da}(y',x)G^{-1}_{bc}(x',y)+G^{-1}_{ac}(x,y)G^{-1}_{db}(y',x'), \tag{4.6}$$

$$N^{int}_{abcd}(x,x',y,y')=\lambda^2[G^{eg}G^{fh}](x,x')c_{acef}c_{bdgh}\times\delta(x-y)\,\delta(x'-y')-i\lambda c_{abcd}\times\delta(x-x')\,\delta(y-y').$$

B. Free fields

Let us begin by asking whether for free fields the quantum fluctuations Eq. (4.5) go into anything like the classical result Eq. (2.21) in the kinetic theory limit. There is no obvious reason why it should be so, since the physical basis for either formula is at first sight totally different. As we saw in the Introduction, Eq. (4.5) simply reproduces the full quantum fluctuations, computed in terms of the propagators themselves on the assumption that Wick theorem holds (which is an assumption on the allowed initial states of the field, see [10])

$$\langle\Delta^{ab}\Delta^{cd}\rangle=i\left[\frac{\delta^2\Gamma}{\delta G^{ab}\delta G^{cd}}\right]^{-1}=G^{ac}G^{db}+G^{da}G^{bc}, \tag{4.7}$$

while the classical autocorrelation Eq. (2.21) has been found by applying Einstein's formula to the phenomenological entropy Eq. (2.13). The only clear point of contact between both approaches is that both assume Bose statistics.

Introducing the Wigner transform of the fluctuations

$$\Delta^{ab}(x,x')=\int\frac{d^4k}{(2\pi)^4}e^{ik(x-x')}\Delta^{ab}(X,k),\quad X=\frac{1}{2}(x+x'), \tag{4.8}$$

we observe that in this case we are not entitled to assume that the dependence of the Wigner transform on X is weak. Equation (4.8) has a formal inverse

$$\Delta^{ab}(X,k)=\int du e^{iku}\Delta^{ab}\left(X+\frac{u}{2},X-\frac{u}{2}\right), \tag{4.9}$$

and from Eq. (4.7) we get

$$\langle\Delta^{ab}(X,p)\Delta^{cd}(Y,q)\rangle=\int dudv e^{i(pu+qv)}K[X,Y,u,v],$$

where

$$K[X,Y,u,v]=G^{ac}\left(X+\frac{u}{2},Y+\frac{v}{2}\right)G^{bd}\left(X-\frac{u}{2},Y-\frac{v}{2}\right)+G^{ad}\left(X+\frac{u}{2},Y-\frac{v}{2}\right)G^{bc}\left(X-\frac{u}{2},Y+\frac{v}{2}\right).$$

The propagators in the right hand side are equilibrium ones, and so we can use the representation Eq. (3.11)

$$K[X,Y,u,v]=\int\frac{d^4r}{(2\pi)^4}\frac{d^4s}{(2\pi)^4}\int dudv e^{i(pu+qv)}K[X,Y,r,s],$$

$$K[X,Y,r,s]=e^{ir[X-Y+(1/2)(u-v)]}e^{is[X-Y-(1/2)(u-v)]}G^{ac}(r)G^{bd}(s)+e^{ir[X-Y+(1/2)(u+v)]}e^{is[X-Y-(1/2)(u+v)]}\times G^{ad}(r)G^{bc}(s).$$

Now integrate over u, v and s

$$\langle\Delta^{ab}(X,p)\Delta^{cd}(Y,q)\rangle=16\int d^4re^{i2(r+p)(X-Y)}\times[\delta(p+q)G^{ac}(r)G^{bd}(r+2p)+\delta(p-q)G^{ad}(r)G^{bc}(r+2p)]. \tag{4.10}$$

We have 16 different quantum autocorrelations to compare against a single classical result, so we can only expect real agreement in the large occupation number limit, where all propagators converge to the same expression. With this proviso in mind, we can choose any combination of indices to continue the calculation. The most straightforward choice (to a certain extent suggested by the structure of the closed-time-path; see [31,32]) is $a=b=1$, $c=d=2$; we are thus seeking the correlations among the fluctuations in the Feynman and Dyson propagators

$$\begin{aligned}\langle\Delta^{11}(X,p)\Delta^{22}(Y,q)\rangle &= 16(2\pi)^2[\delta(p+q)+\delta(p-q)]\\ &\quad\times\int d^4r e^{i2(r+p)(X-Y)}\delta[r^2+M^2]\delta[(r+2p)^2+M^2]\\ &\quad\times[\theta(-r^0)+f_{eq}(r)][\theta(-r^0-2p^0)+f_{eq}(r+2p)].\end{aligned}$$

The arguments of the delta functions can be simplified

$$\begin{aligned}\langle\Delta^{11}(X,p)\Delta^{22}(Y,q)\rangle &= 4(2\pi)^2[\delta(p+q)+\delta(p-q)]\int d^4r e^{i2(r+p)(X-Y)}\\ &\quad\times\delta[r^2+M^2]\times\delta[rp+p^2][\theta(-r^0)+f_{eq}(r)]\\ &\quad\times[\theta(-r^0-2p^0)+f_{eq}(r+2p)]. \end{aligned} \tag{4.11}$$

A difference from the classical case already stands out here: in the quantum case, a fluctuation in the number of particles with momentum p correlates not only with itself, but also with the corresponding fluctuation in the number of antiparticles with momentum $-p$. This is unavoidable, given the symmetries of the propagators in this theory.

Let us stress that we are trying to push the quasiparticle (kinetic) description of quantum field dynamics beyond the calculation of mean values (of such quantities as particle number or energy density), to account for their fluctuations. The calculation of the fluctuations of the distribution function for on-shell particles gives a crucial consistency check on such an attempt. Indeed, we know that each on-shell mode of the free field contributes an amount [cf. Eqs. (3.15), (3.16) and (3.21)] $\rho_k\sim\omega_k(1/2+f_{eq})$ to the mean energy density, where f_{eq} is the equilibrium distribution function Eq. (2.15). The fluctuations of this quantity at equilibrium will be given by $\langle\delta\rho_k^2\rangle=T^2(\partial\rho_k/\partial T)\sim\omega_k^2 f_{eq}(1+f_{eq})$. So, if these fluctuations are still described by a distribution function consistent with ordinary statistical mechanics, then this distribution function must fluctuate like in Eq. (2.21). (This may in the face of it be a rather big *if*.)

For large M^2, the condition that p is nearly on-shell means that the spatial components are much smaller than the time component, and we may approximate

$$\delta[r^2+M^2]\delta[rp+p^2]\sim\delta[p^2+M^2]\frac{1}{|p^0|}\delta(r^0+p^0),$$

thus obtaining

$$\begin{aligned}\langle\Delta^{11}(X,p)\Delta^{22}(Y,q)\rangle|_{\text{on-shell}} &= \frac{1}{2\omega_p}(2\pi)^5[\delta(p+q)+\delta(p-q)]f_{eq}(p)\\ &\quad\times(1+f_{eq}(p))\delta[p^2+M^2]\delta(\vec{X}-\vec{Y}).\end{aligned}$$

To finish the comparison, assume, e.g., that $p^0\geqslant 0$, then

$$\begin{aligned}\delta(p-q)\delta[p^2+M^2] &= \delta(q^0-\omega_q)\delta(\vec{q}-\vec{p})\delta[p^2+M^2]\\ &= 2\omega_q\theta(q^0)\delta(\vec{q}-\vec{p})\delta[p^2+M^2]\\ &\quad\times\delta[q^2+M^2]. \end{aligned} \tag{4.12}$$

This result suggests writing

$$\Delta^{11}(X,p)=2\pi\delta f(X,\vec{p})\delta[p^2+M^2]+\text{off-shell terms}. \tag{4.13}$$

Taking p^0 and q^0 to be positive, this yields

$$\begin{aligned}\langle\delta f(t,\vec{X},\vec{p})\delta f(t,\vec{Y},\vec{q})\rangle &= (2\pi)^3\delta(\vec{q}-\vec{p})\delta(\vec{X}-\vec{Y})f_{eq}(p)\\ &\quad\times(1+f_{eq}(p)), \end{aligned} \tag{4.14}$$

which is identical to Eq. (2.21). This is one of the most important results of this paper, as it gives a whole new meaning to the phenomenological entropy Eq. (2.13)

We have thus completed our proof, and obtained new independent confirmation of the validity of our scheme for introducing stochastic kernels as a way to describe the quantum fluctuations in the dynamics of correlations.

C. Interacting fields and the Boltzmann-Langevin equation

The results of the previous section already imply that the full stochastic Dyson equation will go over to the Boltzmann-Langevin equation in the kinetic limit. Indeed, the structure of the fluctuations does not change drastically when interactions are switch on, and since they become identical in the classical limit, the noise in the Dyson equation necessary to sustain the fluctuations at the quantum level must go over to the noise in the Boltzmann equation, which plays the same role in the classical theory. Nevertheless, it is worth identifying exactly which part of the quantum source autocorrelation goes into the classical one in the correspondence limit.

Concretely, our aim is to begin with the stochastic (perturbative) Dyson equation

$$\begin{aligned}&\frac{-i}{2}\mathbf{G}_{ab}^{-1}-\frac{1}{2}\left[c_{ab}(-\Box+m^2)+\frac{\lambda}{2}c_{abcd}\mathbf{G}^{cd}(x,x)\right]\delta(x,x')\\ &\quad+\frac{i\lambda^2}{12}c_{ac}c_{bd}[\mathbf{G}^{cd}(x,x')]^3=\frac{-1}{2}\kappa_{ab}, \end{aligned} \tag{4.15}$$

where the noise autocorrelation is given by Eq. (4.6). We then identify the forces appearing in Eqs. (3.36) and (3.37)

$$F^{ab}(x,x')=i\int d^4y c^{ac}\kappa_{cd}(x,y)G^{db}(y,x'),$$
$$\tilde{F}^{ab}(x,x')=i\int d^4y G^{ac}(x,y)\kappa_{cd}(y,x')c^{db}. \quad (4.16)$$

In condensed notation,

$$H^{ab}\equiv\frac{1}{2}[F-\tilde{F}]^{ab}=\frac{i}{2}\{c^{ac}\kappa_{cd}G^{db}-G^{ac}\kappa_{cd}c^{db}\}, \quad (4.17)$$

whose Wigner transform plays the role of random force in the kinetic equation (3.39). Restricting ourselves to on-shell fluctuations, we can compute the autocorrelation of this force and compare the result to the classical expectation Eq. (2.30).

Let us observe from the outset that the classical result involves the expression ν^2 [Eq. (2.26)], which, through l_+ [Eq. (2.27)], is related to the Fourier transform of the cube of a propagator. In Eq. (4.6), the first term N contains the inverse propagators, which in turn is related to the cube of propagators through the Dyson equations (4.2). The other term N^{int} contains no such thing. Thus it is clear that our only chance lies in the first term, the other one contributing to sustaining the nonclassical correlations already present in the free field case. Correspondingly, we shall ignore N^{int} in what follows.

We thus approximate

$$\langle\kappa_{ab}\kappa_{cd}\rangle=G^{-1}_{da}G^{-1}_{bc}+G^{-1}_{ac}G^{-1}_{db}, \quad (4.18)$$

leading to

$$\langle H^{ab}H^{cd}\rangle=\frac{-1}{4}[c^{ae}G^{fb}-G^{ae}c^{fb}][c^{cg}G^{hd}-G^{cg}c^{hd}][G^{-1}_{he}G^{-1}_{fg}+G^{-1}_{eg}G^{-1}_{hf}]$$
$$=\frac{1}{4}[-c^{ae}G^{fb}+G^{ae}c^{fb}][\delta^d_eG^{-1c}_f+\delta^d_fG^{-1c}_e-\delta^c_fG^{-1d}_e-\delta^c_eG^{-1d}_f]$$
$$=\frac{1}{4}[G^{ad}G^{-1bc}+G^{-1ad}G^{bc}-G^{bd}G^{-1ac}-G^{-1bd}G^{ac}+2(c^{ac}c^{bd}-c^{ad}c^{bc})].$$

For the same reasons as above, we shall disregard the propagator-independent terms.

Next, we write [recalling Eq. (3.25)]

$$H^{ab}(x,x')=\int\frac{d^4k}{(2\pi)^4}e^{ik(x-x')}H^{ab}(X,k),$$
$$H^{ab}(X,k)=\int d^4u e^{-iku}H^{ab}\left(X+\frac{u}{2},X-\frac{u}{2}\right) \quad (4.19)$$

to get

$$\langle H^{ab}(X,p)H^{cd}(Y,q)\rangle=\frac{1}{4}\int d^4ud^4v e^{-i(pu+qv)}[J-K], \quad (4.20)$$

where, using the translation invariance of the equilibrium propagators,

$$J=\int\frac{d^4r}{(2\pi)^4}\frac{d^4s}{(2\pi)^4}\exp\left[ir\left(X-Y+\frac{u+v}{2}\right)\right]\exp\left[is\left(X-Y-\frac{u+v}{2}\right)\right]\times\{G^{ad}(r)G^{-1bc}(s)+G^{-1ad}(r)G^{bc}(s)\},$$
$$K=\int\frac{d^4r}{(2\pi)^4}\frac{d^4s}{(2\pi)^4}\exp\left[ir\left(X-Y-\frac{u-v}{2}\right)\right]\exp\left[is\left(X-Y+\frac{u-v}{2}\right)\right]\times\{G^{bd}(r)G^{-1ac}(s)+G^{-1bd}(r)G^{ac}(s)\}.$$

Upon integration over u and v, the K term gives a contribution proportional to $\delta^{(4)}(p+q)$. This is unrelated to the noise autocorrelation, being only a cross correlation between the positive and negative frequency components of the source, and we shall not analyze it further. We also restrict ourselves to the case where $a=b=1$, $c=d=2$. Using the reflection symmetry of the equilibrium propagators, obtaining the inverse propagators from Eq. (4.2) and retaining only the dominant term in the correspondence limit, we find

$$\langle H^{11}(X,p)H^{22}(Y,q)\rangle \sim \frac{4\lambda^2}{3}\delta(p-q)\int d^4s \cos[2(s+p)\times(X-Y)]\Sigma^{12}(s+2p)G^{12}(s). \tag{4.21}$$

An analysis of this expression shows that in the high temperature limit the correlation length is of order M^{-1}. This is a microscopic scale, much smaller than the macroscales of relevance to the kinetic limit (if this limit exists). Therefore we are justified in writing

$$\langle H^{11}(X,p)H^{22}(Y,q)\rangle \sim \gamma\delta(X-Y). \tag{4.22}$$

We compute γ by simply integrating Eq. (4.21) over X

$$\gamma=\frac{\lambda^2}{12}(2\pi)^4\Sigma^{12}(p)G^{21}(p)\delta(p-q). \tag{4.23}$$

From Eqs. (3.27) and (2.26) we get

$$\gamma=\omega_p(2\pi)^2\delta(p-q)\delta(p^2+M^2)\nu^2. \tag{4.24}$$

Assuming $p^0,q^0\geqslant 0$, this is

$$\gamma=2\omega_p^2(2\pi)^2\delta(\vec{p}-\vec{q})\delta(q^2+M^2)\delta(p^2+M^2)\nu^2. \tag{4.25}$$

So, writing

$$H(X,k)=2\pi\omega_k\delta(k^2+M^2)j(X,\vec{k})+\text{off-shell}, \tag{4.26}$$

we get the final result

$$\langle j(X,\vec{p})j(Y,\vec{q})\rangle=2\delta(\vec{p}-\vec{q})\delta(X-Y)\nu^2, \tag{4.27}$$

which agrees with the classical result, Eq. (2.30).

We have shown that there is a piece of the full quantum noise which can be identified with the classical source j. Clearly j does not account for the full quantum noise, the difference being due among other things to the role of negative frequency in the quantum theory.

Finally, we note that Abe *et al.* [15] have given a nonrelativistic derivation of the Boltzmann-Langevin equation, while ours is fully relativistic, being also immune to the reservations expressed by Greiner and Leupold [17].

V. MASTER EFFECTIVE ACTION

So far in the paper we have referred several times to the possibility of conceiving the low order correlations of a quantum field as the field variables of an open quantum system, interacting with the environment provided by the higher correlations. The goal of this final section is to present a formalism, the master effective action, built on this perspective. In particular, in this formalism the usual Dyson equations are seen to emerge from the averaging over higher correlations. As a simple consequence and illustration, we derive Eq. (4.1).

A. The low order effective actions

The simplest application of functional methods in quantum field theory concerns the dynamics of the expectation value of the field [53]. The expectation value or mean field may be deduced from the generating functional $W[J]$

$$\exp\{iW[J]\}=\int D\Phi\exp\left\{iS[\Phi]+i\int d^4xJ(x)\Phi(x)\right\}, \tag{5.1}$$

$$\phi(x)=\left.\frac{\delta W}{\delta J}\right|_{J=0}. \tag{5.2}$$

We obtain the dynamics from the effective action, which is the Legendre transform of W

$$\Gamma[\phi]=W[J]-\int d^4xJ(x)\phi(x). \tag{5.3}$$

The physical equation of motion is

$$\frac{\delta\Gamma}{\delta\phi}=0. \tag{5.4}$$

In a causal theory, we must adopt Schwinger's CTP formalism. The point x may therefore lie on either branch of the closed time path, or equivalently we may have two background fields $\phi^a(x)=\phi(x^a)$. The classical action is defined as

$$S[\Phi^a]=S[\Phi^1]-S[\Phi^2]^*, \tag{5.5}$$

which automatically accounts for all sign reversals. We also have two sources

$$\int d^4xJ_a(x)\Phi^a(x)=\int d^4x[J^1(x)\Phi^1(x)-J^2(x)\Phi^2(x)]$$

and obtain two equations of motion

$$\frac{\delta\Gamma}{\delta\phi^a}=0. \tag{5.6}$$

However, these equations always admit a solution where $\phi^1=\phi^2=\phi$ is the physical mean field, and after this identification, they become a real and causal equation of motion for ϕ.

The functional methods we have used so far to derive the dynamics of the mean field may be adapted to investigate more general operators. In order to find the equations of motion for two-point functions, for example, we add a nonlocal source $K_{ab}(x,x')$ [36,10]

$$\exp\{iW[J_a,K_{ab}]\}=\int D\Phi^a \exp i\left\{S[\Phi^a]+\int d^4x J_a\Phi^a + \frac{1}{2}\int d^4x d^4x' K_{ab}\Phi^a\Phi^b\right\}. \quad (5.7)$$

It follows that

$$\frac{\delta W}{\delta K_{ab}(x,x')}=\frac{1}{2}[\phi^a(x)\phi^b(x')+G^{ab}(x,x')].$$

Therefore the Legendre transform, the so-called 2PI effective action,

$$\Gamma[\phi^a,G^{ab}]=W[J_a,K_{ab}]-\int d^4x J_a\phi^a - \frac{1}{2}\int d^4x d^4x' K_{ab}[\phi^a\phi^b+G^{ab}] \quad (5.8)$$

generates the equations of motion

$$\frac{\delta\Gamma}{\delta\phi^a}=-J_a-K_{ab}\phi^b;\quad \frac{\delta\Gamma}{\delta G^{ab}}=-\frac{1}{2}K_{ab}. \quad (5.9)$$

The goal of this section is to show these two examples as just successive truncations of a single object, the master effective action.

B. Formal construction

In this section, we shall proceed with the formal construction of the master effective action, a functional of the whole string of Green functions of a field theory whose variation generates the Dyson-Schwinger hierarchy. Since we are using Schwinger-Keldish techniques, all fields are to be defined on a closed time path. Also we adopt DeWitt's condensed notation [54].

We consider then a scalar field theory whose action

$$S[\Phi]=\frac{1}{2}S_2\Phi^2+S_{int}[\Phi] \quad (5.10)$$

decomposes into a free part and an interaction part

$$S_{int}[\Phi]=\sum_{n=3}^{\infty}\frac{1}{n!}S_n\Phi^n. \quad (5.11)$$

Here and after, we use the shorthand

$$K_n\Phi^n\equiv\int d^dx_1\ldots d^dx_n K_{na^1\ldots a^n}(x_1,\ldots x_n) \times\Phi^{a^1}(x_1)\ldots\Phi^{a^n}(x_n), \quad (5.12)$$

where the kernel K is assumed to be totally symmetric.

Let us define also the ''source action''

$$J[\Phi]=J_1\Phi+\frac{1}{2}J_2\Phi^2+J_{int}[\Phi], \quad (5.13)$$

where $J_{int}[\Phi]$ contains the higher order sources

$$J_{int}[\Phi]=\sum_{n=3}^{\infty}\frac{1}{n!}J_n\Phi^n \quad (5.14)$$

and define the generating functional

$$Z[\{J_n\}]=e^{iW[\{J_n\}]}=\int D\Phi e^{iS_t[\Phi,\{J_n\}]}, \quad (5.15)$$

where

$$S_t[\Phi,\{J_n\}]=J_1\Phi+\frac{1}{2}(S_2+J_2)\Phi^2+S_{int}[\Phi]+J_{int}[\Phi]. \quad (5.16)$$

We shall also call

$$S_{int}[\Phi]+J_{int}[\Phi]=S_I. \quad (5.17)$$

As it is well known, the Taylor expansion of Z with respect to J_1 generates the expectation values of path-ordered products of fields

$$\frac{\delta^n Z}{\delta J_{1a^1}(x_1)\ldots\delta J_{1a^n}(x_n)}=\langle P\{\Phi^{a^1}(x_1)\ldots\Phi^{a^n}(x_n)\}\rangle \equiv F_n^{a^1\ldots a^n}(x_1,\ldots x_n), \quad (5.18)$$

while the Taylor expansion of W generates the ''connected'' Green functions (''linked cluster theorem'' [4])

$$\frac{\delta^n W}{\delta J_{1a^1}(x_1)\ldots\delta J_{1a^n}(x_n)}=\langle P\{\Phi^{a^1}(x_1)\ldots\Phi^{a^n}(x_n)\}\rangle_{\text{connected}} \equiv C_n^{a^1\ldots a^n}(x_1,\ldots x_n). \quad (5.19)$$

Comparing these last two equations, we find the rule connecting the F's with the C's. First, we must decompose the ordered index set $(i_1,\ldots i_n)$ $[i_k=(x_k,a^k)]$ into all possible clusters P_n. A cluster is a partition of $(i_1,\ldots i_n)$ into N_{P_n} ordered subsets $p=(j_1,\ldots j_r)$. Then

$$F_n^{i_1\ldots i_n}=\sum_{P_n}\prod_p C_r^{j_1\ldots j_r}. \quad (5.20)$$

Now from the obvious identity

$$\frac{\delta Z}{\delta J_{ni_1\ldots i_n}}\equiv\frac{1}{n!}\frac{\delta^n Z}{\delta J_{i_1}\ldots\delta J_{i_n}} \quad (5.21)$$

we obtain the chain of equations

$$\frac{\delta W}{\delta J_{ni_1\ldots i_n}}\equiv\frac{1}{n!}\sum_{P_n}\prod_p C_r^{j_1\ldots j_r}. \quad (5.22)$$

We can invert these equations to express the sources as functionals of the connected Green functions, and define the master effective action (MEA) as the full Legendre transform of the connected generating functional

$$\Gamma_\infty[\{C_r\}]=W[\{J_n\}]-\sum_n \frac{1}{n!}J_n\sum_{P_n}\prod_p C_r. \qquad (5.23)$$

The physical case corresponds to the absence of external sources, whereby

$$\frac{\delta\Gamma_\infty[\{C_r\}]}{\delta C_s}=0. \qquad (5.24)$$

This hierarchy of equations is equivalent to the Dyson-Schwinger series.

C. The background field method

The master effective action just introduced becomes more manageable if one applies the background field method (BFM) [53] approach. We first distinguish the mean field and the two point functions

$$C_1^i\equiv\phi^i, \qquad (5.25)$$

$$C_2^{ij}\equiv G^{ij}, \qquad (5.26)$$

We then perform the Legendre transform in two steps: first with respect to ϕ and G only, and then with respect to the rest of the Green functions. The first (partial) Legendre transform yields

$$\Gamma_\infty[\phi,G,\{C_r\}]\equiv\Gamma_2[\phi,G,\{J_n\}]-\sum_{n\geqslant 3}\frac{1}{n!}J_n\sum_{P_n}\prod_p C_r. \qquad (5.27)$$

Here Γ_2 is the two particle-irreducible (2PI) effective action [36]

$$\Gamma_2[\phi,G,\{J_n\}]=S[\phi]+\frac{1}{2}G^{jk}S_{,jk}-\frac{i}{2}\ln\,\mathrm{Det}\,G+J_{int}[\phi]+\frac{1}{2}G^{jk}J_{int,jk}+W_2 \qquad (5.28)$$

and W_2 is the sum of all 2PI vacuum bubbles of a theory whose action is

$$S'[\varphi]=\frac{i}{2}G^{-1}\varphi^2+S_Q[\varphi], \qquad (5.29)$$

$$S_Q[\varphi]=S_I[\phi+\varphi]-S_I[\phi]-S_I[\phi]_{,i}\varphi^i-\frac{1}{2}S_I[\Phi]_{,ij}\varphi^i\varphi^j, \qquad (5.30)$$

where φ is the fluctuation field around ϕ, i.e., $\Phi=\phi+\varphi$. Decomposing S_Q into source-free and source-dependent parts, and Taylor expanding with respect to φ, we may define the background-field dependent coupling and sources where

$$\sigma_{ni_1\ldots i_n}=\sum_{m\geqslant n}\frac{1}{(m-n)!}S_{mi_1\ldots i_nj_{n+1}\ldots j_m}\phi^{j_{n+1}}\ldots\phi^{j_m}, \qquad (5.31)$$

$$\chi_{ni_1\ldots i_n}=\sum_{m\geqslant n}\frac{1}{(m-n)!}J_{mi_1\ldots i_nj_{n+1}\ldots j_m}\phi^{j_{n+1}}\ldots\phi^{j_m}. \qquad (5.32)$$

Now, from the properties of the Legendre transformation, we have, for $n>2$,

$$\left.\frac{\delta W}{\delta J_n}\right|_{J_1,J_2}\equiv\left.\frac{\delta\Gamma_\infty}{\delta J_n}\right|_{\phi,G}. \qquad (5.33)$$

Computing this second derivative explicitly, we conclude that

$$\left.\frac{\delta W}{\delta J_n}\right|_{J_1,J_2}\equiv\frac{1}{n!}\phi^n+\frac{1}{2(n-2)!}G\phi^{n-2}+\sum_{m=3}^{n}\frac{\delta\chi_m}{\delta J_n}\frac{\delta W_2}{\delta\chi_m}. \qquad (5.34)$$

Comparing this equation with

$$\frac{\delta W}{\delta J_{ni_1\ldots i_n}}\equiv\frac{1}{n!}\sum_{P_n}\prod_p C_r^{j_1\ldots j_r}, \qquad (5.35)$$

we obtain the identity

$$\frac{\delta W_2}{\delta\chi_{ni_1\ldots i_n}}\equiv\frac{1}{n!}\sum_{P_n}^{*}\prod_p C_r^{j_1\ldots j_r}, \qquad (5.36)$$

where the asterisk above the sum means that clusters containing one element subsets are deleted. This and

$$\sum_{n\geqslant 3}\frac{1}{n!}J_n\sum_{P_n}\prod_p C_r=J_{int}[\phi]+\frac{1}{2}G^{ij}\frac{\delta J_{int}[\phi]}{\delta\phi^i\delta\phi^j}+\sum_{n\geqslant 3}\frac{1}{n!}\chi_n\sum_{P_n}^{*}\prod_p C_r \qquad (5.37)$$

allow us to write

$$\Gamma_\infty[\phi,G,\{C_r\}]\equiv S[\phi]+\left(\frac{1}{2}\right)G^{ij}\frac{\delta S[\phi]}{\delta\phi^i\delta\phi^j}-\frac{i}{2}\ln\,\mathrm{Det}\,G+\left\{W_2[\phi,\{\chi_n\}]-\sum_{n\geqslant 3}\frac{1}{n!}\chi_n\sum_{P_n}^{*}\prod_p C_r\right\}. \qquad (5.38)$$

This entails an enormous simplification, since it implies that to compute Γ_∞ it is enough to consider W_2 as a functional of the χ_n, without ever having to decompose these background dependent sources in terms of the original external sources.

D. Truncation and slaving: Loop expansion and correlation order

After obtaining the formal expression for Γ_∞, and thereby the formal hierarchy of Dyson-Schwinger equations, we should proceed with it much as with the BBGKY hierarchy in statistical mechanics [46], namely, truncate it and close the lower-order equations by constraining the high order correlation functions to be given (time-oriented) functionals of the lower correlations. Truncation proceeds by discarding the higher correlation functions and replacing them by given functionals of the lower ones, which represent the dynamics in some approximate sense [2]. The system which results is an open system and the dynamics becomes an effective dynamics.

It follows from the above that truncations will be generally related to approximation schemes. In field theory we have several such schemes available, such as the loop expansion, large N expansions, expansions in coupling constants, etc. For definiteness, we shall study the case of the loop expansion, although similar considerations will apply to any of the other schemes.

Taking then the concrete example of the loop expansion, we observe that the nonlocal χ sources enter into W_2 in as many nonlinear couplings of the fluctuation field φ. Now, W_2 is given by a sum of connected vacuum bubbles, and any such graph satisfies the constraints

$$\sum nV_n=2i, \tag{5.39}$$

$$i-\sum V_n=l-1, \tag{5.40}$$

where i,l,V_n are the number of internal lines, loops, and vertices with n lines, respectively. Therefore,

$$l=1+\sum \frac{n-2}{2}V_n \tag{5.41}$$

we conclude that χ_n only enters the loop expansion of W_2 at order $n/2$. At any given order l, we are effectively setting $\chi_n\equiv 0,\ n>2l$. Since W_2 is a function of only χ_3 to χ_{2l}, it follows that the C_r's cannot be all independent. Indeed, the equations relating sources to Green functions

$$\frac{\delta W_2}{\delta\chi_{n i_1\dots i_n}}\equiv\frac{1}{n!}\sum_{P_n}^{*}\prod_p C_r^{j_1\dots j_r} \tag{5.42}$$

have now turned, for $n>2l$, into the algebraic constraints

$$\sum_{P_n}^{*}\prod_p C_r^{j_1\dots j_r}\equiv 0. \tag{5.43}$$

In other words, the constraints which make it possible to invert the transformation from sources to Green functions allow us to write the higher Green functions in terms of lower ones. In this way, we see that the loop expansion is by itself a truncation in the sense above and hence any finite loop or perturbation theory is intrinsically an effective theory.

Actually, the number of independent Green functions at a given number of loops is even smaller than $2l$. It follows from the above that W_2 must be linear on χ_n for $l+2\leqslant n\leqslant 2l$. Therefore the corresponding derivatives of W_2 are given functionals of the $\chi_m,\ m\leqslant l+1$. Writing the lower sources in terms of the lower order Green functions, again we find a set of constraints on the Green functions, rather than new equations defining the relationship of sources to functions. These new constraints take the form

$$\sum_{P_n}^{*}\prod_p C_r^{j_1\dots j_r}=f_n(G,C_3,\dots C_{l+1}) \tag{5.44}$$

for $l+2\leqslant n\leqslant 2l$. In other words, to a given order l in the loop expansion, only ϕ, G and C_r, $3\leqslant r\leqslant l+1$, enter into Γ_∞ as independent variables. Higher correlations are expressed as functionals of these by virtue of the constraints implied by the loop expansion on the functional dependence of W_2 on the sources.

However, these constraints are purely algebraic, and therefore do not define an arrow of time. The dynamics of this lower order functions is unitary. Irreversibility appears only when one makes a time-oriented ansatz in the form of the higher correlations, such as the "weakening of correlations" principle invoked in the truncation of the BBGKY hierarchy [2]. This is done by substituting some of the allowed correlation functions at a given number of loops l, by solutions of the l-loop equations of motion. Observe that even if we use exact solutions, the end result is an irreversible theory, because the equations themselves are only an approximation to the true Dyson-Schwinger hierarchy.

To summarize, the truncation of the MEA in a loop expansion scheme proceeds in two stages. First, for a given accuracy l, an l-loop effective action is obtained which depends only on the lowest $l+1$ correlation functions, say, $\{\phi,G,C_3,\dots C_{l+1}\}$. This truncated effective action generates the l-loop equations of motion for these correlation functions. In the second stage, these equations of motion are solved (with causal boundary conditions) for some of the correlation functions, say $\{C_k,\dots C_{l+1}\}$, and the result is substituted into the l loop effective action. (We say that $\{C_k,\dots C_{l+1}\}$ have been slaved to $\{\phi,G,C_3,\dots C_{k-1}\}$.) The resulting truncated effective action is generally complex and the mean field equations of motion it generates will come out to be dissipative, which indicates that the effective dynamics is stochastic.

E. Example: The three-loop 2PI EA

We shall conclude this paper by explicitly computing the 2PI CTP EA for a $\lambda\phi^4$ self-interacting scalar field theory, out of the corresponding MEA. We carry out our analysis at three loops order, this being the lowest order at which the dynamics of the correlations is nontrivial, in the absence of a symmetry breaking background field [10].

To this accuracy, we have room for four nonlocal sources besides the mean field and the two point correlations, namely χ_3, χ_4, χ_5, and χ_6. However, the last two enter linearly in the generating functional. Thus the three-loop effective action only depends nontrivially on the mean field and the two, three and four point correlations. By symmetry, there must be a solution where the mean field and the three point function remain identically zero, which we shall assume.

Our first step is to compute Eq. (5.38), which now reads

$$\Gamma_4[G,C_4]\equiv\left(\frac{-1}{2}\right)c_{ij}(-\Box+M^2)G^{ij}-\frac{i}{2}\ln\text{ Det }G$$
$$+W_2[\phi,\{\chi_n\}]-\frac{1}{24}\chi_{4ijkl}$$
$$\times[C_4^{ijkl}+G^{ij}G^{kl}+G^{ik}G^{jl}+G^{il}G^{jk}], \tag{5.45}$$

where W_2 denotes the sum of 2PI vacuum bubbles of a quantum field theory with quartic self interaction and a coupling constant $\lambda-\chi_4$ [see Eqs. (5.28) and (5.30)] up to three loops

$$W_2=\left(\frac{-1}{8}\right)(\lambda-\chi_4)_{ijkl}G^{ij}G^{kl}+\left(\frac{i}{48}\right)(\lambda-\chi_4)_{ijkl}$$
$$\times(\lambda-\chi_4)_{pqrs}G^{ip}G^{jq}G^{kr}G^{ls}. \tag{5.46}$$

Equation (5.36) yields

$$C_4^{ijkl}=-i(\lambda-\chi_4)_{pqrs}G^{ip}G^{jq}G^{kr}G^{ls}. \tag{5.47}$$

Inverting and substituting back in Eq. (5.45), we obtain

$$\Gamma_4[G,C_4]\equiv\left(\frac{-1}{2}\right)c_{ij}(-\Box+M^2)G^{ij}-\frac{i}{2}\ln\text{ Det }G$$
$$-\left(\frac{1}{8}\right)\lambda_{ijkl}G^{ij}G^{kl}-\left(\frac{1}{24}\right)\lambda_{ijkl}C_4^{ijkl}$$
$$+\left(\frac{i}{48}\right)C_4^{ijkl}[G_{ip}^{-1}G_{jq}^{-1}G_{kr}^{-1}G_{ls}^{-1}]C_4^{pqrs}. \tag{5.48}$$

This functional generates the self-consistent, time reversal invariant dynamics of the two and four particle Green functions to three loop accuracy. To reduce it further to the dynamics of the two point functions alone, we must slave the four point functions. Consider the three loops equation of motion for C_4

$$[G_{ip}^{-1}G_{jq}^{-1}G_{kr}^{-1}G_{ls}^{-1}]C_4^{pqrs}=-i\lambda_{ijkl}. \tag{5.49}$$

Solving for this equation with causal boundary conditions yields

$$G_4^{ijkl}=-i\lambda_{pqrs}G^{ip}G^{jq}G^{kr}G^{ls} \tag{5.50}$$

(in other words, $\chi_4=0$) and substituting back in Eq. (5.48) we obtain

$$\Gamma[G]\equiv\left(\frac{-1}{2}\right)c_{ij}(-\Box+M^2)G^{ij}-\frac{i}{2}\ln\text{ Det }G$$
$$-\left(\frac{1}{8}\right)\lambda_{ijkl}G^{ij}G^{kl}+\left(\frac{i}{48}\right)\lambda_{ijkl}G^{ip}G^{jq}G^{kr}G^{ls}\lambda_{pqrs}, \tag{5.51}$$

which is seen to be equivalent to Eq. (4.1). This effective action leads to a dissipative and, as we have seen, also stochastic dynamics, which results from the slaving of the four point functions.

VI. DISCUSSIONS

In this paper we have introduced a new object, the stochastic correlation function **G**, whose expectation value reproduces the usual propagators (Green functions), but whose fluctuations are designed to account for the quantum fluctuations in the binary product of (operator) fields. We have introduced the dynamical equation for **G** which takes the form of an explicitly stochastic Dyson equation, and showed that in the kinetic limit, both the fluctuations in **G** become the classical fluctuations in the one particle distribution function, and the dynamic equation for **G**'s Wigner transform becomes the Boltzmann-Langevin equation. Each of these results has interest of its own. *A priori*, there is no simple reason why the fluctuations derived from quantum field theory should have a physical meaning corresponding to a phenomenological entropy flux and Einstein's relation.

The notion that Green functions (and indeed, higher correlations as well) may or even ought to be seen as possessing fluctuating characters (when placed in the larger context of the whole hierarchy) with clearly discernable physical meanings is likely to have an impact on the way we perceive the statistical properties of field theory. For example, we are used to fixing the ambiguities of renormalization theory by demanding certain Green functions to take on given values under certain conditions (conditions which should resemble the physical situation of interest as much as possible, as discussed by O'Connor and Stephens [55]). If the Green functions themselves are to be regarded as fluctuating, then the same ought to hold for the renormalized coupling constants defined from them, and for the renormalization group (RG) equations describing their scale dependence.

While the application of renormalization group methods to stochastic equations is presented in well-known monographs [56], our proposal here goes beyond these results in at least two ways. First, in our approach the noise is not put in by hand or brought in from outside (e.g., the environment of an open system), as in the usual Langevin equation approach, but it follows from the (quantum) dynamics of the system itself. Actually, the possibility of learning about the system from the noise properties (whether it is white or colored, additive or multiplicative, etc.)—unraveling the noise, or treating noise creatively—is a subtext in our program. Second, our result suggests that stochasticity may, or should, not only appear at the level of equations of motion, but also the level of the RG equations, as they describe the running of ''constants'' which are themselves fluctuating.

Indeed, the possibility of a nondeterministic renormalization group flow is even clearer if we think of the RG as encoding the process of eliminating irrelevant degrees of freedom from our description of a system [57]. These elimination processes lead as a rule to dissipation and noise, the noise and dissipation in the influence action and the CTP-effective action are but a particular case. If the need for such an enlarged RG has not been felt so far, the groundbreaking work on the dynamical RG by Ma, Mazenko, Hohenberg, Halperin, and many significant others notwithstanding, it is probably due to the fact that the bulk of RG research has been focused on equilibrium, stationary properties rather than far-from-equilibrium dynamics [58]. An attempt to constructing a RG theory for nonequilibrium processes from these considerations is currently under way [59].

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APPENDIX: CLOSED TIME PATH CONVENTIONS

The closed time path (CTP) or Schwinger-Keldysh technique [34] is a bookkeeping device to generate diagrammatic expansions for true expectation values (as opposed to IN-OUT matrix elements) of certain quantum operators. The basic idea is that any expectation value of the form

$$\langle IN|\tilde{T}[\phi(x_1)\dots\phi(x_n)]T[\phi(x_{n+1})\dots\phi_m]|IN\rangle, \quad \text{(A1)}$$

where $|IN\rangle$ is a suitable initial quantum state, x_1 to x_m are space time points, ϕ is the field operator, T stands for time ordering and $\tilde{T}$ for anti time ordering, may be thought of as a path ordered expectation value on a closed time path ranging from $t=-\infty$ to ∞ and back. These path ordered products are generated by path integrals of the form

$$\int D\phi^1 D\phi^2[\phi^2(x_1)\dots\phi^2(x_n)\phi^1(x_{n+1})\dots\phi^1_m] \times e^{i[S(\phi^1)-S^*(\phi^2)]}, \quad \text{(A2)}$$

where ϕ^1 is a field configuration in the forward leg of the path, and ϕ^2 likewise on the return leg. These configurations match each other on a spacelike surface at the distant future. The boundary conditions at the distant past depend on the initial state $|IN\rangle$; for example, if this is a vacuum, then we add a negative imaginary part to the mass. We shall not discuss these boundary conditions further, except to note that we assume the validity of Wick's theorem (see [10]).

In general we shall use a latin index $a,b,\dots.$ taking values 1 or 2 to denote the CTP branches. Where the space time position is not specified, it must be assumed that it has been subsummed within the CTP upper index. Also we shall refer to the expression $S(\phi^a)=S(\phi^1)-S^*(\phi^2)$ as the CTP action. We always use the Einstein sum convention, and if not explicit, integration over space time must be understood as well.

It is convenient to introduce a CTP metric tensor $c_{ab}=\mathrm{diag}(1,-1)$ to keep track of sign inversions. Thus $c_{ab}J^a\phi^b=J^1\phi^1-J^2\phi^2$. In general, we write an expression like this as $J_a\phi^a$, where $J_a=c_{ab}J^b$; the index a has been lowered by means of the metric tensor. The opposite operation of raising an index is accomplished with the inverse metric tensor $c^{ab}=(c^{-1})^{ab}=\mathrm{diag}(1,-1)$. Thus $J^a=c^{ab}J_b$.

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