Vacuum effects in an asymptotically uniformly accelerated frame with a constant magnetic field

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In the present article we solve the Dirac-Pauli and Klein-Gordon equations in an asymptotically uniformly accelerated frame when a constant magnetic field is present. We compute, via the Bogoliubov coefficients, the density of scalar and spin-1/2 particles ''created.'' We discuss the role played by the magnetic field and the thermal character of the spectrum.

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I. INTRODUCTION

The study of quantum effects in nonintertial frames of reference has been thoroughly discussed in the literature. The pioneering articles of Fulling and Unruh [1,2] showing the nonequivalence of the quantization of scalar fields in Rindler and Minkowski coordinates and the thermal character of the radiation were the origin of a large body of articles devoted to analyze quantum measurement processes in uniformly accelerated frames and possible interpretations thereof.

The advantage of considering Rindler coordinates are many. They can be associated with a uniformly accelerated observer. They also possess a global timelike Killing vector and the (massive and massless) Klein-Gordon as well as the Dirac equations are separable in the Rindler coordinates (see Fig. 1). The Rindler coordinates can be extended to cover the whole space time and thermal effects can be related, via the equivalence principle, with the Hawking effect [3,4].

The study of quantum effects in nonuniformly accelerated frames of reference presents, at first glance, different technical problems. Among them we can mention that the system of coordinates associated with non-Rindler kinematics do not possess in general a timelike Killing vector, and therefore a standard interpretation of positive and negative frequency solutions is absent. The complete separation of variables of the Klein-Gordon and the Dirac equations is possible only in a restricted set of coordinates, and those coordinates allowing separability sometimes present coordinate singularities; therefore the quantization scheme fails.

Among the nonstatic coordinate systems where the Klein-Gordon and Dirac equations separate we have [5,6]

$$t+x=\frac{2}{\omega}\sinh\omega(T+X),\quad t-x=\frac{-1}{\omega}e^{-\omega(T-X)},\quad y=y,\quad z=z. \tag{1.1}$$

The line element associated with the coordinate transformation (1.1) is

$$ds^2=(e^{-2\omega T}+e^{2\omega X})(dX^2-dT^2)+dy^2+dz^2. \tag{1.2}$$

The separability of he Klein-Gordon equation in Eq. (1.1) has been discussed by Kalnins [5], one the authors has accomplished a complete separability of the Dirac [7] equation in Eq. (1.1).

The kinematics associated with the coordinates (1.1) have been exhaustively analyzed by Costa [6,8]. The proper time along $X=X_0$ is

$$s=\frac{1}{\omega}e^{-2\omega T}+e^{2\omega X_0}+\frac{1}{\omega}e^{\omega X_0}\sinh^{-1}e^{\omega(T+X_0)}.$$

Considering T as the evolution parameter, we have that an observer co-moving to the system (1.1) has a four-velocity given by

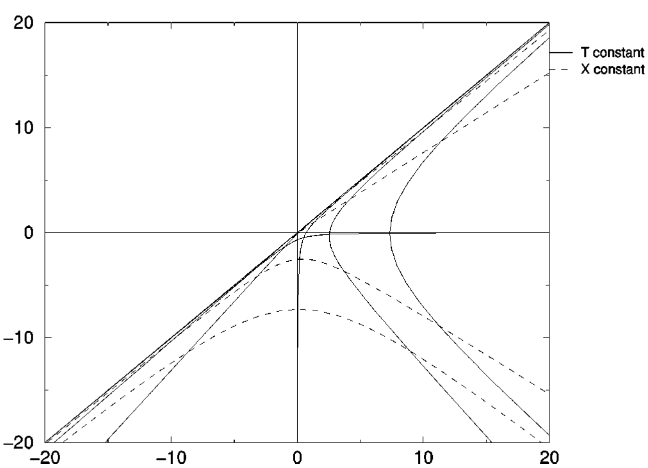

FIG. 1. The accelerated coordinates with $\omega=1$. The solid lines correspond to the spacelike $T=$const Cauchy surfaces. The dashed lined correspond to the timelike curves $X=$const.

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$$\mathbf{V}=\frac{(\cosh\omega(X+T)+e^{\omega(X-T)}/2,\cosh\omega(X+T)-e^{\omega(X-T)}/2,0,0)}{\sqrt{e^{-2\omega T}+e^{2\omega X}}} \tag{1.3}$$

and experiences an acceleration whose components are

$$a_{t,x}=\frac{\omega\sinh\omega(X+T)\mp\omega[e^{\omega(X-T)}/2]}{e^{-2\omega T}+e^{2\omega X}}+\frac{\cosh\omega(X+T)\pm[e^{\omega(X-T)}/2]}{(e^{-2\omega T}+e^{2\omega X})^2}\omega e^{-2\omega T},$$

$$a_y=0,\quad a_z=0. \tag{1.4}$$

From the absolute value of the acceleration, we readily obtain that $a=|a^{\mu}a_{\mu}|^{1/2}$ takes the form

$$a=\omega e^{2\omega X}(e^{-2\omega T}+e^{2\omega X})^{-3/2}. \tag{1.5}$$

From Eq. (1.5) we obtain that the accelerated frame becomes an ''inertial'' Milne system [9,10] as $T\to-\infty$, and on the other hand it evolves toward an uniformly accelerated frame as $T\to+\infty$.

Quantum effects in the noninertial frame (1.1) have been discussed by Costa [6] and by Percoco and Villalba [11] for Dirac particles. The nonexistence of a global timelike Killing vector for the line element (1.2) precludes making a straightforward identification of the positive and negative frequency solutions of the scalar and Dirac wave equations. In order to circumvent this difficulty, we identify positive and negative frequency modes comparing the asymptotic solutions of the wave equations with those obtained for the relativistic Hamilton-Jacobi equation.

Recently, Bautista [12], has discussed, in a Rindler accelerated frame, vacuum effect associated with a spin-1/2 particle with anomalous magnetic moment in a constant magnetic field directed along the acceleration. The author obtains a Planckian distribution of created particles that depends on the magnetic field via the nonminimal anomalous coupling. The author also discusses deviation of the energy density from a thermal distribution due to the magnetic field. As a preliminary step towards a deeper understanding of quantum processes in noninertial frames of reference, in the present paper we analyze vacuum effects associated with scalar and spin-1/2 particles in the coordinates (1.1), when a constant magnetic field is also present.

The article is structured as follows. In Sec. II, we solve the relativistic Hamilton-Jacobi equation with a constant magnetic field in the accelerated frame (1.1). In Sec. III, we solve the Klein-Gordon equation and compute the rate of scalar particles created. In Sec. IV, we solve the Dirac equation with anomalous magnetic moment and compute the density of particles related by the magnetic field in the non inertial frame (1.1). Finally, we discuss the results obtained in this article in Sec. V.

II. SOLUTION OF THE HAMILTON-JACOBI EQUATION

The relativistic Hamilton-Jacobi equation coupled to an electromagnetic field can be written as [6,13]

$$g^{\alpha\beta}(\partial_{\alpha}S-eA_{\alpha})(\partial_{\beta}S-eA_{\beta})+m^2=0, \tag{2.1}$$

where here and elsewhere we adopt the units where $c=1$, and $\hbar=1$. The vector potential associated with a constant magnetic field $\vec{B}=B_x\hat{x}$ directed along the acceleration (1.4) has the form

$$\mathbf{A}=(0,0,B_xz,0). \tag{2.2}$$

It is not difficult to verify that Eq. (2.2) satisfies the conditions $\nabla_{\mu}A^{\mu}=0$ and $F^{\alpha\beta}F_{\alpha\beta}=2B_x^2$.

Substituting the line element (1.2) into Eq. (2.1) we obtain

$$\frac{1}{e^{2\omega X}+e^{-2\omega T}}[(\partial_X S)^2-(\partial_T S)^2]+(\omega_0 z-\partial_y S)^2+(\partial_z S)^2+m^2=0, \tag{2.3}$$

where $\omega_0=eB_x$. The solution of Eq. (2.3) has the form

$$S(X,y,z,T)=-k_y y\pm i\int\sqrt{m^2-\lambda^2+(k_y+\omega_0 z)^2}dz\pm\int\sqrt{\epsilon^2-\lambda^2 e^{2\omega X}}dX\pm\int\sqrt{\epsilon^2+\lambda^2 e^{-2\omega T}}dT, \tag{2.4}$$

which in the asymptotic limit as $X\to\infty$ and $T\to-\infty$ reduces to

$$S(X,y,z,T)=-k_y y\pm i\int\sqrt{m^2-\lambda^2+(k_y+\omega_0 z)^2}dz\pm i\frac{\lambda}{\omega}e^{\omega X}\pm\frac{\lambda}{\omega}e^{-\omega T}. \tag{2.5}$$

The wave function $u(X,y,z,T)=e^{iS}$ gives the quasiclassical asymptotes of the solutions of the Klein-Gordon and Dirac equations. In the remote past, as $T\to-\infty, u_{-\infty}(X,y,z,T)$ takes the form

$$u_{-\infty}(X,y,z,T)=C(y,z)e^{-(\lambda/\omega)e^{\omega X}}c_{\mp}e^{\pm i(\lambda/\omega)e^{-\omega T}}, \tag{2.6}$$

where in Eq. (2.6) the upper and lower signs correspond, respectively, to positive and negative frequency modes.

Analogously, we have that as $X\to\infty$ and $T\to\infty$ Eq. (2.4) reduces to

$$S(X,y,z,T)=-k_y y\pm i\int \sqrt{m^2-\lambda^2+(k_y+\omega_0 z)^2}dz \pm i\frac{\lambda}{\omega}e^{\omega X}\pm\epsilon T \tag{2.7}$$

and, consequently,

$$u_\infty(X,y,z,T)=C'(y,z)e^{-(\lambda/\omega)e^{\omega X}}c'_{\mp}e^{\mp i\epsilon T}, \tag{2.8}$$

where, in the present case, the upper sign corresponds to positive frequency modes and the lower sign to negative modes. The results (2.6) and (2.8) give the quasiclassical asymptotic behaviors of the relativistic wave equations in the accelerated coordinates (1.1).

III. SOLUTION OF THE KLEIN-GORDON EQUATION

In this section we solve the Klein-Gordon equation, coupled to a constant magnetic field, in the accelerated coordinates with the line element given by Eq. (1.2). The covariant generalization of the Klein-Gordon equation is [4,14]

$$g^{\alpha\beta}(\nabla_\alpha-ieA_\alpha)(\nabla_\beta-ieA_\beta)\Phi-m^2\Phi=0, \tag{3.1}$$

where $\nabla_\alpha=\partial_\alpha-\Gamma_\alpha$ is the covariant derivative, and A_α is the vector potential given by Eq. (2.2).

Substituting Eq. (1.2) into (3.1) we readily obtain

$$\left[\frac{1}{e^{-2\omega T}+e^{2\omega X}}(\partial_T^2-\partial_X^2)-(\partial_y^2+\partial_z^2)+i2eB_x z\partial_y + e^2B_x^2z^2-m^2\right]\psi=0. \tag{3.2}$$

Since Eq. (3.2) commutes with the operator $-i\partial_y$ we can look for a solution of the form $\psi=\phi(X,z,T)e^{ik_y y}$ which reduces Eq. (3.2) to

$$\left\{\frac{1}{e^{-2\omega T}+e^{2\omega X}}t(\partial_T^2-\partial_X^2)+(eB_xz-k_y)^2-\partial_z^2-m^2\right\}\phi=0. \tag{3.3}$$

Equation (3.3) can be separated in the form $\phi(X,z,T)=\eta(X,T)f(z)$. The resulting equations are

$$\left[\frac{1}{e^{-2\omega T}+e^{2\omega X}}(\partial_T^2-\partial_X^2)+\lambda^2\right]\eta=0, \tag{3.4}$$

$$\frac{d^2f}{dz^2}=[(\omega_0 z-k_y)^2+(-m^2-\lambda^2)]f(z), \tag{3.5}$$

where λ is a separation constant. Variables X and T can be separated in Eq. (3.4) after making the substitution $\eta(X,T)=f_X(X)f_T(T)$. The resulting equations for X and T are

$$\frac{d^2f_X}{dX^2}=-(-\lambda^2e^{2\omega X}+\epsilon^2)f_X(X), \tag{3.6}$$

$$\frac{d^2f_T}{dT^2}=-(+\lambda^2e^{-2\omega T}+\epsilon^2)f_T(T), \tag{3.7}$$

where ϵ is a constant of separation. Equation (3.6) takes a more familiar form in terms of Bessel functions [15,16] after introducing the variable $u=e^{\omega X}$

$$u^2\frac{d^2f_X}{du^2}+u\frac{df_X}{du}+\left(\frac{\epsilon^2}{\omega^2}-\frac{\lambda^2}{\omega^2}u^2\right)f_u=0, \tag{3.8}$$

whose solution are the modified Bessel functions [15] $I_{i\nu}(z)$ and $K_{i\nu}(z)$

$$f_X(X)=AI_{\pm i\nu}\left(\frac{\lambda}{\omega}e^{\omega X}\right)+BK_{i\nu}\left(\frac{\lambda}{\omega}e^{\omega X}\right), \tag{3.9}$$

where A,B are arbitrary constants and $\nu=\epsilon/\omega$. The solution of Eq. (3.7) can be obtained in the same manner. Introducing the change of variables $v=e^{-\omega T}$ in Eq. (3.7) we get the Bessel equation

$$v^2\frac{d^2f_T}{dv^2}+v\frac{df_T}{dv}+\left(\frac{\epsilon^2}{\omega^2}+\frac{\lambda^2}{\omega^2}v^2\right)f_v=0, \tag{3.10}$$

whose solutions can be expressed in terms of the Hankel functions [15] $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$

$$f_T(T)=A'H_{i\nu}^{(1)}\left(\frac{\lambda}{\omega}e^{-\omega T}\right)+B'H_{i\nu}^{(2)}\left(\frac{\lambda}{\omega}e^{-\omega T}\right), \tag{3.11}$$

where A' and B' are arbitrary constants.

In order to solve Eq. (3.5), we introduce the new variable $x=\sqrt{2/\omega_0}(\omega_0z+k_y)$. Equation (3.5) takes the form

$$\frac{d^2f}{dx^2}-\left[\frac{x^2}{4}+a\right]f(x)=0, \tag{3.12}$$

where $a=-(m^2+\lambda^2)/2\omega_0$. Equation (3.12) is the Parabolic cylinder equation [15]. The solution of Eq. (3.5) regular at $x=0$ can be expressed in terms of the function $U(a,x)$ [15]

$$f(x)=U\left(a,\sqrt{\frac{2}{\omega_0}}(\omega_0z+k_y)\right). \tag{3.13}$$

Let us analyze the asymptotic behavior of the solutions of Eq. (3.2). As $X\to\infty$ and $T\to-\infty$ we obtain that

$$\eta_{-\infty}(X,T)=K_{i\nu}\left(\frac{\lambda}{\omega}e^{\omega X}\right)\left[A_{-\infty}H_{i\nu}^{(1)}\left(\frac{\lambda}{\omega}e^{-\omega T}\right)+B_{-\infty}H_{i\nu}^{(2)}\left(\frac{\lambda}{\omega}e^{-\omega T}\right)\right] = e^{-(\lambda/\omega)e^{\omega X}}[A'_{-\infty}e^{i(\lambda/\omega)e^{-\omega T}}+B'_{-\infty}e^{-i(\lambda/\omega)e^{-\omega T}}]. \tag{3.14}$$

Comparing Eq. (3.14) with Eq. (2.6) we identify the first term on the right-hand side as a positive frequency mode, and the second term as a negative frequency. On the other hand when $X\to\infty$ and $T\to\infty$ we obtain

$$\eta_\infty(X,T)=K_{i\nu}\left(\frac{\lambda}{\omega}e^{\omega X}\right)\left[A_\infty J_{i\nu}\left(\frac{\lambda}{\omega}e^{-\omega T}\right)+B_\infty J_{-i\nu}\left(\frac{\lambda}{\omega}e^{-\omega T}\right)\right]$$
$$=e^{-(\lambda/\omega)e^{\omega X}}[A'_\infty(e^{-\omega T})^{i\nu}+B'_\infty(e^{-\omega T})^{-i\nu}]$$
$$=e^{-(\lambda/\omega)e^{\omega X}}(A'_\infty e^{-i\epsilon T}+B'_\infty e^{i\epsilon T}). \tag{3.15}$$

Also, comparing Eq. (3.15) with Eq. (2.8) we can identify the first and second terms on the right-hand side in Eq. (3.15) as positive and negative frequency modes, respectively.

Now, we are going to express an inertial positive frequency mode ($T\to-\infty$)

$$\eta_{\rm inertial}(X,T)=C_0K_{i\nu}\left(\frac{\lambda}{\omega}e^{\omega X}\right)H^{(1)}_{i\nu}\left(\frac{\lambda}{\omega}e^{-\omega T}\right) \tag{3.16}$$

in terms of the accelerated modes in the asymptotic future ($T\to+\infty$)

$$\eta_{\rm acc}(X,T)=C_1K_{i\nu}\left(\frac{\lambda}{\omega}e^{\omega X}\right)J_{i\nu}\left(\frac{\lambda}{\omega}e^{-\omega T}\right), \tag{3.17}$$

where C_1 and C_0 are normalization constants according to the standard inner product [3] $\langle\eta_i,\eta_j\rangle=-i\int(\eta_i\vec{\partial}_s\eta_j^*-\eta_j\vec{\partial}_s\eta_i^*)dS^s$ for the Klein-Gordon equation. The relation between $H^{(1)}_{i\nu}(z)$ and $J_{i\nu}(z)$ permits one to express $\eta_{\rm inertial}(X,T)$ in terms of $\eta_{\rm acc}(X,T)$ as follows:

$$\eta_{\rm inertial}(X,T)=\frac{C_0}{C_1}\left[\frac{e^{\pi\nu}}{\sinh\pi\nu}\eta_{\rm acc}(X,T)-\frac{1}{\sinh\pi\nu}\eta^*_{\rm acc}(X,T)\right]. \tag{3.18}$$

Since the inertial and accelerated modes are related via the Bogoliubov coefficients [3,4] α and β,

$$\eta_{i(\rm inertial)}=\sum_j \alpha_{ij}\eta_{j(\rm acc)}+\beta_{ij}\eta^*_{j(\rm acc)}, \tag{3.19}$$

we have that Eq. (3.18) gives immediately the values of α_{ij} and β_{ij}:

$$\alpha_{ij}=\frac{C_0}{C_1}\frac{e^{\pi\nu}}{\sinh\pi\nu}\delta_{ij}=\alpha\delta_{ij},$$
$$\beta_{ij}=-\frac{C_0}{C_1}\frac{1}{\sinh\pi\nu}\delta_{ij}=\beta\delta_{ij}. \tag{3.20}$$

Since $|\alpha|^2-|\beta|^2=1$ and

$$\left|\frac{\beta}{\alpha}\right|=e^{-\pi\nu} \tag{3.21}$$

the density of created particles [17] has the form

$$\langle 0_{\rm acc}|N|0_{\rm acc}\rangle=|\beta|^2=\frac{1}{e^{2\pi\nu}-1} \tag{3.22}$$

which can be identified as a Planck distribution with a temperature

$$\mathcal{T}_0=\frac{\omega}{2\pi K_B}. \tag{3.23}$$

The temperature measured by the accelerated observer can be obtained using the relation [3,18]:

$$\mathcal{T}=(g_{00})^{-1/2}\mathcal{T}_0 \tag{3.24}$$

In the asymptotic limit as $T\to+\infty$ we have $\lim_{T\to+\infty}(g_{00})^{-1/2}=e^{-\omega X}$ and the temperature T takes the value

$$\mathcal{T}=\frac{\omega e^{-\omega X}}{2\pi K_B}=\frac{a_{+\infty}}{2\pi K_B}. \tag{3.25}$$

Then we find that the temperature is proportional to the asymptotic value of the acceleration.

IV. SOLUTION OF THE DIRAC-PAULI EQUATION

In this section we solve the Dirac equation with anomalous magnetic moment in the accelerated coordinates (1.1) when a constant magnetic field is present. The covariant generalization of the Dirac-Pauli equation in curvilinear coordinates is [12,14]

$$\left\{\gamma^\alpha(\partial_\alpha-\Gamma_\alpha)+\frac{\mu_0}{2}\gamma^\alpha\gamma^\beta F_{\alpha\beta}+m\right\}\Psi=0, \tag{4.1}$$

where γ^α are the curvilinear Dirac matrices satisfying the anticommutation relations $\{\gamma^\alpha,\gamma^\beta\}_+=2g^{\alpha\beta}$,$\Gamma_\alpha$ are the spinor connections, and $F_{\alpha\beta}$ is the electromagnetic tensor. The curvilinear γ^α matrices are related to the constant Minkowski $\tilde{\gamma}^i$ matrices with $\{\tilde{\gamma}^i,\tilde{\gamma}^j\}_+=2\eta^{ij}$ via the tetrad ${h^\alpha}_i$:

$$\gamma^\alpha=h^\alpha_i\tilde{\gamma}^i. \tag{4.2}$$

In order to write the curvilinear Dirac matrices in Eq. (4.1) we have to choose a tetrad ${h^\alpha}_i$. Here we are going to work in the diagonal tetrad gauge, where ${h^\alpha}_i$ takes the form

$${h^\alpha}_i=\begin{pmatrix}\frac{1}{\sqrt{e^{-2\omega T}+e^{2\omega X}}} & 0 & 0 & 0\\ 0 & \frac{1}{\sqrt{e^{-2\omega T}+e^{2\omega X}}} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{4.3}$$

It is easy to verify that $g^{ik}=\eta^{jl}h^{i}{}_{j}h^{k}{}_{l}$. In this tetrad gauge, the curvilinear Dirac matrices can be expressed in terms of the constant $\tilde{\gamma}^i$ as

$$\gamma^0=\frac{\bar{\tilde{\gamma}}^0}{\sqrt{e^{-2\omega T}+e^{2\omega X}}},\qquad \gamma^1=\frac{\bar{\tilde{\gamma}}^1}{\sqrt{e^{-2\omega T}+e^{2\omega X}}},$$

$$\gamma^2=\bar{\tilde{\gamma}}^2,\qquad \gamma^3=\bar{\tilde{\gamma}}^3. \tag{4.4}$$

In the Diagonal tetrad gauge (4.3) the spinor connections, defined by the relation [19]:

$$\Gamma_\mu=\frac{1}{4}g_{\lambda\alpha}\left(\frac{\partial h_\nu^{\ i}}{\partial x^\mu}h^\alpha_{\ .i}-\Gamma^\alpha_{\nu\mu}\right)s^{\lambda\nu} \tag{4.5}$$

with $s^{\lambda\nu}=\frac{1}{2}(\gamma^\lambda\gamma^\nu-\gamma^\nu\gamma^\lambda)$, take the form

$$\Gamma_0=-\frac{1}{2}\omega e^{2\omega X}(e^{-2\omega T}+e^{2\omega X})^{-1}\bar{\gamma}^1\bar{\gamma}^0,$$

$$\Gamma_1=\frac{1}{2}\omega e^{-2\omega T}(e^{-2\omega T}+e^{2\omega X})^{-1}\bar{\gamma}^1\bar{\gamma}^0,\qquad \Gamma_2=0,\qquad \Gamma_3=0. \tag{4.6}$$

Substituting Eqs. (4.6) , (4.4), and (2.2) into Eq. (4.1) we obtain

$$\frac{\bar{\gamma}^1}{\sqrt{e^{2\omega X}+e^{-2\omega T}}}\left[\frac{\partial\Psi}{\partial X}+\frac{\omega e^{2\omega X}}{2(e^{2\omega X}+e^{-2\omega T})}\Psi\right]+\bar{\gamma}^2\left(\frac{\partial\Psi}{\partial y}-ieB_xz\Psi\right)+\bar{\gamma}^3\frac{\partial\Psi}{\partial z}+\frac{\bar{\gamma}^4}{\sqrt{e^{2\omega X}+e^{-2\omega T}}}\times\left[\frac{\partial\Psi}{\partial T}-\frac{\omega e^{-2\omega T}}{2(e^{2\omega X}+e^{-2\omega T})}\Psi\right]+m\Psi+i\mu_0B_x\bar{\gamma}^2\bar{\gamma}^3\Psi=0. \tag{4.7}$$

Introducing the spinor Φ

$$\Psi=\frac{\Phi}{(e^{2\omega X}+e^{-2\omega T})^{1/4}}, \tag{4.8}$$

we eliminate the contribution terms due to the spinor connections. The Dirac equation takes the form

$$\frac{\bar{\gamma}^1(\partial\Phi/\partial X)+\bar{\gamma}^4(\partial\Phi/\partial T)}{\sqrt{e^{2\omega X}+e^{-2\omega T}}}+\bar{\gamma}^2\left(\frac{\partial\Phi}{\partial y}-ieB_xz\Phi\right)+\bar{\gamma}^3\frac{\partial\Phi}{\partial z}+m\Phi+i\mu_0B_x\bar{\gamma}^2\bar{\gamma}^3\Phi=0. \tag{4.9}$$

In order to solve Eq. (4.9) we proceed to separate variables using the algebraic method of separation developed by Shishkin and Villalba [7,20–22] The idea behind the method is to reduce Eq. (4.9) to a sum of two commuting first order differential operators

$$[\hat{K}_1(T,X)+\hat{K}_2(y,z)]\psi=0,\qquad [\hat{K}_1(T,X),\hat{K}_2(y,z)]_-=0, \tag{4.10}$$

where in the present case $\psi=\bar{\gamma}^3\bar{\gamma}^2\Phi$ and

$$\hat{K}_1=\frac{\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^3\partial_X+\bar{\gamma}^4\bar{\gamma}^2\bar{\gamma}^3\partial_T}{\sqrt{e^{2\omega X}+e^{-2\omega T}}},\qquad \hat{K}_2=\bar{\gamma}^3(\partial_y-ieB_xz)-\bar{\gamma}^2\partial_z+\bar{\gamma}^2\bar{\gamma}^3m-i\mu_0B_x. \tag{4.11}$$

Since $\hat{K}_1$ and $\hat{K}_2$ satisfy Eq. (4.10) they satisfy the eigenvalue equations

$$K_1\psi=-i\lambda\psi,\qquad K_2\psi=i\lambda\psi. \tag{4.12}$$

Now, we proceed to solve equation $K_1\psi=-i\lambda\psi$:

$$(\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^3\partial_X+\bar{\gamma}^4\bar{\gamma}^2\bar{\gamma}^3\partial_T)\psi=-i\lambda\sqrt{e^{2\omega X}+e^{-2\omega T}}\psi. \tag{4.13}$$

Applying the transformation $\psi=S\phi$ defined by

$$S=e^{\Xi(X,T)}e^{[i\bar{\gamma}^1\bar{\gamma}^4\Theta(X,T)]} \tag{4.14}$$

we reduce Eq. (4.13) to the form

$$(\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^3\partial_X+\bar{\gamma}^4\bar{\gamma}^2\bar{\gamma}^3\partial_T)\phi+\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^3[\partial_X\Xi(X,T)+i\partial_T\Theta(X,T)]\phi+\bar{\gamma}^4\bar{\gamma}^2\bar{\gamma}^3[\partial_T\Xi(X,T)+i\partial_X\Theta(X,T)]\phi=-i\lambda\sqrt{e^{2\omega X}+e^{-2\omega T}}e^{2i\bar{\gamma}^1\bar{\gamma}^4\Theta(X,T)}\phi. \tag{4.15}$$

In order to separate variables in Eq. (4.15) we demand that the terms inside the brackets vanish, i.e.,

$$\partial_X\Xi(X,T)=-i\partial_T\Theta(X,T),\qquad \partial_T\Xi(X,T)=-i\partial_X\Theta(X,T). \tag{4.16}$$

The solution of Eq. (4.16) is

$$\Xi(X,T)=\frac{i}{2}\arctan(e^{\omega(X+T)}),$$

$$\Theta(X,T)=\frac{1}{2}\arctan(e^{-\omega(X+T)}) \tag{4.17}$$

which determines the spinor transformation S Eq. (4.14). Equation (4.15) reduces to

$$(\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^3\partial_X+\bar{\gamma}^4\bar{\gamma}^2\bar{\gamma}^3\partial_T)\phi=i\lambda(e^{\omega X}+i\bar{\gamma}^1\bar{\gamma}^4e^{-\omega T})\phi. \tag{4.18}$$

Let L_1 and L_2 be the commuting operators

$$L_1=\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^3\partial_X-i\lambda e^{\omega X},\qquad L_2=\bar{\gamma}^1\bar{\gamma}^2\bar{\gamma}^3\partial_T+\lambda e^{-\omega T},$$

$$[L_1,L_2]=0. \tag{4.19}$$

Equation (4.18) can be expressed in terms of L_1 and L_2 as follows:

$$(L_1+L_2\overline{\gamma^1}\,\overline{\gamma^4})\phi=0. \tag{4.20}$$

In order to solve Eq. (4.20) we introduce the auxiliary spinor

$$\phi=(\overline{\gamma^1}\,\overline{\gamma^4}L_1-L_2)W, \tag{4.21}$$

Substituting Eq. (4.21) into Eq. (4.20) we have that $(L_1\overline{\gamma^1}\,\overline{\gamma^4}L_1-L_2\overline{\gamma^1}\,\overline{\gamma^4}L_2)W=0$. This allows separation of variables as follows:

$$(\partial_X^2-\lambda^2e^{2\omega X}+i\lambda\omega e^{\omega X}\overline{\gamma^1}\,\overline{\gamma^2}\,\overline{\gamma^3})W=-\epsilon^2W, \tag{4.22}$$

$$(\partial_T^2+\lambda^2e^{-2\omega T}+\lambda\omega e^{-\omega T}\overline{\gamma^1}\,\overline{\gamma^2}\,\overline{\gamma^3})W=-\epsilon^2W, \tag{4.23}$$

where ϵ is a constant of separation.

When we choose the following representation for the Dirac matrices [23] $\tilde{\gamma}^i$:

$$\tilde{\gamma}^1=\begin{pmatrix}0 & \sigma_1\\ \sigma_1 & 0\end{pmatrix}\quad \tilde{\gamma}^2=\begin{pmatrix}0 & \sigma_2\\ \sigma_2 & 0\end{pmatrix},$$

$$\tilde{\gamma}^3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\quad \tilde{\gamma}^4=\begin{pmatrix}0 & i\sigma_3\\ i\sigma_3 & 0\end{pmatrix}. \tag{4.24}$$

Then the spinor W has the structure

$$W=\begin{pmatrix}\alpha(X)A(T)\\ \beta(X)B(T)\\ \gamma(X)C(T)\\ \delta(X)D(T)\end{pmatrix}. \tag{4.25}$$

Substituting Eq. (4.25) into Eq. (4.22) we obtain

$$\begin{gathered}(d_X^2-\lambda^2e^{2\omega X}-\lambda\omega e^{\omega X}+\epsilon^2)\begin{pmatrix}\alpha(X)\\ \delta(X)\end{pmatrix}=0,\\ (d_X^2-\lambda^2e^{2\omega X}+\lambda\omega e^{\omega X}+\epsilon^2)\begin{pmatrix}\beta(X)\\ \gamma(X)\end{pmatrix}=0;\end{gathered} \tag{4.26}$$

analogously, Eq. (4.23) reduces to

$$\begin{gathered}(\partial_T^2+\lambda^2e^{-2\omega T}+i\lambda\omega e^{-\omega T}+\epsilon^2)\begin{pmatrix}A(T)\\ D(T)\end{pmatrix}=0,\\ (\partial_T^2+\lambda^2e^{-2\omega T}-i\lambda\omega e^{-\omega T}+\epsilon^2)\begin{pmatrix}B(T)\\ C(T)\end{pmatrix}=0.\end{gathered} \tag{4.27}$$

In order to solve the system of equations (4.26), it suffices to solve the second order equation

$$u^2\frac{d^2f}{du^2}+u\frac{df}{du}+\left(-\frac{\lambda^2}{\omega^2}u^2\pm\frac{\lambda}{\omega}u+\frac{\epsilon^2}{\omega^2}\right)f=0, \tag{4.28}$$

where $u=e^{\omega X}$.

Analogously, we have that solving the system of equations (4.27) is equivalent to solving the Whittaker [15] differential equation

$$v^2\frac{d^2g}{dv^2}+v\frac{dg}{dv}+\left(\frac{\lambda^2}{\omega^2}v^2\pm\frac{i\lambda}{\omega}v+\frac{\epsilon^2}{\omega^2}\right)g=0, \tag{4.29}$$

where we have introduced the change of variable $v=e^{-\omega T}$. The solution of Eq. (4.28) can be expressed in terms of a combination of Whittaker functions

$$f(u)=\frac{C_1}{\sqrt{u}}M_{\pm 1/2,\mu}\left(\frac{2\lambda}{\omega}u\right)+\frac{C_2}{\sqrt{u}}W_{\pm 1/2,\mu}\left(\frac{2\lambda}{\omega}u\right), \tag{4.30}$$

where $\mu=i\epsilon/\omega$ and C_1 and C_2 are arbitrary constants. Analogously we have that the solution of Eq. (4.29) is

$$g(v)=\frac{C_3}{\sqrt{v}}M_{\pm 1/2,\mu}\left(\frac{2i\lambda}{\omega}v\right)+\frac{C_4}{\sqrt{v}}W_{\pm 1/2,\mu}\left(\frac{2i\lambda}{\omega}v\right) \tag{4.31}$$

where C_3 and C_4 are arbitrary constants.

The spinor ϕ can be computed with the help of the inverse transformation (4.21):

$$\phi=(\overline{\gamma^1}\,\overline{\gamma^4}L_1-L_2)W=\begin{pmatrix}-i\alpha(X)(\partial_T-i\lambda e^{-\omega T})A(T)-B(T)(\partial_X+\lambda e^{\omega X})\beta(X)\\ i\beta(X)(\partial_T+i\lambda e^{-\omega T})B(T)-A(T)(\partial_X-\lambda e^{\omega X})\alpha(X)\\ i\gamma(X)(\partial_T+i\lambda e^{-\omega T})C(T)+D(T)(\partial_X-\lambda e^{\omega X})\delta(X)\\ -i\delta(X)(\partial_T-i\lambda e^{-\omega T})D(T)+C(T)(\partial_X+\lambda e^{\omega X})\gamma(X)\end{pmatrix}. \tag{4.32}$$

The system of equations (4.26) can be written as a coupled system of ordinary differential equations

$$(\partial_X-\lambda e^{\omega X})\alpha(X)=i\epsilon\beta(X),\quad (\partial_X+\lambda e^{\omega X})\beta(X)=i\epsilon\alpha(X),$$

$$(\partial_X+\lambda e^{\omega X})\gamma(X)=\epsilon\delta(X),\quad (\partial_X-\lambda e^{\omega X})\delta(X)=-\epsilon\gamma(X). \tag{4.33}$$

Analogously, we have that the Eq. (4.27) is equivalent to

$$\begin{aligned}(\partial_T-i\lambda e^{-\omega T})A(T)&=i\epsilon B(T),\\ (\partial_T+i\lambda e^{-\omega T})B(T)&=i\epsilon A(T),\\ (\partial_T+i\lambda e^{-\omega T})C(T)&=\epsilon D(T),\\ (\partial_T-i\lambda e^{-\omega T})D(T)&=-\epsilon C(T).\end{aligned} \tag{4.34}$$

Substituting Eqs. (4.33) and (4.34) into Eq. (4.32) we arrive at

$$\phi=\epsilon\begin{pmatrix}(-i+1)\alpha(X)B(T)\\ (-i-1)\beta(X)A(T)\\ (i-1)\gamma(X)D(T)\\ (i+1)\delta(X)C(T)\end{pmatrix}, \tag{4.35}$$

where the solutions, convergent at large values of X, are

$$\alpha(u)=\frac{c_1}{\sqrt{u}}W_{-1/2,\mu}\left(\frac{2\lambda}{\omega}u\right),\quad \beta(u)=\frac{ic_1}{\sqrt{u}}\frac{\omega}{\epsilon}W_{1/2,\mu}\left(\frac{2\lambda}{\omega}u\right),$$

$$\gamma(u)=\frac{c_3}{\sqrt{u}}W_{1/2,\mu}\left(\frac{2\lambda}{\omega}u\right),\quad \delta(u)=\frac{c_3}{\sqrt{u}}\frac{\epsilon}{\omega}W_{-1/2,\mu}\left(\frac{2\lambda}{\omega}u\right), \tag{4.36}$$

and

$$A(T)=\frac{d_1}{\sqrt{v}}W_{-1/2,\mu}\left(\frac{2i\lambda}{\omega}v\right)+\frac{d_2}{\sqrt{v}}M_{-1/2,\mu}\left(\frac{2i\lambda}{\omega}v\right),$$

$$B(T)=\frac{Id_1\epsilon}{\sqrt{v}\,\omega}W_{1/2,\mu}\left(\frac{2i\lambda}{\omega}v\right)+\frac{d_2}{\sqrt{v}}M_{1/2,\mu}\left(\frac{2i\lambda}{\omega}v\right),$$

$$C(T)=\frac{d_3}{\sqrt{v}}W_{1/2,\mu}\left(\frac{2i\lambda}{\omega}v\right)+\frac{d_4}{\sqrt{v}}M_{1/2,\mu}\left(\frac{2i\lambda}{\omega}v\right),$$

$$D(T)=\frac{d_3\epsilon}{\sqrt{v}\,\omega}W_{-1/2,\mu}\left(\frac{2i\lambda}{\omega}v\right)+\frac{Id_4}{\sqrt{v}}M_{-1/2,\mu}\left(\frac{2i\lambda}{\omega}v\right). \tag{4.37}$$

Using the recurrence relations for the Whittaker functions we find that ϕ has the form

$$\phi=\begin{pmatrix}a_1(i-1)f_1\\ -a_1(i+1)f_2\\ a_2(i+1)f_2\\ a_2(i-1)f_1\end{pmatrix}, \tag{4.38}$$

where the regular solutions f_1 and f_2 as $T\to-\infty$ are

$$f_1=\frac{1}{\sqrt{uv}}W_{-1/2,\mu}\left(\frac{2\lambda}{\omega}u\right)W_{1/2,\mu}\left(\frac{2i\lambda}{\omega}v\right),$$

$$f_2=\frac{1}{\sqrt{u\nu}}W_{1/2,\mu}\left(\frac{2\lambda}{\omega}u\right)W_{-1/2,\mu}\left(\frac{2i\lambda}{\omega}v\right), \tag{4.39}$$

and the corresponding f_1 and f_2 regular as $T\to0$ take the form

$$f_1=\frac{1}{\sqrt{uv}}W_{-1/2,\mu}\left(\frac{2\lambda}{\omega}u\right)M_{1/2,\mu}\left(\frac{2i\lambda}{\omega}v\right),$$

$$f_2=\frac{1}{\sqrt{u\nu}}W_{1/2,\mu}\left(\frac{2\lambda}{\omega}u\right)M_{-1/2,\mu}\left(\frac{2i\lambda}{\omega}v\right). \tag{4.40}$$

Recalling that $\psi=S\phi$ where S is given by Eq. (4.14) we have that ψ takes the form

$$\psi=\begin{pmatrix}a_1\cos\Theta(i-1)f_1-a_1\sin\Theta(i+1)f_2\\ -a_1\sin\Theta(i-1)f_1-a_1\cos\Theta(i+1)f_2\\ a_2\cos\Theta(i+1)f_2+a_2\sin\Theta(i-1)f_1\\ -a_2\sin\Theta(i+1)f_2+a_2\cos\Theta(i-1)f_1\end{pmatrix}e^{\Xi}. \tag{4.41}$$

Now we proceed to solve the equations (4.12) governing the dependence of the spinor solution of the Dirac equation on the coordinates y and z

$$(\overline{\gamma}^3(\partial_y-ieB_xz)-\overline{\gamma}^2\partial_z+\overline{\gamma}^2\overline{\gamma}^3m-i\mu_0B_x)\psi=i\lambda\psi. \tag{4.42}$$

Since $[\hat{K}_2,-i\partial_y]_-=0$, we have that ψ can be written as

$$\psi=e^{ik_yy}\varphi(z), \tag{4.43}$$

substituting Eq. (4.43) into Eq. (4.42) we obtain

$$[-i\overline{\gamma}^3(-k_y+\omega_0z)-\overline{\gamma}^2\partial_z+\overline{\gamma}^2\overline{\gamma}^3m]\varphi=i(\lambda+\mu_0B_x)\varphi, \tag{4.44}$$

where $\omega_0=eB_x$. Introducing the spinor $\varphi=\Sigma\theta$ with

$$\Sigma=\frac{1-\overline{\gamma}^1\overline{\gamma}^3}{\sqrt{2}} \tag{4.45}$$

we find that Eq. (4.44) reduces to

$$[-i\overline{\gamma}^1(-k_y+\omega_0z)-\overline{\gamma}^2\partial_z+\overline{\gamma}^2\overline{\gamma}^1m]\theta=i(\lambda+\mu_0B_x]\theta; \tag{4.46}$$

substituting into Eq. (4.46) the Dirac matrices in the representation (4.24) and considering a spinor with the structure

$$\theta=\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\end{pmatrix}, \tag{4.47}$$

we reduce our problem to that of solving the following system of partial differential equations:

$$[\partial_z-(-k_y+\omega_0 z)]\theta_4=(m+\lambda+\mu_0)\theta_1,$$
$$[\partial_z+(-k_y+\omega_0 z)]\theta_1=(m-\lambda-\mu_0 B_x)\theta_4, \tag{4.48}$$
$$[\partial_z-(-k_y+\omega_0 z)]\theta_2=(m+\lambda+\mu_0)\theta_3,$$
$$[\partial_z+(-k_y+\omega_0 z)]\theta_3=(m-\lambda-\mu_0 B_x)\theta_2. \tag{4.49}$$

Looking at Eqs. (4.48) and (4.49) we see that $\theta_1\sim\theta_3$ and $\theta_2\sim\theta_4$ and therefore it is only necessary solve one of the systems of coupled equations. Making the change of variable $x=\sqrt{2/\omega_0}(\omega_0 z-k_y)$ we have

$$\frac{d^2}{dx^2}\theta_1-\left[\frac{x^2}{4}+a\right]\theta_1=0, \tag{4.50}$$

$$\frac{d^2}{dx^2}\theta_4-\left[\frac{x^2}{4}+(a+1)\right]\theta_4=0, \tag{4.51}$$

where $a=(m^2-\lambda^2)/2\omega_0-\frac{1}{2}$. Equations (4.50) and (4.51) are Parabolic cylinder equations [15] and their solutions are

$$\theta_1=d_1 U\left(a,\sqrt{\frac{2}{\omega_0}}(\omega_0 z-k_y)\right),$$
$$\theta_3=d_3 U\left(a,\sqrt{\frac{2}{\omega_0}}(\omega_0 z-k_y)\right),$$
$$\theta_4=d_4 U\left(a+1,\sqrt{\frac{2}{\omega_0}}(\omega_0 z-k_y)\right),$$
$$\theta_2=d_2 U\left(a+1,\sqrt{\frac{2}{\omega_0}}(\omega_0 z-k_y)\right) \tag{4.52}$$

with the help of the recurrence relations for the parabolic cylinder equation and the equations (4.48),(4.49) , we can find the relation between the coefficients d_i:

$$\frac{d_1}{d_4}=-\frac{\sqrt{2\omega_0}}{m+\lambda+\mu_0 B_x} \tag{4.53}$$

and

$$\frac{d_3}{d_2}=-\frac{\sqrt{2\omega_0}}{m+\lambda+\mu_0 B_x}, \tag{4.54}$$

consequently,

$$\theta=\begin{pmatrix}-\dfrac{\sqrt{2\omega_0}}{m+\lambda+\mu_0 B_x}d_4 U(a,x)\\ d_2 U(a+1,x)\\ -\dfrac{\sqrt{2\omega_0}}{m+\lambda+\mu_0 B_x}d_2 U(a,x)\\ d_4 U(a+1,x)\end{pmatrix}. \tag{4.55}$$

The spinor φ is obtained using the matrix transformation Σ(4.45):

$$\varphi=\Sigma\theta=\frac{1}{\sqrt{2}}\begin{pmatrix}\theta_1+\theta_4\\ \theta_2+\theta_3\\ -\theta_2+\theta_3\\ -\theta_1+\theta_4\end{pmatrix}. \tag{4.56}$$

From Eqs. (4.45) and (4.43) we obtain

$$\psi=\frac{1}{\sqrt{2}}\begin{pmatrix}d_4\left(U(a+1,x)-\dfrac{\sqrt{2\omega_0}}{m+\lambda+\mu_0 B_x}U(a,x)\right)\\ d_2\left(U(a+1,x)-\dfrac{\sqrt{2\omega_0}}{m+\lambda+\mu_0 B_x}U(a,x)\right)\\ -d_2\left(U(a+1,x)+\dfrac{\sqrt{2\omega_0}}{m+\lambda+\mu_0 B_x}U(a,x)\right)\\ d_4\left(U(a+1,x)+\dfrac{\sqrt{2\omega_0}}{m+\lambda+\mu_0 B_x}U(a,x)\right)\end{pmatrix}e^{ik_y y}. \tag{4.57}$$

Combining Eq. (4.41) with Eq. (4.57) we find that the spinor ψ is

$$\psi=\frac{1}{\sqrt{2}}\begin{pmatrix}[\cos\Theta(i-1)f_1-\sin\Theta(i+1)f_2] & \left(U(a+1,x)-\dfrac{\sqrt{2\omega_0}}{m+\lambda+\mu_0 B_x}U(a,x)\right)\\ [-\sin\Theta(i-1)f_1-\cos\Theta(i+1)f_2] & \left(U(a+1,x)-\dfrac{\sqrt{2\omega_0}}{m+\lambda+\mu_0 B_x}U(a,x)\right)\\ [\cos\Theta(i+1)f_2+\sin\Theta(i-1)f_1] & \left(U(a+1,x)+\dfrac{\sqrt{2\omega_0}}{m+\lambda+\mu_0 B_x}U(a,x)\right)\\ [-\sin\Theta(i+1)f_2+\cos\Theta(i-1)f_1] & \left(U(a+1,x)+\dfrac{\sqrt{2\omega_0}}{m+\lambda+\mu_0 B_x}U(a,x)\right)\end{pmatrix}e^{ik_y y}e^{\Xi}. \tag{4.58}$$

Now, we proceed to analyze the asymptotic limits as $T\to-\infty$ and $T\to+\infty$. We will confine our attention to the solutions of the spinor (4.58). In the asymptotes we obtain a time dependent term multiplied by a factor depending on space variables. Here, we proceed as we did in Sec. III for the scalar case.

The relation between $W_{\lambda,\mu}(z)$ and $M_{\lambda,\mu}(z)$ [24] will be helpful

$$W_{\lambda,\mu}(z)=\frac{\Gamma(-2\mu)}{\Gamma(1/2-\mu-\lambda)}M_{\lambda,\mu}(z)+\frac{\Gamma(2\mu)}{\Gamma(1/2+\mu-\lambda)}M_{\lambda,-\mu}(z). \tag{4.59}$$

Taking into account expression (4.59) we find that the solutions of Eq. (4.34) are related as follows:

$$W_{\varkappa,\mu}\left(-\frac{2i\lambda}{\omega}v\right)=\frac{\Gamma(-2\mu)}{\Gamma(1/2-\mu-\varkappa)}M_{\varkappa,\mu}\left(-\frac{2i\lambda}{\omega}v\right)+\frac{\Gamma(2\mu)}{\Gamma(1/2+\mu-\varkappa)}e^{i\pi(\mu-1/2)}\times M_{-\varkappa,-\mu}\left(\frac{2i\lambda}{\omega}v\right). \tag{4.60}$$

Looking at the quasiclassical behavior given by Eq. (2.8), recalling the behavior of $M_{\varkappa,\mu}(z)$ as $z\to 0$, and the relation between Whittaker and Bessel functions, we identify the positive and negative accelerated modes as

$$\psi_+^{\rm acc}=N_1M_{1/2,\mu}\left(-\frac{2i\lambda}{\omega}v\right),\quad \psi_-^{\rm acc}=N_1M_{-1/2,-\mu}\left(\frac{2i\lambda}{\omega}v\right), \tag{4.61}$$

where N_1 is a normalization constant. Also an inertial ($T\to-\infty$) positive frequency mode $\psi_+^{\rm ine}$ is given by

$$\psi_+^{\rm ine}=N_3W_{1/2,\mu}\left(-\frac{2i\lambda}{\omega}v\right), \tag{4.62}$$

where N_3 is a normalization constant.

Looking at Eq. (4.60) and recalling that the inertial modes can be expressed in terms of the accelerated positive and negative modes via the Bogoliubov coefficients

$$\psi_+^{\rm ine}=\alpha\psi_+^{\rm acc}+\beta\psi_-^{\rm acc}, \tag{4.63}$$

we get that

$$\left|\frac{\beta}{\alpha}\right|=e^{-\epsilon\pi/\omega} \tag{4.64}$$

and taking into account that $|\alpha|^2+|\beta|^2=1$, we find that the density of particles created is [17]

$$\langle 0_{\rm acc}|N|0_{\rm acc}\rangle=|\beta|^2=\frac{1}{1+e^{2\pi(\epsilon/\omega)}}, \tag{4.65}$$

a result that can be identified as a Fermi-Dirac distribution of particles associated with a temperature

$$\mathcal{T}_0=\frac{\omega}{2\pi K_B} \tag{4.66}$$

and, consequently, the temperature detected in the accelerated frame is [18]

$$\mathcal{T}=(g_{00})^{-1/2}\mathcal{T}_0 \tag{4.67}$$

that in the asymptotic limit as $T\to+\infty$ takes the form

$$\mathcal{T}=\frac{\omega e^{-\omega X}}{4\pi K_B},$$

showing that the temperature is proportional to the asymptotic value of the acceleration.

V. DISCUSSION OF THE RESULTS

In the present paper we have separated variables and solved the Klein-Gordon and Dirac equations in the curvilinear coordinates (1.1) when a constant magnetic field (2.2) is present. The algebraic method of separation [7,20–22] has been applied to reduce Dirac-Pauli equation to a system of coupled ordinary differential equations. Using the obtained exact solutions we calculated the density of scalar and spin-1/2 particles detected by an accelerated observer associated with system of coordinates (1.1) when a constant magnetic field in the direction of acceleration is present. The identification of positive and negative frequency modes was carried out comparing the relativistic solutions with the quasiclassical Hamilton-Jacobi solutions (2.6) and (2.8). The results obtained in Secs. III and IV indicate that the magnetic field does not modify the thermal character of spectrum. The temperature associated with the thermal bath is not modified by B. This result has a classical counterpart: If a magnetic field B is collinear to the motion of a particle, then it does not accelerate the particle and no radiation is caused by it. The presence of a nonminimal coupling in Eq. (4.1) does not affect the density (4.65). The proportionality between temperature and accelerations remains valid even if uniform accelerations are reached asymptotically. The role of the anomalous magnetic moment in the energy spectrum density will be discussed in a forthcoming publication.

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