Covariant gauge fixing and Kuchař decomposition

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The symplectic geometry of a broad class of generally covariant models is studied. The class is restricted so that the gauge group of the models coincides with the Bergmann-Komar group and the analysis can focus on the general covariance. A geometrical definition of gauge fixing at the constraint manifold is given; it is equivalent to a definition of a background (spacetime) manifold for each topological sector of a model. Every gauge fixing defines a decomposition of the constraint manifold into the physical phase space and the space of embeddings of the Cauchy manifold into the background manifold (Kuchař decomposition). Extensions of every gauge fixing and the associated Kuchař decomposition to a neighborhood of the constraint manifold are shown to exist.

I. INTRODUCTION

In 1971, Bergmann and Komar [1] wrote
''…in general relativity the identity of a world point is not preserved under the theory's widest invariance group. This assertion forms the basis for the conjecture that some physical theory of the future may teach us how to dispense with world points as the ultimate constituents of space-time altogether.''

We share this view and we are going to support it by revealing some of the underlying mathematical structure.

The formulation of general relativity and, in fact, of any generally covariant model, is based on the mathematical theory of (pseudo) Riemannian manifolds. There is, however, a catch: in the mathematics, even a naked manifold has well-defined, distinguishable points. In the physics, points are defined and distinguished *only* by values of physical fields or as positions of physical objects. Attempts to take naked points seriously lead to well-known paradoxes and problems. The first paradox of this kind was constructed by Einstein (the ''hole'' argument [2]); a more recent example is due to Fredenhagen and Haag [3]. Any clean separation between spacetime points on one hand and physical fields on the other violates the diffeomorphism invariance (for an extended discussion of this point, see Stachel [4] and Isham [5]). From the mathematical point of view, the space that one works with is the space of geometries $\text{Riem}\mathcal{M}/\text{Diff}\mathcal{M}$ on a manifold $\mathcal{M}$ rather than the space of metric fields $\text{Riem}\mathcal{M}$ on the manifold $\mathcal{M}$. In the space of geometries, points of the manifold $\mathcal{M}$ are entangled with the metric fields and it is impossible to reconstruct (disentangle) them in any natural, unique, way.

Accordingly, Einstein dynamics is not a field dynamics on any manifold. This does not mean, however, that one cannot reduce it to such a field theory. For example, the dynamics is reformulated as a system of partial differential equations for some fields on a fixed background manifold in the study of the Cauchy problem (see, e.g., a recent review [6]). This reduction is based on choices of gauge (coordinate conditions). The choice of gauge plays, in such a way, a two-fold role for generally covariant models: (1) it renders the dynamics unique (as in any gauge field theory), and (2) it defines the background manifold points. It is also well known that the gauge group of such models is much larger that just the diffeomorphism group of one fixed manifold [1].

The definition of background manifolds by means of gauge choices does not violate the gauge invariance of the full theory, if one can show at the end that the measurable results are independent of the choice; this has indeed been possible for many problems of classical physics. Another popular method of defining background manifolds is to expand a certain sector of a given model around a classical spacetime (such as, e.g., the Minkowski spacetime). A special role given to a fixed classical spacetime enables one to use this particular spacetime as a background, and to select the diffeomorphism group of this spacetime as the remaining gauge symmetry. This is a strong restriction of the original symmetry. The procedure might be justified, if e.g. some kind of WKB approximation is valid in the situation considered and the corresponding metric is a part of a classical solution from which the iterative steps of the WKB method start.

In the present paper, we are going to study the symplectic geometry of quite a general class of diffeomorphically invariant models. We shall concentrate on those properties that are relevant to gauge fixing, gauge transformations, and physical degrees of freedom. The main ideas are covariant gauge fixing [7] and the Kuchař decomposition [8]; we shall give a complete description of these ideas and their interconnection. The plan of the paper is as follows.

In Sec. II, we analyze in some detail gauge choices using very simple examples from general relativity. We try to separate the two aspects of gauge fixing—the point definition and the coordinate choice—to motivate our notion of covariant gauge fixing. We also briefly recapitulate Kuchař ''third way'' [8].

In Sec. III, we describe the properties of generally covariant models that are needed for subsequent constructions.

We present a list of properties that can be considered as a kind of definition of the generally covariant models. However, rather than attempting to identify a minimal set of independent properties, we just collect all assumptions that will be necessary for the proofs. For the sake of simplicity we also exclude all gauge fields (such as Yang-Mills fields) so that we can focus on the issue of general covariance.

Section IV contains the constructions that are necessary for our definition of covariant gauge fixing on the constraint manifold of the model. The fixing identifies spacetime points belonging to different spacetime solutions. In this way, a unique background manifold results and everything is manifestly invariant with respect to coordinate transformations on this manifold. The transformation between two covariant gauge fixings can be described as a set of diffeomorphisms, one for each solution; such a set of transformations is an element of the Bergmann-Komar group [1]. A covariant gauge fixing is thus defined in a geometrical, coordinate free and general manner. Still, it has a close relation to the usual way of choosing gauge: a ''nice'' coordinate condition leads to a special case of such a covariant gauge fixing.

The local existence of covariant gauge fixings is equivalent to the following statement. For the sectors that are spatially compact, any open subset of the generic part of the constraint surface on which the gauge fixing works is a subset of a fiber bundle: its basis manifold is the physical phase space, its typical fiber is the space of embeddings of the Cauchy surface into the background manifold, and its group is that of diffeomorphisms of the background. For the sectors that are not spatially compact, this description is to be modified (see Sec. IV). Each gauge fixing is equivalent to a trivialization of this bundle, i.e. to a decomposition of the constraint surface into a Cartesian product of the base and the typical fiber. Existence of such decompositions has been first observed by Kuchař [9]; we shall call them *Kuchař decompositions*. In this way, we establish a connection between covariant gauge fixings and Kuchař decompositions.

The main result of this paper is described in Sec. V, where we extend the Kuchař decomposition to a whole neighborhood of the constraint surface. The construction is based on the Darboux-Weinstein theorem [10] and it shows explicitly that there are many such extensions. As the construction is based on an existence theorem, it will not be practically viable in most cases of interest. However, Kuchař decompositions have as yet been explicitly constructed only for very few cases, cylindrical waves [9] and the Schwarzschild family [11], and even the question of existence was not clear. For most purposes (as, e.g., for quantization), the explicit form of the decomposition outside the constraint surface is not needed.

The mathematical language which is used in this paper and which enables concise and effective formulations is that of vector bundles and symplectic geometry of infinite dimensional manifolds modeled on Banach spaces (see, e.g., Abraham, Marsden and Ratiu [12], Libermann and Marle [13] and Lang [14]). Typically, all these manifolds are modeled on Sobolev spaces H^s (see Marsden [15]) but there is no universal functional analytic framework for field theory: it seems that each particular theory needs its own choice of the

class of functions to which we restrict our search of solution of field equations. Unfortunately, our results cannot help to make this choice. Nevertheless, they are rigorous. What we prove is the following statement: whenever a generally covariant field theory is equipped with a *correct* functional analytic structure [''correct'' means that (1) the space of non-constraint Cauchy data is a Banach manifold, (2) the constraint surface is its regular submanifold and (3) the gauge orbits form a regular foliation of the latter] then this space is locally isomorphic to a Cartesian product of the physical phase space and the cotangent bundle of embeddings of the Cauchy surface into the background manifold. Each such local isomorphism is connected with a covariant gauge fixing.

II. GAUGE IN GENERAL RELATIVITY

In this section we analyze the gauge choice in general relativity and review the original Kuchař decomposition.

To discuss the gauge choice, we use a strongly simplified model. This will motivate our subsequent definitions and constructions.

Consider the Schwarzschild solutions to the Einstein equations in the future of the influence (white hole) horizon. They form a one-dimensional family and the value of the Schwarzschild mass $M \in (0,\infty)$ distinguishes different elements of the family from each other. The metric can be given the form

$$ds^2 = -\left(1 - \frac{2M}{R}\right) dW^2 + 2 dW\, dR + R^2\, ds_2^2, \qquad (1)$$

where ds_2^2 is the metric of a 2-sphere of radius 1; W and R are the advanced Eddington-Finkelstein coordinates with the domains

$$R \in (0,\infty), \quad W \in (-\infty,\infty). \qquad (2)$$

Nothing seems to prevent us from considering Eq. (1) as a one-dimensional *set of metric fields* on a fixed background manifold $\mathcal{M}_1 = \mathbf{R}^2 \times S^2$ in the coordinate chart W, R, ϑ and φ [of course, at least two charts (ϑ,φ) and (ϑ',φ') are necessary to cover S^2]. The same metric can, however, also be given another form, if we pass to the Kruskal coordinates U, V, ϑ and φ,

$$ds^2 = -\frac{16M^2}{\kappa(-UV)} e^{-\kappa(-UV)} dU\, dV + 4M^2 \kappa^2(-UV)\, ds_2^2, \qquad (3)$$

where $\kappa:(-1,\infty) \mapsto (0,\infty)$ is the well-known Kruskal function defined by its inverse, $\kappa^{-1}(x) = (x-1)e^x$ for $x \in (0,\infty)$; the coordinates U and V are restricted to the domains

$$U \in (-\infty,\infty), \quad V \in (0,\infty), \qquad (4)$$

in order that the same parts of the spacetimes as given by Eq. (2) are covered.

Let us look carefully at the transformation between the Eddington-Finkelstein and Kruskal coordinates:

$$U = \left(\frac{R}{2M} - 1\right) \exp\left(\frac{R}{2M}\right) \exp\left(-\frac{W}{4M}\right), \quad V = \exp\left(\frac{W}{4M}\right) \tag{5}$$

(the transformation of the angular coordinates is trivial). Equation (5) do not represent any coordinate transformation on $\mathcal{M}_1$, because they are solution dependent: the right-hand sides are non-trivial functions of M. They can only be interpreted as coordinate transformations, if we view Eq. (1) together with the manifold $\mathcal{M}_1$ as a *family of solutions* $\{(\mathcal{M}_1, g_M)\}$ rather that a family of metric fields $\{g_M\}$ on a background manifold $\mathcal{M}_1$.

Equations (1) and (2) express the Schwarzschild family in two different gauges. We can see from the above that a gauge transformation in general relativity is a *set of coordinate transformations*, one transformation for each solution (cf. Bergmann and Komar [1]), rather than a coordinate transformation on one manifold.

The illusion of a background manifold only arises, if one pastes together all solution manifolds in such a way that points with the same value of coordinates representing some gauge are considered to be identical. Thus, the background manifold $\mathcal{M}_1$ results, if we identify all points that have the same values of the Eddington-Finkelstein coordinates W and R; an analogous background manifold $\mathcal{M}_2$ for metrics (3) is constructed by identifying all points with the same values of Kruskal coordinates U and V. It should be clear that, in spite of the fact that both manifolds are formally diffeomorphic to each other, they nevertheless represent two very different localizations of geometrical properties of the Schwarzschild family. For example, the position of the event horizon on $\mathcal{M}_1$, which is given by the equation $R = 2M$, is not well-defined (fuzzy): the horizon of each solution lies at different points of $\mathcal{M}_1$. On the contrary, the position of the event horizon on $\mathcal{M}_2$, which is given by $U = 0$, is well-defined (sharp), because it is solution independent.

This all is well known and rather trivial. Still, the detailed form of the above analysis simplifies the understanding of the following point. A choice of gauge in general relativity mixes two different things: (1) it defines how points of different solution manifolds are to be identified so that a background manifold can be constructed; (2) it chooses definite coordinates on the background manifold. It is already the first step alone that delivers what we require from a gauge fixing: a *unique* metric field on a background manifold, say $\mathcal{M}$ for any solution (determined by the value of M in our example). This metric field can be given in any coordinate system on the background manifold; that is, all coordinate transformations on $\mathcal{M}$ are allowed (these should be M-independent for our example). Everything can be made manifestly covariant with respect to such transformations, in spite of the clear fact that the gauge has been fixed.

We do not know if this observation has ever been put forward in its full generality, but it surely has been done for the perturbative approach to general relativity by DeWitt [16]. Let us explain DeWitt idea in more detail in order to prevent misunderstanding. DeWitt chooses a particular classical *background spacetime* $(\mathcal{M}, g)$ (his method has, in fact, been called *background field method*). All other spacetimes

in some vicinity of $(\mathcal{M}, g)$ are described by small disturbances δg around g. Two kinds of gauge fixing is now possible: The first kind is just a choice of coordinates x^μ on $\mathcal{M}$; with respect to x^μ, the metric g and the disturbance δg have components $g_{\mu\nu}(x)$ and $\delta g_{\mu\nu}(x)$ and they transform as two tensor fields with respect to changes of these coordinates. A gauge transformation of the second kind is a small diffeomorphism $\delta \xi$ on $\mathcal{M}$. The background metric field changes then by a Lie derivative $\mathcal{L}_\xi g$; in coordinates, $g_{\mu\nu}(x) \mapsto g_{\mu\nu}(x) + \delta \xi_{\mu;\nu} + \delta \xi_{\nu;\mu}$, where the semicolon denotes the covariant derivative defined by the metric g. Such a change is not considered as a change of a physical state, so disturbances of the form $\delta g_{\mu\nu} = \delta \xi_{\mu;\nu} + \delta \xi_{\nu;\mu}$ for any small vector field $\delta \xi$ on $\mathcal{M}$ are considered as ''pure gauges.'' To fix a gauge of second kind, DeWitt requires a *supplementary condition* that is covariant with respect to coordinate transformations on $\mathcal{M}$ (gauge of the first kind). Thus, the background field method becomes covariant; even the field equations after the gauge of the second kind has been fixed are covariant in his formalism.

DeWitt's supplementary condition hinders gauge transformations of the second kind; these form a group, namely the group Diff$\mathcal{M}$ of diffeomorphisms of the background manifold $\mathcal{M}$. On the other hand, the gauge transformations that we are considering form the much larger Bergmann-Komar group [1]. Thus, there is an analogy, but not complete equivalence between the two ideas of gauge fixing.

Covariant gauge fixing is connected [7] to an idea due to Kuchař [8]. Let us describe this briefly in the rest of this section (for more details, see Kuchař [17,18]).

The Hamiltonian formalism for general relativity has been described in an elegant 3-covariant form by Arnowitt, Deser and Misner [19]. The action depends on the Arnowitt-Deser-Misner (ADM) variables $g_{kl}(x)$ and $\pi^{kl}(x)$ as follows:

$$S = \int dt \int_\Sigma d^3x (\pi^{kl}(x)\dot{g}_{kl}(x) - \mathcal{N}(x)\mathcal{H}(x) - \mathcal{N}^k(x)\mathcal{H}_k(x)),$$

where Σ is a three-dimensional manifold, $\mathcal{H}[g_{kl}, \pi_{kl}; x]$ and $\mathcal{H}_k[g_{kl}, \pi_{kl}; x]$ are the constraints [functionals of $g_{kl}(x)$ and $\pi^{kl}(x)$ and functions of x], and $\mathcal{N}(x)$ and $\mathcal{N}^k(x)$ are Lagrange multipliers [19].

Kuchař observed that one can sometimes make a canonical transformation,

$$g_{kl}(x), \pi^{kl}(x) \mapsto X^\mu(x), P_\mu(x), q^\alpha(x), p_\alpha(x) \tag{6}$$

so that the action acquires the form

$$S = \int dt \int_\Sigma d^3x (p_\alpha(x)\dot{q}^\alpha(x) + P_\mu(x)\dot{X}^\mu(x)$$

$$- \mathcal{N}^\mu(x)\mathcal{H}_\mu(x)),$$

where $\mathcal{H}_\mu(x)$ are linear combinations of the original constraints $\mathcal{H}(x)$ and $\mathcal{H}_k(x)$. The new constraints read $\mathcal{H}_\mu(x) = P_\mu(x) + H_\mu[X, q, p; x]$, where $H_\mu[X, q, p; x]$ are ''true Hamiltonians.''

The variables $X^\mu(x)$ describe embeddings of the three-dimensional Cauchy surface Σ with coordinates x^k into a four-dimensional background manifold $\mathcal{M}$ with coordinates X^μ. The function $\int_\Sigma d^3x \mathcal{N}^\mu(x)\mathcal{H}_\mu(x)d\epsilon$ generates an infinitesimal canonical transformation in the phase space that describes the dynamical evolution from the slice defined by the embedding $X^\mu(x)$ to the slice defined by the embedding $X^\mu(x) + \mathcal{N}^\mu(x)d\epsilon$. In this way, the dynamics is made completely independent from any additional structure on $\mathcal{M}$ such as a particular foliation.

If a transformation (6) exists, one can go a step further and pass to what Kuchař [18] called the Heisenberg picture (see also Kijowski [20]). This is another canonical transformation,

$$X^\mu(x), P_\mu(x), q^\alpha(x), p_\alpha(x) \mapsto X^\mu(x), \mathcal{H}_\mu(x), q_0^\alpha(x), p_{0\alpha}(x),$$
(7)

where $q_0^\alpha(x)$ and $p_{0\alpha}(x)$ are values of $q^\alpha(x)$ and $p_\alpha(x)$ at some particular embedding $X_0^\mu(x)$. Clearly, $q_0^\alpha(x)$ and $p_{0\alpha}(x)$ are constants of motion:

$$\{q_0^\alpha(x), \mathcal{H}_\mu(y)\} = \{p_{0\alpha}(x), \mathcal{H}_\mu(x)\} = 0,$$

so that the action becomes

$$S = \int dt \int_\Sigma d^3x (p_{0\alpha}(x)\dot{q}_0^\alpha(x) + \mathcal{H}_\mu(x)\dot{X}^\mu(x)$$
$$- \mathcal{N}^\mu(x)\mathcal{H}_\mu(x));$$
(8)

this *is* a special case of the form of the action after the first transformation, but the true Hamiltonians are zero and the P's are identical to the constraints now. It is the transformations (6) and (7) and the corresponding variables that we shall call *Kuchař decomposition*.

It is clear that Kuchař decomposition must implicitly include a gauge fixing not only because it leads to a well-defined background manifold $\mathcal{M}$, but also to a fixed coordinate system X^μ on it. Indeed, Kuchař decomposition also defines a particular set of metric fields on $\mathcal{M}$ by one of the canonical transformation equations, namely that of the form

$$g_{kl}(x) = g_{\mu\nu}(q(x), p(x), X(x))X_{,k}^\mu(x)X_{,l}^\nu(x)$$

for any embedding $X^\mu(x)$, where $g_{\mu\nu}(q, p, X)$ is a metric field for any value of the variables $q^\alpha(x)$ and $p_\alpha(x)$ (see, e.g. Kuchař, Romano and Varadarajan [21]). The KRV metric $g_{\mu\nu}(q, p, X)$ is clearly an analog of the metric (1) [or (3)]: q and p play the role of the Schwarzschild mass M, and X that of the Eddington-Finkelstein (Kruskal) coordinates.

In the present paper, we shall describe Kuchař decomposition in geometric (that is, coordinate-free) terms.

III. THE GENERALLY COVARIANT MODELS

We shall consider a class of constrained dynamical systems that are in certain respects similar to general relativity. As examples, general relativity, possibly coupled to matter fields, $2 + 1$ gravity [22], possibly with particle-like sources, and the spherically symmetric gravitating thin shell [23] can be mentioned. In this section, we define the class by a list of properties. For some of the models named above, not all of these properties have been fully established yet.

A. The form of dynamical trajectories

A dynamical trajectory—or classical solution—of each such model consists of two parts. The first part is a spacetime $(\mathcal{M}, g)$, where $\mathcal{M}$ is a manifold of dimension D and g is a metric of (Lorentzian) signature $D - 2$. Each such spacetime will be called a *solution spacetime*. Second, any dynamical trajectory may contain additional fields and branes (submanifolds of $\mathcal{M}$ carrying other fields—they play the role of trajectories of particles, strings, shells etc.) on $\mathcal{M}$, which we shall describe by the symbol ϕ; thus a dynamical trajectory can be denoted by $(\mathcal{M}, g, \phi)$. Just for the sake of simplicity, we assume that there are no gauge fields within ϕ, but this restriction can be removed easily.

B. Diffeomorphism invariance

The dynamical equations of each such model are generally covariant [24,39,40] with respect to all coordinate transformations on $\mathcal{M}$. This implies that any system g and ϕ of fields and branes satisfying the dynamical equations on a manifold $\mathcal{M}$ can be pushed forward by any diffeomorphism $\varphi \in \text{Diff}\mathcal{M}$ to a different set $\varphi_* g$ and $\varphi_* \phi$ on $\mathcal{M}$, which also satisfies the dynamical equations. Indeed, if X are any coordinates on $\mathcal{M}$, and $g(X)$ and $\phi(X)$ the components of all fields and branes with respect to X, then $\varphi_* g$ and $\varphi_* \phi$ have exactly the same components with respect to the pushed-forward coordinates $X' := X \circ \varphi^{-1}$. They satisfy, therefore, the dynamical equations of exactly the same form. Observe that even the spinor fields can be pushed forward in this way, because the metric is, so the push-forward of any D-frame that is orthonormal with respect to the metric g will be orthonormal with respect to $\varphi_* g$.

Hence, if $(\mathcal{M}, g, \phi)$ is a dynamical trajectory, then $(\mathcal{M}, \varphi_* g, \varphi_* \phi)$ is also one for any $\varphi \in \text{Diff}\mathcal{M}$. This feature is called *diffeomorphism invariance*. In general, the set $(\varphi_* g, \varphi_* \phi)$ of fields and branes on $\mathcal{M}$ is different from the set (g, ϕ). However, we are going to treat them as physically equivalent if only $\varphi \in \text{Diff}_\infty\mathcal{M}$, where $\text{Diff}_\infty\mathcal{M}$ is a subgroup of $\text{Diff}\mathcal{M}$ composed of those diffeomorphisms that are "trivial at infinity." For example, if the solution spacetime is asymptotically flat, the elements of $\text{Diff}_\infty\mathcal{M}$ must move neither the points at the infinity nor the frames at these points. For $\mathcal{M}$ spatially compact, there is no "infinity" and $\text{Diff}_\infty\mathcal{M}$ simply coincides with the entire diffeomorphism group Diff $\mathcal{M}$. Thus, the physical state of the system under consideration is always described by a whole class of equivalent trajectories *modulo* the action of the group $\text{Diff}_\infty\mathcal{M}$. We denote such a class $\{(\mathcal{M}, g, \phi)\}$, where $(\mathcal{M}, g, \phi)$ is a particular set of fields and branes on $\mathcal{M}$ satisfying the dynamical equations.

Even if the whole group $\text{Diff}\mathcal{M}$ (i.e., also those diffeomorphisms which are non-trivial "at infinity") forms the symmetry group of the theory, the gauge group of the model will be constructed only from the subgroup $\text{Diff}_\infty\mathcal{M}$. The

reason for this decision is obvious if we think e.g. about special-relativistic mechanics of a free particle: considering two motions as being physically equivalent if they only differ by the action of an element of a symmetry group (in that case it would be the Poincaré group) would be an abuse of the very notion of symmetry in physics. The physical phase space resulting from such a construction would consist of a single equivalence class, composed of all possible physical situations. In such a zero-dimensional space no non-trivial dynamics is possible.

C. The initial data

We assume further that each model determines a class of $(D-1)$-dimensional manifolds; each such manifold Σ is called *initial manifold*. Further, it also determines a class of a system of some fields and membranes γ, ψ, $\dot{\gamma}$ and $\dot{\phi}$ on Σ. The object $(\Sigma, \gamma, \psi, \dot{\gamma}, \dot{\psi})$ built up from the elements of the classes is then called an *initial datum* of the model. For example, in general relativity, any three-dimensional Riemannian manifold can serve as an initial manifold; the field γ is a Riemann metric on it, $\dot{\gamma}$ is a symmetric tensor field K_{kl} on Σ and there is no ψ and $\dot{\psi}$.

An important connection of initial data to the dynamical trajectories is the following. Let $(\mathcal{M}, g, \phi)$ be a dynamical trajectory; then, $(D-1)$-dimensional submanifolds of $\mathcal{M}$ satisfying certain requirements are called *Cauchy surfaces* in $\mathcal{M}$. Each Cauchy surface is a possible initial manifold from the class above. Let Σ be a Cauchy surface for the dynamical trajectory $(\mathcal{M}, g, \phi)$; then the fields and branes g and ϕ determine a unique initial datum $(\Sigma, \gamma, \psi, \dot{\gamma}, \dot{\psi})$. For example, γ and ψ are the pull-backs of the fields to, and intersections of the branes with, the surface Σ; $\dot{\gamma}$ and $\dot{\phi}$ are some geometrical quantities on Σ constructed from the fields and their first derivatives at Σ, and from projections into Σ of the normalized D-velocities of the branes. Such an initial datum is called *induced* on the Cauchy surface Σ by the dynamical trajectory $(\mathcal{M}, g, \phi)$. Then the dynamical equations (and the asymptotic conditions for g and ϕ) of the model imply some asymptotic conditions for γ, ψ, $\dot{\gamma}$ and $\dot{\phi}$ and some relations between them on Σ that are called *constraints*.

D. The existence and uniqueness of dynamical trajectories

We assume further that the dynamical equations and all initial data have the following property. For each initial datum $(\Sigma, \gamma, \psi, \dot{\gamma}, \dot{\psi})$ that satisfies the constraints there is a *unique* $\mathrm{Diff}_{\infty}(\Sigma \times \mathbf{R})$-class $\{(\Sigma \times \mathbf{R}, g, \phi)\}$ of maximal dynamical trajectories such that each element $(\Sigma \times \mathbf{R}, g, \phi)$ of the class contains a Cauchy surface on which the induced datum is diffeomorphic to $(\Sigma, \gamma, \psi, \dot{\gamma}, \dot{\psi})$. This implies that the set of objects defining initial data must be complete in a certain sense.

The uniqueness of the maximal dynamical trajectory is understood in the sense of Choquet-Bruhat and Geroch [25]. It has been shown for general relativity that the solution spacetimes of maximal dynamical trajectories are globally hyperbolic; we will assume the same property for all our models. According to a theorem of Geroch [26], each globally hyperbolic spacetime in general relativity can be completely foliated by spacelike hypersurfaces, each of them being diffeomorphic to Σ. This leads us to assume that $\mathcal{M} = \Sigma \times \mathbf{R}$.

The uniqueness implies the following property of the dynamical equations. Suppose that $(\mathcal{M}, g, \phi)$ is a maximal dynamical trajectory for the initial datum $(\Sigma, \gamma, \psi, \dot{\gamma}, \dot{\psi})$ [so $(\Sigma, \gamma, \psi, \dot{\gamma}, \dot{\psi})$ satisfies the constraints and the asymptotic conditions]. Let Σ' be an arbitrary Cauchy surface in $\mathcal{M}$, and the initial datum $(\Sigma', \gamma', \psi', \dot{\gamma}', \dot{\psi}')$ be induced by the dynamical trajectory $(\mathcal{M}, g, \phi)$ on Σ'. Then $(\Sigma', \gamma', \psi', \dot{\gamma}', \dot{\psi}')$ satisfies the constraints and the asymptotic conditions, and any representative $(\mathcal{M}, g', \phi')$ of the unique maximal dynamical trajectory corresponding to $(\Sigma', \gamma', \psi', \dot{\gamma}', \dot{\psi}')$ is diffeomorphic to $(\mathcal{M}, g, \phi)$.

E. The phase space

Let us consider the set Γ_1' of all initial data, and the subset Γ_1 of those data that satisfy the constraints (and asymptotic conditions). In the case of spatially compact sectors, we exclude all data from Γ_1' that lie in Γ_1 and determine maximal dynamical trajectories admitting any symmetry (or any symmetry that is higher than the symmetry following from the definition of the model). It seems that this deleting is not necessary [15] for sectors that are not spatially compact. Thus, not only (global) Killing vector fields in the solution spacetimes $(\mathcal{M}, g)$ are forbidden, but also any finite (discrete) symmetry. Let us denote the resulting sets by Γ_2' and Γ_2. We assume that a subset Γ' of Γ_2' has been organized in such a way that Γ' is a manifold modeled on a Banach space and $\Gamma := \Gamma' \cap \Gamma_2$ is a closed submanifold of Γ'. In general, Γ' is an open submanifold of the phase space of the model. In general relativity, Γ was shown to be a C^{∞}-submanifold (for reviews, see Fischer and Marden [27] and Marsden [15]) The condition of no symmetry (even a discrete one) for the spatially compact sectors is necessary for the construction of the covariant gauge fixing in the next section to work.

We assume that each model defines a symplectic form Ω' on Γ' (this may be determined by the variational principle of the model under study—for more discussion, see Kijowski and Tulczyjew [28]). For example, in general relativity, $\Omega' = d\Theta'$ and $\Theta' = \int_{\Sigma} d^3x \, \pi^{kl}(x) d\gamma_{kl}(x)$, where $\pi^{kl} := \mathrm{Det}(\gamma_{mn})^{1/2}(\gamma^{kl}\gamma^{ij} - \gamma^{ki}\gamma^{lj})K_{ij}$ is a super-local functional of γ_{kl} and K_{kl}. We assume further that Γ is a coisotropic [13] submanifold of Γ' with respect to Ω'. That is the following property. Let $p \in \Gamma$ and let $L_p(\Gamma)$ be a subspace of $T_p(\Gamma')$ defined by

$$L_p(\Gamma) := \{v \in T_p(\Gamma') | \Omega'(v, u) = 0 \quad \forall u \in T_p(\Gamma)\}. \quad (9)$$

One can write alternatively [13] $L_p(\Gamma) = \mathrm{orth}_{\Omega'} T_p(\Gamma)$. Γ is coisotropic if $L_p(\Gamma) \subset T_p(\Gamma)$. Hence, the pull-back Ω of Ω' to Γ is a presymplectic form on Γ. The space $L_p(\Gamma)$ is called the *characteristic space* of Ω.

The subbundle $L(\Gamma) := \{(p, L_p(\Gamma)) | p \in \Gamma\}$ of the tangent bundle $T(\Gamma)$ is an integrable subbundle, because Ω is

closed; it is called the *characteristic bundle* of Ω. Let us call the maximal integral manifolds of $L(\Gamma)$ *c-orbits*.

In the case of infinite-dimensional models, additional, model-specific assumptions [15,27] are needed for the proofs that Γ is a submanifold and that c-orbits with suitable properties exist.

F. Relation between the c-orbits and the maximal dynamical trajectories

We assume that there is a relation between c-orbits and dynamical trajectories of the model as follows. Let o be a c-orbit and $p \in o$. Let p be the initial datum $(\Sigma, \gamma, \psi, \dot{\gamma}, \dot{\psi})$ and let $(\mathcal{M}, g, \phi)$ be a maximal development of $(\Sigma, \gamma, \psi, \dot{\gamma}, \dot{\psi})$. Then the initial datum $(\Sigma', \gamma', \psi', \dot{\gamma}', \dot{\psi}')$ corresponding to any point $q \in o$ defines a unique Cauchy surface Σ' in $\mathcal{M}$, by the condition that the dynamical trajectory $(\mathcal{M}, g, \phi)$ induces the initial datum $(\Sigma', \gamma', \psi', \dot{\gamma}', \dot{\psi}')$ on Σ'. Moreover, Σ' can be obtained from Σ by the action of some $\varphi \in \mathrm{Diff}_\infty \mathcal{M}$. In general, all Cauchy surfaces that correspond to points in the c-orbit o form an open subset of $\mathcal{M}$.

If the Cauchy surfaces are not compact, then they must satisfy certain boundary conditions. For example, in the case of asymptotically flat solution spacetimes, the Cauchy surfaces must be asymptotically flat and all *coincide* with each other at infinity in order to define points of the same c-orbit in Γ. We can also say: let p and q be any two points from the same c-orbit o. Let the corresponding initial data determine dynamical trajectories $(\mathcal{M}, g, \phi)$ and $(\mathcal{M}', g', \phi')$. Then these two dynamical trajectories are diffeomorphic to each other. In this way, any c-orbit o determines a class of $\mathrm{Diff}_\infty \mathcal{M}$-equivalent maximal dynamical trajectories.

Observe that a dynamical trajectory $(\mathcal{M}, g, \phi)$ corresponding to the datum $(\Sigma, \gamma, \psi, \dot{\gamma}, \dot{\psi})$ contains *exactly* one surface $\Sigma \subset \mathcal{M}$ such that the datum induced on Σ coincides with $(\Sigma, \gamma, \psi, \dot{\gamma}, \dot{\psi})$. Indeed, two different Cauchy surfaces carrying initial data that are $\mathrm{Diff}_\infty \mathcal{M}$-equivalent to each other would imply existence of a non-trivial symmetry of the dynamical trajectory $(\mathcal{M}, g, \phi)$, and such points have been excluded from Γ.

G. The physical phase space

The last important property we assume is that the set of the c-orbits in the constraint surface form a *quotient manifold* [14] Γ/o with the natural projection $\pi : \Gamma \mapsto \Gamma/o$ being a *submersion* [14]. Furthermore, there is a unique symplectic form $\tilde{\Omega}$ on Γ/o such that $\Omega = \pi^* \tilde{\Omega}$, where π^* is the pull-back of forms by π. Our reduced symplectic space $(\Gamma/o, \tilde{\Omega})$ is, in general, an open subset of the *physical phase space*. For example, in general relativity, some aspects of the physical phase space are discussed by Marsden [15] and Fischer and Moncrief [29].

We maintain that all information about physical properties of the models is contained in the physical phase spaces. One is, however, forced to use the extended structures Γ and Γ', because it is often difficult in practice to perform the reduction to the physical phase space and to find an explicit parametrization of it.

IV. COVARIANT GAUGE FIXINGS

The general structure described in the previous section enables us to work out a geometric definition of gauge fixing based on the ideas of the previous paper [7]. This will concern only the diffeomorphism group—as explained, we assume that there are no other gauge groups acting.

Let o be an arbitrary c-orbit, Σ_o be the manifold structure of the corresponding Cauchy surfaces and $\{(\mathcal{M}, g, \phi)\}_o$ the diffeomorphism class of the maximal dynamical trajectories determined by o. Let us choose one fixed representative $(\mathcal{M}_o, g_o, \phi_o)$ from this class for each o. Consider the set $\mathrm{Emb}_c(\Sigma_o, \mathcal{M}_o)$ of all embeddings of Σ_o in $\mathcal{M}_o$ such that the embedded submanifold is a Cauchy surface; we call such embeddings *Cauchy embeddings*. If Σ_o is not compact, we restrict the space $\mathrm{Emb}_c(\Sigma_o, \mathcal{M}_o)$ to a class of embeddings that satisfy the boundary conditions formulated in Sec. III F. We assume that $\mathrm{Emb}_c(\Sigma_o, \mathcal{M}_o)$ is an open subset of the space $\mathrm{Emb}(\Sigma_o, \mathcal{M}_o)$ of all (smooth) embeddings of Σ_o in $\mathcal{M}_o$ (that also satisfy the boundary conditions in the noncompact case). Discussion of this point for the compact cases is given in Isham and Kuchař [30]. Then it follows from the assumptions in Sec. III F that there is an injection

$$\rho_o : o \mapsto \mathrm{Emb}_c(\Sigma_o, \mathcal{M}_o)$$

such that each point p of o is mapped onto that Cauchy surface $h(\Sigma_o)$ in $\mathcal{M}_o$, $h \in \mathrm{Emb}_c(\Sigma_o, \mathcal{M}_o)$, on which the initial datum p is induced by $(\mathcal{M}_o, g_o, \phi_o)$. Such a map ρ_o is not unique. It depends on the chosen representative $(\mathcal{M}_o, g_o, \phi_o)$, but any two possible ρ_o's differ by a diffeomorphism $\varphi \in \mathrm{Diff}_\infty \mathcal{M}_o$, $\rho_o' = \varphi \circ \rho_o$.

We assume that Γ' and $\mathrm{Emb}_c(\Sigma_o, \mathcal{M}_o)$ has been given differentiable structure such that the map ρ_o together with its inverse become differentiable. This implies the following properties. Let $h : \Sigma_o \mapsto \mathcal{M}_o$ be a point of $\mathrm{Emb}_c(\Sigma_o, \mathcal{M}_o)$ that lies in the range of ρ_o. Then the elements of the tangent space $T_h \mathrm{Emb}_c(\Sigma_o, \mathcal{M}_o)$ to $\mathrm{Emb}_c(\Sigma_o, \mathcal{M}_o)$ at h are vector fields $V(x)$ along $h(\Sigma_o)$ in $\mathcal{M}_o$, where $x \in h(\Sigma_o)$ (they may have to satisfy some suitable smoothness and boundary conditions [30]); all such vector fields form the tangent space $T_h \mathrm{Emb}_c(\Sigma_o, \mathcal{M}_o)$. In particular, the following conditions are to be satisfied:

(1) There is a smooth family of smooth curves $C_x(\lambda)$ in $\mathcal{M}_o$ such that $C_x(0) = x$ and $C_x'(0) = V(x)$ for each $x \in h(\Sigma_o)$.

(2) There is an $\epsilon > 0$ such that $C_x(\lambda)$ is well defined for each $x \in h(\Sigma_o)$ and each λ such that $|\lambda| < \epsilon$.

(3) For any fixed λ such that $|\lambda| < \epsilon$, the expression $C_x(\lambda)$ defines a map $c_\lambda : x \mapsto \mathcal{M}_o$ for all $x \in h(\Sigma_o)$ by $c_\lambda(x) = C_x(\lambda)$; then we require that $c_\lambda \circ h : \Sigma_o \mapsto \mathcal{M}_o$ belongs to $\mathrm{Emb}_c(\Sigma_o, \mathcal{M}_o)$ for all λ, or $c_\lambda(h(\Sigma_o))$ is a Cauchy surface in $\mathcal{M}_o$ for all $\lambda \in (-\epsilon, \epsilon)$.

If there is such a family, then there are many.

Consider the map $\rho_o^{-1}(c_\lambda(h(\Sigma_o))) : \lambda \mapsto o$. It defines a curve through the point $p = \rho_o^{-1}(h(\Sigma_o))$ in o. The differen-

tiability requirements on ρ_o mean that it is a smooth curve in o with a well-defined tangential vector v at $p \in o$; v is non-zero if $V(x) \neq 0$, is tangential to o and depends only on the vector field $V(x)$, not on a particular family of curves $C_x(\lambda)$.

In the opposite direction, let $p \in o$ and v be a vector at p tangential to o. Then there is a curve $\widetilde{C}(\lambda)$ in o for $|\lambda| < \epsilon$, $\epsilon > 0$, such that $\widetilde{C}(0) = p$ and $\widetilde{C}'(0) = v$ and the map ρ_o determines a family of Cauchy surfaces $\{\Sigma_\lambda\}$ in $\mathcal{M}_o$ by $\Sigma_\lambda = \rho_o(\widetilde{C}(\lambda))(\Sigma_o)$. In particular, any fixed point $x \in \Sigma_o$ will be mapped by the Cauchy embedding $\rho_o(\widetilde{C}(\lambda))$ to a point that we denote by $C_x(\lambda)$ in a neighborhood of $\rho_o(p)(x)$. These curves have tangent vectors for each x at $\lambda = 0$ that we denote by $V(x)$; hence

$$V(x) = (\rho_o(C(\lambda))(x))'|_{\lambda = 0}.$$

Again, $V(x)$ must be from $T_{\rho_o(p)}\mathrm{Emb}(\Sigma_o, \mathcal{M}_o)$, non-zero if $v \neq 0$ and independent of a particular curve $\widetilde{C}(\lambda)$ chosen in o. For example, in general relativity, this differentiability has been shown by Moncrief [31].

Consider the submanifold Γ_Σ containing all points of Γ that correspond to a fixed initial manifold Σ. As a rule, different Γ_Σ's are topologically separated in Γ, and can be called (topological) *sectors* of the model. A *covariant gauge fixing* in Γ_Σ is the set of maps $\{\iota_o\}$ that satisfies two requirements:

(1) For each c-orbit $o \in \Gamma_\Sigma/o$, $\iota_o:\mathcal{M}_o \mapsto \Sigma \times \mathbf{R}$ is a well-defined differentiable injection with differentiable inverse.

Any such map induces a differentiable injection $\widetilde{\iota}_o:\mathrm{Emb}(\Sigma, \mathcal{M}_o) \mapsto \mathrm{Emb}(\Sigma, \Sigma \times \mathbf{R})$ with differentiable inverse by $\widetilde{\iota}_o h := \iota_o \circ h$ for any $h \in \mathrm{Emb}(\Sigma, \mathcal{M}_o)$. Define $\sigma:\Gamma_\Sigma \mapsto \mathrm{Emb}(\Sigma, \Sigma \times \mathbf{R})$ by $\sigma|_o := \widetilde{\iota}_o \circ \rho_o$ for all $o \in \Gamma_\Sigma/o$. Then

(2) σ is a differentiable map of Γ_Σ into $\mathrm{Emb}(\Sigma, \Sigma \times \mathbf{R})$.

Observe that the maps ρ_o are arbitrary and are not supposed to have any relations for different o's so that the dynamical trajectories $(\mathcal{M}_o, g_o, \phi_o)$ and $(\mathcal{M}_{o'}, g_{o'}, \phi_{o'})$ need not converge to each other when $o' \to o$. However, $(\mathcal{M}_{o'}, g_{o'}, \phi_{o'})$ will converge to some dynamical trajectory that is $\mathrm{Diff}_\infty \mathcal{M}_o$-equivalent to $(\mathcal{M}_o, g_o, \phi_o)$ as $o' \to o$ and we can include such diffeomorphisms into ι_o's to make σ differentiable.

The condition (1) and the discussion before it imply that $\sigma|_o$ is a differentiable injection with differentiable inverse for any $o \in \Gamma/o$. The condition (2) implies then that the maps π and σ are transversal to each other. That is the following property. Let p be an arbitrary point of Γ. Then any vector $v \in T_p(\Gamma)$ that satisfies both equations $d\sigma(v) = 0$ and $d\pi(v) = 0$ must be the zero vector.

We shall call the pair (π, σ) of maps a *Kuchař* Γ-*decomposition* (recall that π is the projection defined in Sec. III G). Thus, for any $p \in \Gamma_\Sigma$, $\pi(p)$ is a point in the physical phase space of the model, and $\sigma(p)$ is some embedding, $\sigma(p):\Sigma \mapsto \Sigma \times \mathbf{R}$, giving the position of "the slice" Σ in "the background manifold" $\Sigma \times \mathbf{R}$.

The image of Γ_Σ under $\pi \times \sigma$ is a subset $(\pi \times \sigma)(\Gamma_\Sigma)$ of the Cartesian product manifold $K_\Sigma := (\Gamma_\Sigma/o) \times \mathrm{Emb}(\Sigma,$

$\Sigma \times \mathbf{R})$. $\sigma(o)$ is some open subset of $\mathrm{Emb}(\Sigma, \Sigma \times \mathbf{R})$ for each o. These subsets can vary with varying $o \in \Gamma_\Sigma/o$. Indeed, consider an embedding $h:\Sigma \mapsto (\Sigma \times \mathbf{R})$ and study its pull-back $h':\Sigma \mapsto \mathcal{M}_o$ by ι_o defined by $h' := \iota_o^{-1} \circ h$. Such a pull-back need not exist, because ι_o is only injection so there may be points in $\Sigma \times \mathbf{R}$ that do not lie in the range of ι_o and $h(\Sigma)$ can contain such points. Even if the pull-back exists, it need not define a spacelike surface in $(\mathcal{M}_o, g_o)$, and even if the surface is spacelike, it need not be a Cauchy surface. For a different c-orbit o', $o' \neq o$, the $\iota_{o'}$-pull-back of the same embedding h can have quite different properties as to its existence and as to its being a Cauchy surface in $(\mathcal{M}_{o'}, g_{o'})$. Thus, the projection of the subset $(\pi \times \sigma)(\Gamma_\Sigma)$ of K_Σ to $\mathrm{Emb}(\Sigma, \Sigma \times \mathbf{R})$ may be empty; on the other hand, its projection onto Γ_Σ/o coincides with Γ_Σ/o.

From our definitions and assumptions, it follows that the map $\pi \times \sigma:\Gamma_\Sigma \mapsto K_\Sigma$ is a differentiable injection with differentiable inverse. Indeed, it is an injection because all points that have the same projection by the submersion π form a c-orbit o, o is mapped by the bijection ρ_o to $\mathrm{Emb}_c(\Sigma_o, \mathcal{M}_o)$, and this set by the injection $\widetilde{\iota}_o$ induced by ι_o to $\sigma(o) \subset \mathrm{Emb}(\Sigma, \Sigma \times \mathbf{R})$ (of course, Σ is diffeomorphic to Σ_o). It is differentiable, because both maps, π and σ, are. It has a differentiable inverse because π and σ are transversal.

Then there is a well-defined presymplectic form Ω_K on $(\pi \times \sigma)(\Gamma_\Sigma)$, $\Omega_K := (\pi \times \sigma)_* \Omega$, the push-forward by the map $\pi \times \sigma$ of the presymplectic form Ω on Γ_Σ. It is easy to see what the structure of Ω_K is: Its characteristic subspaces are tangential to $\mathrm{Emb}(\Sigma, \Sigma \times \mathbf{R})$ at any point $(\pi(p), \sigma(p))$, where o is the c-orbit through p, $o = \pi^{-1}(\pi(p))$. Its pull-back to Γ_Σ/o coincides with the form $\widetilde{\Omega}$ defined in Sec. III G.

We can go further and consider the space K_Σ to be a trivialization of a fiber bundle E_Σ with the base space Γ/o, the typical fiber $\mathrm{Emb}(\Sigma, \Sigma \times \mathbf{R})$ and the group $\mathrm{Diff}_\infty(\Sigma \times \mathbf{R})$. Each σ can be decomposed into a direct map $\kappa:\Gamma_\Sigma \mapsto E_\Sigma$, which is a differentiable injection with differentiable inverse, a trivialization $\tau:E_\Sigma \mapsto K_\Sigma$, and the projection $\eta:K_\Sigma \mapsto \mathrm{Emb}(\Sigma, \Sigma \times \mathbf{R})$, so that $\sigma = \eta \circ \tau \circ \kappa$. κ is independent of covariant gauge fixings; each fixing, however, defines σ, and so a trivialization τ of E_Σ.

The covariant gauge fixing determines also a unique set of fields and branes with a domain in the background manifold $\Sigma \times \mathbf{R}$ for any dynamical trajectory, that is for any c-orbit $o \in \Gamma/o$; let us denote this set by $(g(o), \phi(o))$, where $o \in \Gamma/o$. This can be seen as follows. For each orbit o, we have a definite representative $(\mathcal{M}_o, g_o, \phi_o)$; the spacetime manifold $\mathcal{M}_o$ is mapped by the diffeomorphism ι_o into the background manifold $\mathcal{M}$; hence, we can define

$$g(o) := \iota_{o*} g_o, \quad \phi(o) := \iota_{o*} \phi_o, \tag{10}$$

where ι_{o*} is the push-forward defined by the map ι_o. $g(o)$ is the coordinate-free version of the Kuchař, Romano and Varadarajan metric [21] mentioned in Sec. IV.

Observe that the unique set $(g(o), \phi(o))$ is exactly what one expects a gauge fixing to deliver: a unique set of fields and branes on the background manifold $\Sigma \times \mathbf{R}$ for each class

$\{(\mathcal{M}_o, g_o, \phi_o)\}$. In fact, the set $(g(o), \phi(o))$ represents what can be called a locally *complete solution* to the dynamical equations: each dynamical trajectory of an open set is obtained by a suitable choice of o.

In this section, we have deliberately left the map σ completely general. Of course, one can easily construct a map σ if one knows a *complete* coordinate condition that works in a neighborhood of a *whole* Cauchy surface and admits a sufficiently large set of initial data on this surface to cover a *whole* open set $\tilde{U}$ in the physical phase space. A ''complete'' condition defines a unique coordinate system in certain domain in each maximal dynamical trajectory corresponding to a point of $\tilde{U}$. For example, the harmonic coordinate condition is not complete in this sense. Some conditions work even globally. An example for general relativity coupled to special kind of continuous matter [32,33] has been given. For pure gravity, suppose e.g. that all spatially compact maximal solutions of Einstein's equations admit a unique and complete foliation by surfaces Σ_K of constant mean external curvature K, $K := q^{kl} K_{kl}$ (this is a form of the well-known CMC hypothesis). Then one can stipulate that the surfaces Σ_K are mapped onto (Σ, K) in $(\Sigma \times \mathbf{R})$ by each ι_o and hope that the map ι_o can be completed suitably inside of each Σ_K. Then the corresponding covariant gauge fixing is also global. In general, it seems plausible that any construction of a particular map σ can be based on a set of differentiable sections of the submersion π on Γ.

The simplest construction of σ, however, would starts from a locally complete solution $(\mathcal{M}, g(o), \phi(o))$, if such is known. Each $o \in \Gamma/o$ is determined by values of a suitable set of constants of motion and π is trivial. For each o and each Cauchy embedding $h: \Sigma \mapsto \mathcal{M}$, one then calculates the initial datum $(\Sigma, \gamma, \psi, \dot{\gamma}, \dot{\psi})$ that the fields and branes $g(o)$ and $\phi(o)$ induce on $h(\Sigma)$. This defines $(\sigma_o)^{-1}$. We shall use this construction in subsequent papers.

V. EXTENSIONS OF KUCHAŘ DECOMPOSITIONS FROM Γ TO Γ'

A covariant gauge fixing described in the previous section defines a division of variables into two groups: the set of dynamical variables that determine points of the physical phase space Γ/o and the set of kinematical variables that describe an embedding of the Cauchy surface Σ into a background manifold $\mathcal{M}$; this division is done without use of coordinates in any of these manifolds. It is, however, not yet a full Kuchař decomposition as outlined in Sec. II, because it works only inside the constraint surface Γ, whereas the original Kuchař decomposition holds in a neighborhood of Γ in the phase space Γ'. In the present section, we shall extend gauge fixings and Kuchař decompositions from Γ to Γ'. We shall work with a fixed Σ-sector of the model, and we leave out the corresponding index Σ everywhere.

Let us first describe what exactly is the problem. Clearly the components of the symplectic form Ω' on Γ' with respect to Kuchař coordinates are

$$\Omega' = \int_\Sigma d^3x (d\mathcal{H}_\mu(x) \wedge dX^\mu(x) + dp_\alpha(x) \wedge dq_\alpha(x)).$$

This can be written as $\Omega' = \Omega_1 \oplus_\perp \Omega_2$, where

$$\Omega_1 := d \int_\Sigma d^3x \, p_\alpha(x) \wedge dq^\alpha(x)$$

is a symplectic form on Γ/o with coordinates $p_\alpha(x)$ and $q^\alpha(x)$; in the form

$$\Omega_2 := d \int_\Sigma d^3x \, \mathcal{H}_\mu(x) \wedge dX^\mu(x) \tag{11}$$

we clearly recognize the canonical symplectic structure of $T^*(\text{Emb}(\Sigma, \Sigma \times \mathbf{R}))$, where $X^\mu(x)$ is a point of the manifold $\text{Emb}(\Sigma, \Sigma \times \mathbf{R})$ and $\mathcal{H}_\mu(x)$ is a cotangent vector at $X^\mu(x)$.

The set $(\pi \times \sigma)(\Gamma)$ is a submanifold of K, which is in turn a submanifold of $K' := (\Gamma/o) \times T^*(\text{Emb}(\Sigma, \Sigma \times \mathbf{R}))$ that is determined by the equations $\mathcal{H}_\mu(x) = 0$ in K'. Hence, $\pi \times \sigma$ injects Γ symplectically into K'. What we are looking for is, therefore, a symplectic injection φ that maps a neighborhood U' of Γ in Γ' into K' so that $\varphi|_\Gamma = \pi \times \sigma$.

We shall show the existence of such an extension φ in three steps. The proof will be given in a form that is immediately valid only for finite-dimensional manifolds. After each step, however, we shall discuss the points that do not admit a straightforward generalization to infinite dimensional cases and show how the argument can be improved.

A. Extension of $\pi \times \sigma$ to the tangent space of Γ in Γ'

First, we extend $\pi \times \sigma$ just ''to the first order'' at Γ, that is, we construct a map $\varphi_1 : T'(\Gamma) \mapsto T(K')$, where $T'(\Gamma)$ is the vector bundle with the base space Γ whose fibers are the spaces $T_p(\Gamma')$ tangent to Γ' at all $p \in \Gamma$; it is a subbundle of $T(\Gamma')$ which could also be denoted by $T_p(\Gamma')|_\Gamma$. The map φ_1 must have the following properties: (i) φ_1 is a vector bundle morphism, (ii) $\varphi_1|_{T(\Gamma)} = d(\pi \times \sigma)$, and (iii) φ_1 is symplectic. Because of (i), φ_1 can be decomposed [14] into a set of maps containing a base-space map $\varphi_{1b} : \Gamma \mapsto K$, and fiber maps $\varphi_{1fp} : T'(\Gamma) \mapsto T_{\varphi_{1b}(p)}(K')$ for each $p \in \Gamma$; φ_{1b} is a differentiable injection and φ_{1fp} is a linear isomorphism for each $p \in \Gamma$. Because of (ii), $\varphi_{1b} = \pi \times \sigma$ and

$$\varphi_{1fp}|_{T_p(\Gamma)} = d(\pi \times \sigma)|_{T_p(\Gamma)}. \tag{12}$$

Finally, because of (iii), φ_{1fp} is a symplectic isomorphism at each $p \in \Gamma$.

The map φ_{1fp} is, therefore, already determined on the subspace $T_p(\Gamma)$ of $T'_p(\Gamma)$, and we have to specify it only on a subspace, say, $N_p(\Gamma)$ of $T'_p(\Gamma)$ such that $T'_p(\Gamma) = N_p(\Gamma) \oplus T_p(\Gamma)$.

The symplectic forms Ω' and the pull-back $\varphi_{1fp}^* \Omega_{K'}$ must, moreover, coincide on $T'_p(\Gamma)$ for all $p \in \Gamma$; they do so already on $T_p(\Gamma)$. This suggests an idea for the construction. The image $T_{\varphi_{1b}(p)}(K')$ splits in a way adapted to the symplectic form $\Omega_{K'}$:

$$T_{\varphi_{1b}(p)}(K') = T_{\varphi_{1b}(p)}(K) \oplus T_{\sigma(p)}(T^*_{\sigma(p)}(\text{Emb}(\Sigma,\Sigma\times\mathbf{R}))),$$

where the space $T_{\sigma(p)}(T^*_{\sigma(p)}(\text{Emb}(\Sigma,\Sigma\times\mathbf{R})))$ is $\Omega_{K'}$-isotropic (i.e., restriction of $\Omega_{K'}$ to this space is a zero form) and $\Omega_{K'}$-orthogonal to $T_{\pi(p)}(\Gamma/o)$. Hence, the pre-image $N_p(\Gamma)$ of this space,

$$N_p(\Gamma) := \varphi_{1fp}^{-1} T_{\sigma(p)}(T^*_{\sigma(p)}(\text{Emb}(\Sigma,\Sigma\times\mathbf{R}))) \qquad (13)$$

must be isotropic in $T'_p(\Gamma)$ with respect to Ω' and Ω'-orthogonal to $Q_p(\Gamma)$, which is the pre-image of $T_{\pi(p)}(\Gamma/o)$, $Q_p(\Gamma) := \varphi_{1fp}^{-1} T_{\pi(p)}(\Gamma/o)$. However, $\Gamma/o \subset K$, so φ_{1fp}^{-1} on Γ/o is $(d(\pi\times\sigma))^{-1}$, and we have finally

$$Q_p(\Gamma) := (d(\pi\times\sigma))^{-1} T_{\pi(p)}(\Gamma/o).$$

$Q(\Gamma)$ must be a smooth vector bundle whose basis is the constraint manifold; our construction starts from this bundle.

The crucial observation now is that any subspace $N_p(\Gamma)$ of $T'_p(\Gamma)$ that satisfies (a) $T'_p(\Gamma) = N_p(\Gamma) \oplus T_p(\Gamma)$, (b) $N_p(\Gamma)$ is Ω'-orthogonal to $Q_p(\Gamma)$ and (c) $N_p(\Gamma)$ is Ω'-isotropic, defines a suitable symplectic map φ_{1fp} by the requirement (13): as φ_{1fp} is linear, and because of the condition (a), the knowledge of φ_{1fp} on $T_p(\Gamma)$ [which is well known, see Eq. (12)] and on $N_p(\Gamma)$ determines φ_{1fp} uniquely.

As is already suggested by the notation, $N(\Gamma)$ is to be a smooth vector bundle in order that φ_1 is a differentiable map. A construction of an example of such an $N(\Gamma)$ would show the existence of φ_1.

The vector bundle $Q(\Gamma)$ is a subbundle of $T(\Gamma)$ which, in turn, is a subbundle of $T'(\Gamma)$. As Γ is a submanifold of Γ', there must be a vector bundle $N(\Gamma)$ such that

$$T'(\Gamma) = N(\Gamma) \oplus T(\Gamma), \qquad (14)$$

where $N(\Gamma)$ is a (smooth) vector bundle. If $N(\Gamma)$ is isotropic and orthogonal to $Q(\Gamma)$, then it is the desired bundle. If $N_p(\Gamma)$ is not orthogonal to $Q_p(\Gamma)$, we can find a continuous linear map $\psi_Q : N_p(\Gamma) \mapsto T_p(\Gamma')$ such that $\psi_Q(N_p(\Gamma))$ is orthogonal to $Q_p(\Gamma)$ and $T_p(\Gamma') = \psi_Q(N_p(\Gamma)) \oplus T_p(\Gamma)$ as follows.

Recall that $L_p(\Gamma)$ is the characteristic subspace, Eq. (9), and that

$$T_p(\Gamma) = L_p(\Gamma) \oplus Q_p(\Gamma) \qquad (15)$$

for all $p \in \Gamma$, because π and σ are transversal to each other. It follows that Ω' must be non-degenerate on $Q_p(\Gamma)$. Indeed, if $v \in Q_p(\Gamma)$ were Ω'-orthogonal to all vectors of $Q_p(\Gamma)$, then it would also be Ω'-orthogonal to all of $T_p(\Gamma)$ and so it would belong to $L_p(\Gamma)$. Then, however, $v=0$ because of Eq. (15).

If Ω' is non-degenerate on $Q_p(\Gamma)$, then there is a unique vector $a \in Q_p(\Gamma)$ for each linear function α on $Q_p(\Gamma)$ such that $\alpha(v) = \Omega'(a,v)$ for all $v \in Q_p(\Gamma)$. Let $w \in T'_p(\Gamma)$; then $\Omega'(w,\cdot)$ is a linear function on $Q_p(\Gamma)$ and it determines, therefore, a unique element $\omega_Q(w) \in Q_p(\Gamma)$ such that

$$\Omega'(w,v) = \Omega'(\omega_Q(w),v) \qquad (16)$$

for all $v \in Q_p(\Gamma)$; the map $\omega_Q : T'_p(\Gamma) \mapsto Q_p(\Gamma)$ is linear. The desired map ψ_Q is then defined by

$$\psi_Q := \text{id}_{N_p(\Gamma)} - \omega_Q|_{N_p(\Gamma)}. \qquad (17)$$

Orthogonality can be shown as follows. Let $n \in N_p(\Gamma)$, and let us calculate $\Omega'(\psi_Q(n),q)$ for any $q \in Q_p(\Gamma)$:

$$\Omega'(n - \omega_Q(n),q) = \Omega'(n,q) - \Omega'(\omega_Q(n),q) = 0$$

because of Eq. (16). Moreover, any $v \in T'_p(\Gamma)$ can be written as $v = n + t$, where $n \in N_p(\Gamma)$ and $t \in T_p(\Gamma)$; then we have also

$$v = \psi_Q(n) + (t + \omega_Q(n)). \qquad (18)$$

From the definition of ω_Q, it follows that $\omega_Q(n) \in Q_p(\Gamma)$ for all $n \in N_p(\Gamma)$, so $(t + \omega_Q(n)) \in T_p(\Gamma)$ and $\psi_Q(N_p(\Gamma)) \cap T_p(\Gamma) = 0$. Then the decomposition (18) is unique and the property follows. Thus, we have proved Eq. (14) with $N_p(\Gamma)$ being everywhere orthogonal to $Q_p(\Gamma)$.

If $N_p(\Gamma)$ is orthogonal but not isotropic, we can improve it further as follows. Consider the space $Q_p^{\perp}(\Gamma)$ defined by $Q_p^{\perp}(\Gamma) := \text{orth}_{\Omega'} Q_p(\Gamma)$. Here, we denote the space of all vectors of $T'_p(\Gamma)$ that are Ω'-orthogonal to $Q_p(\Gamma)$ by $\text{orth}_{\Omega'} Q_p(\Gamma)$. As $\Omega'|_{Q_p(\Gamma)}$ is not degenerate, we must have

$$T'_p(\Gamma) = Q_p(\Gamma) \oplus Q_p^{\perp}(\Gamma). \qquad (19)$$

Indeed, $Q_p^{\perp}(\Gamma)$ can be constructed from any linear complement $V_p(\Gamma)$ of $Q_p(\Gamma)$ in $T'_p(\Gamma)$ by $Q_p^{\perp}(\Gamma) = \psi_Q(V_p(\Gamma))$, where ψ_Q is defined by Eq. (17) and the proof is analogous to that for $\psi_Q(N_p(\Gamma))$.

The spaces $N_p(\Gamma)$ and $L_p(\Gamma)$ are subspaces of $Q_p^{\perp}(\Gamma)$. They are disjoint, $N_p(\Gamma) \cap L_p(\Gamma) = \{0\}$, because $N_p(\Gamma) \cap T_p(\Gamma) = \{0\}$ and $L_p(\Gamma) \subset T_p(\Gamma)$. Moreover, if $v \in Q_p^{\perp}(\Gamma)$, then $v = n + t$, where $n \in N_p(\Gamma)$ and $t \in T_p(\Gamma)$ and also $t = x + q$, where $x \in L_p(\Gamma)$ and $q \in Q_p(\Gamma)$ because of Eq. (15). Thus, we obtain that $v = n + x + q$. Now, $v \in Q_p^{\perp}(\Gamma)$, $n \in Q_p^{\perp}(\Gamma)$, and $x \in Q_p^{\perp}(\Gamma)$, hence also $q \in Q_p^{\perp}(\Gamma)$, and as $Q_p^{\perp}(\Gamma) \cap Q_p(\Gamma) = \{0\}$, we must have $q = 0$, so $v = n + x$. We have thus shown that

$$Q_p^{\perp}(\Gamma) = N_p(\Gamma) \oplus L_p(\Gamma). \qquad (20)$$

From the definition (9) of $L_p(\Gamma)$ it follows that each vector of $Q_p^{\perp}(\Gamma)$ that is Ω'-orthogonal to all of $L_p(\Gamma)$ must lie in $L_p(\Gamma)$. For such a vector is Ω'-orthogonal to both $L_p(\Gamma)$ and $Q_p(\Gamma)$, and so to all of $T_p(\Gamma)$ because of Eq. (15).

Now suppose that $\alpha : N_p(\Gamma) \mapsto \mathbf{R}$ is any linear function on $N_p(\Gamma)$. We can extend such a function to $\bar{\alpha} : Q_p^{\perp}(\Gamma) \mapsto \mathbf{R}$ by requiring

$$\bar{\alpha}|_{L_p(\Gamma)} = 0. \qquad (21)$$

As Ω' is non-degenerate on $Q_p^{\perp}(\Gamma)$, there is a unique vector $b \in Q_p^{\perp}(\Gamma)$ such that $\bar{\alpha}(u) = \Omega'(b,u)$ for all $u \in Q_p^{\perp}(\Gamma)$. However, such a vector b must then lie in $L_p(\Gamma)$ because of

Eq. (21). Hence: for any linear function α on $N_p(\Gamma)$, there is $l \in L_p(\Gamma)$ such that $\alpha(u) = \Omega'(l,u)$ for all $u \in Q_p^\perp(\Gamma)$.

Let $n \in N_p(\Gamma)$; then $\Omega'(v,n)$ can be considered as a linear function on $N_p(\Gamma)$ for any $v \in T_p'(\Gamma)$. There is, therefore, a unique element $\omega_L(v)$ of $L_p(\Gamma)$ such that

$$\Omega'(v,n) = \Omega'(\omega_L(v),n). \tag{22}$$

From its construction, it follows that $\omega_L : T_p'(\Gamma) \mapsto L_p(\Gamma)$ is a linear map.

Consider the linear map $\psi_L : N_p(\Gamma) \mapsto T_p'(\Gamma)$ defined by

$$\psi_L(n) := n - (1/2)\omega_L(n)$$

for all $n \in N_p(\Gamma)$. ψ_L is an injection, because the equation $n = (1/2)\omega_L(n)$ can have only zero solutions. Indeed, the left-hand side is an element of $N_p(\Gamma)$ and the right-hand side is an element of $L_p(\Gamma)$. Moreover, the image, $\psi_L(N_p(\Gamma))$, is isotropic; this can be seen as follows. Using the definition of ψ_L we obtain

$$\Omega'(\psi_L(n_1),\psi_L(n_2)) = \Omega'(n_1,n_2) - (1/2)\Omega'(\omega_L(n_1),n_2)$$
$$+ (1/2)\Omega'(\omega_L(n_2),n_1)$$
$$+ (1/4)\Omega'(\omega_L(n_1),\omega_L(n_2)).$$

The last term is zero, because $\omega_L(n_{1,2}) \in L_p(\Gamma)$ and we finally have from Eq. (22)

$$\Omega'(\psi_L(n_1),\psi_L(n_2)) = \Omega'(n_1,n_2) - (1/2)\Omega'(n_1,n_2)$$
$$+ (1/2)\Omega'(n_2,n_1) = 0.$$

The last property we need is that any vector $q^\perp \in Q_p^\perp(\Gamma)$ can be written as a sum, $q^\perp = z + y$, where $z \in \psi_L(N_p(\Gamma))$ and $y \in L_p(\Gamma)$. However, it holds that $q^\perp = n + x$, where $n \in N_p(\Gamma)$ and $x \in L_p(\Gamma)$. Then $q^\perp = (n - (1/2)\omega_L(n)) + (x + (1/2)\omega_L(n))$ so $z = \psi_L(n)$ and $y = x + \omega_L(n)$ is the desired decomposition.

The restriction of φ_{1fp} to $N_p(\Gamma)$ is uniquely determined by the condition that $\varphi_{1fp} : T_p'(\Gamma) \mapsto T_{\varphi_{1b}(p)}(K')$ is symplectic as follows. Let $n \in N_p(\Gamma)$, then there is a unique linear function $\Omega'(n,\cdot)$ on $L_p(\Gamma)$ defined by n. $L_p(\Gamma)$ is mapped by $\varphi_{1fp}|_{L_p(\Gamma)} = d\sigma$ onto $T_{\sigma(p)}(\text{Emb}(\Sigma,\Sigma \times \mathbf{R}))$, so $\Omega'(n, d\sigma^{-1}\cdot)$ is a linear function on $T_{\sigma(p)}(\text{Emb}(\Sigma, \Sigma \times \mathbf{R}))$. Every linear function α on $T_{\sigma(p)}(\text{Emb}(\Sigma, \Sigma \times \mathbf{R}))$ determines, in turn, a unique element v of $T_{\sigma(p)}(T_{\sigma(p)}^*(\text{Emb}(\Sigma,\Sigma \times \mathbf{R}))) = \varphi_{1fp} N_p(\Gamma)$ such that $\Omega'(v,\cdot) = \alpha$. We must set

$$\varphi_{1fp}(n) := v,$$

or else φ_{1fp} is not symplectic. Hence, the choice of the subspace $N_p(\Gamma)$ determines φ_{1fp}.

The above construction of $N_p(\Gamma)$ that satisfies the requirements (a), (b) and (c) is based on the smooth vector bundles $Q(\Gamma)$ and $L(\Gamma)$ and on the differentiable symplectic form Ω'; the result is, therefore, a smooth vector bundle $N(\Gamma)$.

In the case that the space $T_p'(\Gamma)$ is an infinite-dimensional Banach space, two aspects of our construction become problematical.

(1) If a Banach space B_1 is, as a linear space, a direct sum of two other linear (Banach) spaces, $B_1 = B_2 \oplus B_3$, then the corresponding map between B_1 and $B_2 \times B_3$ need not be a *topological* isomorphism. If it is, one says that B_2 *splits* [14] B_1. Some splittings follow from the assumptions in Secs. III E and III G. For example, Eq. (14) follows from Γ being a submanifold of Γ', and Eq. (15) follows from π being a submersion [14]. We however also need Eqs. (19) and (20).

(2) The norm defining $T_p'(\Gamma)$ can restrict the elements of $T_p'(\Gamma)$ so much that the space of all continuous linear functionals on $T_p'(\Gamma)$—the dual space $T_p'^*(\Gamma)$—contains also functionals that are not of the form $\Omega'(u,\cdot)$ for $u \in T_p'(\Gamma)$, even if $\Omega'(u,\cdot)$ is a non-zero functional for all non-zero u's. Such a Ω' is called a *weakly non-degenerate* or *weak* symplectic form [34]. If Ω' is weak, the definition of the maps ω_Q and ω_L given above does not work.

The way one can cope with these two problems depends on the topology of $T_p'(\Gamma)$. This, however, must be judiciously adapted to the nature of each particular model and there does not seem to be any general method. Still, the following scheme has worked for all examples we have considered as yet. First, one defines certain dense subspaces, $T_{p0}'(\Gamma)$ and $T_{p0}'^*(\Gamma)$, of the Banach spaces $T_p'(\Gamma)$ and $T'^*_p(\Gamma)$; one can take, for example, the spaces of functions all of whose derivatives are smooth and which have compact support. One has to prove that $Q_p(\Gamma) \cap T_{p0}'(\Gamma)$ and $L_p(\Gamma) \cap T_{p0}'(\Gamma)$ are also dense in $Q_p(\Gamma)$ and $L_p(\Gamma)$, and that all functionals from $T'^*_{p0}(\Gamma)$ have the form $\Omega'(u,\cdot)$ where $u \in T_{p0}'(\Gamma)$. Then the construction seems to work for the corresponding dense subspaces such as $Q_{p0}^\perp(\Gamma)$ or $N_{p0}(\Gamma)$—the topology is nowhere needed. Second, one has to show that the projectors onto the subspaces are continuous with respect to the topology. Then the Banach spaces are easily shown to split and everything works.

B. The pull-back of $\Omega_{K'}$ to Γ'

The second step consists of two consecutive pull-backs that bring the form $\Omega_{K'}$ to Γ'.

The restriction to the vector bundle $N(\Gamma)$ of the map φ_1 constructed in the previous subsection maps $N(\Gamma)$ to the vector bundle with the base space $\varphi_{1b}\Gamma \subset K$, and with the fibers $T_{\sigma(p)}(T_{\sigma(p)}^*(\text{Emb}(\Sigma,\Sigma \times \mathbf{R})))$,

$$\varphi_{1fp}N_p(\Gamma) = T_{\sigma(p)}(T_{\sigma(p)}^*(\text{Emb}(\Sigma,\Sigma \times \mathbf{R}))).$$

The cotangent space $T_{\sigma(p)}^*(\text{Emb}(\Sigma,\Sigma \times \mathbf{R}))$ is a linear space, so it can be identified with its tangent space at its zero vector. Hence, with this identification, $\varphi_1|_{N(\Gamma)}$ can be considered as a bundle morphism mapping $N(\Gamma)$ onto the bundle with the basis $\varphi_{1b}\Gamma \subset K$, and the fibers $T_{\sigma(p)}^*(\text{Emb}(\Sigma,\Sigma \times \mathbf{R}))$. However, this vector bundle is nothing but a subbundle of $K' = (\Gamma/o) \times T^*(\text{Emb}(\Sigma,\Sigma \times \mathbf{R}))$. In this way, using the map φ_1, we have constructed a bundle morphism between $N(\Gamma)$ and K'.

Let us denote this morphism by $\varphi_1':N(\Gamma)\mapsto K'$ and the pull-back of $\Omega_{K'}$ by φ_1' to $N(\Gamma)$ by Ω_1, $\Omega_1:=\varphi_1'^*\Omega_{K'}$. Ω_1 is a symplectic form on the manifold $N(\Gamma)$; that is, Ω_1 is a bilinear form in the tangent space $T_P(N(\Gamma))$ at each point $P\in N(\Gamma)$. Let $P=p\in\Gamma$, then $T_p(N(\Gamma))$ can be decomposed as follows:

$$T_p(N(\Gamma))=T_p(\Gamma)\oplus T_p(N_p(\Gamma)).$$

Again, $T_p(N_p(\Gamma))$ can be identified with $N_p(\Gamma)$, so by Eq. (14) $T_p(N(\Gamma))=T_p'(\Gamma)$. From the construction of the map φ_1', it follows that Ω' coincides with Ω_1 at $T_p'(\Gamma)$ for each $p\in\Gamma$, so we can write

$$\Omega_1|_\Gamma=\Omega'|_\Gamma. \tag{23}$$

Observe that Ω_1 is a kind of ''constant'' extension of $\Omega'|_\Gamma$ to the whole bundle $N(\Gamma)$.

The construction of our second pull-back uses the theorem about the existence of tubular neighborhoods [13,14]. A tubular neighborhood is a generalization of the well-known notion of normal coordinate ball. In the case we consider, the theorem states that there is a diffeomorphism $\varphi_2:U_1\mapsto U_2$, where U_1 is a neighborhood of the zero section Γ in $N(\Gamma)$ and U_2 a neighborhood of Γ in Γ' such that $d\varphi_2|_\Gamma=$ id.

It follows that the pull-back $\Omega_2:=\varphi_2^{-1*}\Omega_1$ is a symplectic form on U_2. As $d\varphi_2|_\Gamma$ is an identity, we have

$$\Omega_2|_\Gamma=\Omega_1|_\Gamma. \tag{24}$$

Most constructions of this subsection work for infinite dimensions, if we just replace the word ''non-degenerate'' by ''weakly non-degenerate'' everywhere—the difference is not important here. All necessary splittings can easily be shown. The construction of the tubular neighborhood is mostly straightforward, too. However, if a complicated set has been deleted from Γ_1', then one has to use a smooth partition of unity [14] and it need not be trivial to show its existence. The proof will depend on the properties of the particular model.

C. Application of the Darboux-Weinstein theorem

This step is an application of the Darboux-Weinstein theorem [10,13]. This is a generalization of the well-known Darboux theorem saying roughly that if two symplectic forms Ω_a and Ω_b on a manifold $\mathcal{M}$ coincide at a submanifold $\mathcal{N}\subset\mathcal{M}$, then there is a diffeomorphism $\lambda:\mathcal{M}\mapsto\mathcal{M}$ that, together with its derivative $d\lambda$, is trivial at $\mathcal{N}$, and that $\lambda^*\Omega_2=\Omega_1$ in a neighborhood of $\mathcal{N}$ in $\mathcal{M}$.

Consider the two forms Ω' and Ω_2 in U_2. Equations (24) and (23) imply that $\Omega'|_\Gamma=\Omega_2|_\Gamma$, so the conditions of the Darboux-Weinstein theorem are satisfied. There is, therefore, a diffeomorphism $\varphi_3:U'\mapsto U_3$ of a neighborhood U' of Γ in Γ' with $U_3\subset U_2$ such that $\varphi_3^*\Omega_2=\Omega'$.

Let us finally define the map φ by

$$\varphi:=\varphi_1'\circ\varphi_2^{-1}\circ\varphi_3;$$

it maps the neighborhood U' of Γ in Γ' onto the neighborhood $\varphi U'$ of K in K'. From the constructions above, it follows immediately that $\varphi^*\Omega_{K'}=\Omega'$. The maps φ_2 and φ_3 are identities if restricted to Γ and the restriction of φ_1' to Γ is $\pi\times\sigma$. Thus, $\varphi|_\Gamma=\pi\times\sigma$ and the existence of the extension is shown.

The constructions of this subsection need considerable modification in the case of infinite dimensions. Indeed, the Darboux-Weinstein theorem does not hold for general weak symplectic forms [35]. Marsden has, however, proved an analogous theorem [15] for weak symplectic forms if certain additional conditions are imposed, and the conditions are chosen in such a way that most models met in practice satisfy them. Thus, the modification consists of a proof that the particular model under study satisfies the assumptions of Marsden theorem.

The extension constructed in this section is not unique. Already the step (1) was not unique, because the subspace $N_p(\Gamma)$ is determined only up to a symmetric linear map [13] on $L_p(\Gamma)$. The tubular neighborhood of the step (2) is also quite arbitrary. Finally, the Darboux-Weinstein theorem guarantees just the existence of φ_3, but it says nothing about its uniqueness.

VI. CONCLUSIONS

We have defined a covariant gauge fixing as pointwise identification of different solution spacetimes with each other so that a fixed background manifold has resulted and the dynamics has been reduced to a field dynamics on it. The fixing has first been defined on the constraint manifold of the system; there are very many ways to choose it at least locally; different gauge fixings are related by elements of the huge Bergmann-Komar group.

We have found a connection between covariant gauge fixings and Kuchař Γ-decompositions of the constraint manifold: for any fixing, there is exactly one decomposition. The decomposition itself amounts to a particular choice of (local) foliation of the constraint manifold that is transversal to the c-orbits.

Finally, we have shown that any Kuchař decomposition of the constraint surface can be extended to a whole neighborhood of the constraint surface. This extension is not unique. In this way, the full Kuchař decomposition is doubly non-unique: there are as many Γ-decompositions as covariant gauge fixings, and each Γ-decomposition has many extensions. However, the form of kinematic term of the Kuchař action (8) is always the same, the only interesting and nontrivial part being the algebra of the observables, if we allow for more general [7] algebra than the Heisenberg algebra of q_0^α and $p_{\alpha 0}$ in Eq. (8). The usefulness of the decomposition is based on the enormous simplification it brings about in the description of generally covariant systems.

We would like to make two additional remarks. First, the structure of the weak symplectic manifold (K',Ω') is typical for the so-called *already parametrized theories* such as a parametrized scalar field in flat spacetime (see, e.g. Kuchař [36]). Our construction shows that the generally covariant models are, in general, *not* already parametrized theories for

two quite different reasons. (1) We can prove that only always a part of the symplectic manifold of the system has the structure (K', Ω'), namely just a sector corresponding to a fixed Cauchy surface. Moreover, we have to exclude points in the constraint surface that correspond to dynamical trajectories admitting any symmetry. In fact, Torre [37] has shown that general relativity cannot be considered as already parametrized theory the obstruction coming from points at the constraint surface Γ that represent Cauchy data for spacetimes with Killing vectors; these points are also excised in our paper. (2) For each subsystem that *is* equivalent to an already parametrized system, such an equivalence is not *unique*. There is one Kuchař Γ-decomposition $\pi \times \sigma$ for each covariant gauge fixing, and there are many different, gauge dependent, background manifolds. This is in stark contrast to the structure of an already parametrized system such as in Kuchař [36], where there is a unique background manifold. The points of this manifold are defined by the fixed background metric—the Minkowski metric on it. The constraint manifold of a generally covariant model is just a bundle with *many* different trivializations, unlike that of an already parametrized model, which is a *unique* Cartesian product.

Second, we observe that our construction is closely related to the problem of the so-called abelianization of constraints [38]. Indeed, the new constraints given by the theorem can be taken as components $\mathcal{H}_\mu(x)$ of the cotangent vectors in $T^*(\text{Emb}(\Sigma, \Sigma \times \mathbf{R}))$ with respect to some coordinates on Σ and on $\mathcal{M} = \Sigma \times \mathbf{R}$. All these ''functions'' have vanishing Poisson brackets with each other. Of course, a complete system of Abelian constraint functions still need not exist, because there need not be global coordinates on Σ and $\mathcal{M}$, and the points with symmetries are also excluded.

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