Late time decay of scalar, electromagnetic, and gravitational perturbations outside rotating black holes

Leor Barack
Department of Physics, Technion–Israel Institute of Technology, Haifa, 32000, Israel
(Received 26 July 1999; published 27 December 1999)

We study analytically, via the Newman-Penrose formalism, the late time decay of scalar, electromagnetic, and gravitational perturbations outside a realistic rotating (Kerr) black hole. We find a power-law decay at timelike infinity, as well as at null infinity and along the event horizon (EH). For generic initial data we derive the power-law indices for all radiating modes of the various fields. We also give an exact analytic expression (accurate to leading order in $1/t$) for the r dependence of the late time tail at any r. Some of our main conclusions are the following. (i) For generic initial data, the late time behavior of the fields is dominated by the mode $l=|s|$ (with s being the spin parameter), which dies off at fixed r as $t^{-2|s|-3}$ — as in the Schwarzschild background. (ii) However, other modes admit decay rates slower than in the Schwarzschild case. (iii) For $s>0$ fields, non-axially symmetric modes dominate the late time behavior along the EH. These modes oscillate along the null generators of the EH.

PACS number(s): 04.70.Bw, 04.25.Nx

I. INTRODUCTION

Small perturbations of the Schwarzschild black hole (SBH) geometry die off at late time with an inverse power-law tail. This well-known phenomenon was discovered by Price early in the 1970s. Price explored the dynamics of linear scalar and metric perturbations [1] (and that of all integer-spin fields in the Newman-Penrose formalism [2]) propagating on the SBH background. His analysis provided a detailed description of the relaxation mechanism through which the black hole (BH) exterior settles down at late time into its stationary ''no hair'' state. In particular, Price was able to characterize the actual form of the late time falloff of the perturbations: He found that any radiative multipole mode l,m of an initially compact linear perturbation dies off at late time as t^{-2l-3} (where t is the Schwarzschild time coordinate). In the case there exists an initially static multipole mode l,m it will decay as t^{-2l-2}. These power-law decay tails were found to be the same for all kinds of perturbations, whether scalar, electromagnetic or gravitational (and in this respect, the scalar field model proved to be a useful toy model for more realistic fields).

Price's results were later reproduced using several different approaches, both analytical and numerical [3–8], and were generalized to other spherically symmetric BH spacetimes [4,9–14]. (A brief review of the works on this subject can be found in the Introduction of Ref. [7].) The validity of the perturbative (linear) approach was supported by numerical analyses of the fully nonlinear dynamics [15,11], indicating virtually the same power-law indices for the late time decay.

For a scalar field on the background of a SBH, power-law decay tails were found to be exhibited also at future null infinity [3,4,8] and along the future event horizon [4,8]. It was shown that at null infinity the scalar field dies off with respect to retarded time u as u^{-l-2} (for a compact initial mode) or as u^{-l-1} (for a static initial mode). The decay of the scalar perturbation along the event horizon (EH) was found to be v^{-2l-3} or v^{-2l-2} (for a compact or a static initial mode, respectively), where v is the (Eddington-Finkelstein) advanced-time coordinate.

As was already explained by Price, the late time tails of decay outside spherically symmetric BHs originate from backscattering of the outgoing radiation off spacetime curvature at very large distances. In the framework of a frequency-domain perturbation analysis [3,6,10], these tails are explained in terms of a branch cut in the (frequency domain) Green's function, which is, again, associated with the form of the curvature-induced potential at large distance. This suggests that the form of the decay at late time reflects (and is affected by) merely the large distance structure of spacetime, and may be independent of the existence or absence of an event horizon. This assertion found further support in the analysis by Gundlach *et al.* [15], who studied the purely spherical collapse of a self-gravitating minimally coupled scalar field. It was demonstrated numerically that in this case late time tails form even when the collapse fails to create a black hole. (On the other hand, quasi-normal ringing are found to dominate the early stage of the waves' evolution only if a BH forms.)

Until quite recently, the issue of the late time decay of BH perturbations has been considered only in spherically symmetric models of BHs. It is known, however, that astrophysically realistic BHs are spinning [16], and thus are not spherically symmetric but are rather of the axially symmetric Kerr type. Moreover, it is suggested, in virtue of the ''no hair'' principle [17], that the Kerr black hole (KBH) might be the *only* realistic BH (realistic BHs are not expected to carry a significant amount of net electric charge). Hence, generalization of the above-mentioned analyses to the KBH case seems to be of an obvious importance. Still, such a generalization has awaited almost three decades, as the lack of spherical symmetry in the Kerr background makes both analytical and numerical exploration significantly more complicated.

The ''no hair'' principle for BHs implies that perturbations of the KBH must ''radiate away'' at late time. No further information is available from this general principle as to the actual details of this decay process. A question arises as to what effect rotation has on the form of the late time tails. More basically, does the decay of the perturbing fields still obey a power law? If so, are the power indices the same as in the SBH case?

Such questions were addressed only recently, by several authors. First, Krivan *et al.* carried out a numerical simulation of the evolution of linear scalar [18] and gravitational [19] waves on the background of a Kerr black hole. The first analytic treatment of this problem, for a scalar field, was later presented by Barack and Ori [20,21] (following a preliminary analysis by Ori [22]). Most recently, Hod used a different approach to study the late time decay of both a scalar field [23] and nonzero-spin Newman-Penrose fields [24,25] on the Kerr background. (Hod's analysis, which follows some preliminary considerations by Andersson [6], is carried out in the frequency domain, whereas Barack and Ori use a time domain analysis.) Finally, the analytic progress has motivated a further numerical study by Krivan [26,27].

The above analyses all indicate that power-law tails of decay are exhibited in the Kerr background as well. In the Kerr case, however, the lack of spherical symmetry *couples* between various multipole modes, which results in the power-law indices of specific modes being found to be different, in general, from the ones obtained in spherically symmetric BHs. Another interesting phenomenon caused by rotation (first observed in [22]) is the oscillatory nature of the late time tails along the null generators of the EH of the Kerr BH for nonaxially symmetric perturbation modes.

The purpose of the present paper is to extend the analysis described in Ref. [21] to electromagnetic and gravitational perturbations of the Kerr background, and supply the full technical details of our approach. (In Ref. [21] we merely outlined the application of our technical scheme to a scalar field, and gave a brief description of the results in this case.) The analysis to be described in this paper provides a more complete and accurate picture of the late time decay of physical fields, than already available. Among the results which appear here for the first time:

(i) We derive the form of the late time tail for all radiative modes *anywhere* outside the KBH (i.e. at all distances). We also give an exact analytic expression for the radial dependence of this tail. (In [23–25] Hod only analyzes the asymptotic behavior at very large distance, and along the EH.)

(ii) A careful analysis of the decay along the EH reveals an interesting phenomenon: For $s>0$ fields, it is the oscillatory nonaxially symmetric ($m>0$) modes which dominate the late time behavior there. This result has important implications to the structure of the singularity at the inner horizon of the KBH [29]. In a different paper [28] Barack and Ori further explore and explain this phenomenon, and discuss the reason for the incorrect prediction made in Ref. [24] for the decay rate along the EH in the $s>0$ case. (Recently [25], following the appearance of Ref. [28], Hod has corrected his result.)

For simplicity, we shall refer in this paper only to the (most realistic) situation in which the initial perturbation has a rather generic angular distribution, such that it is composed of all multipole modes and, in particular, the lowest radiatable one. It will be explained why an initial setup in which the lowest modes are missing is more complicated to explore using our approach. Hod's analysis [23–25] provides predictions for this case as well; however, these are not in agreement with recent numerical results by Krivan [26]. It thus seems that further analytic work is needed (using either Hod's method or ours) to clarify this point.

The arrangement of this paper is as follows. In Sec. II we briefly review the subject of perturbations of the Kerr geometry via the Newman-Penrose formalism. We then reduce the master perturbation equation in the time domain to obtain a coupled set of time-radial equations for the various modes. The evolution problem for each of the modes is mathematically formulated as a characteristic initial value problem. In Sec. III we analyze the late time behavior of the various modes at null infinity. To that end we apply the *iterative scheme*, which is, basically, an extension of a technique previously tested for a scalar field in the SBH background [7,8]. In Sec. IV we introduce the *late time expansion* (LTE) scheme, which allows a global treatment of the decay at late time. The late time behavior of fields at a fixed distance outside the EH is dominated by the leading term of the late time expansion, for which we derive an analytic expression in Sec. V. In Sec. VI we then carefully explore the behavior along the EH itself. Finally, in Sec. VII, we use the LTE scheme combined with the results at null infinity, in order to derive the late time decay rates for all modes of the fields at any fixed distance. We conclude (in Sec. VIII) by summarizing our results and discussing their relation to other works.

II. MODE-COUPLED FIELD EQUATION

A. Perturbations of the Kerr geometry via the Newman-Penrose formalism (definitions and a brief review)

The line element in Kerr spacetime reads, in Boyer-Lindquist (BL) coordinates t,r,θ,φ,

$$
\begin{aligned}
ds^2 = & -(1-2Mr/\Sigma)dt^2 + (\Sigma/\Delta)dr^2 + \Sigma d\theta^2 \\
& + (r^2+a^2+2a^2Mr\sin^2\theta/\Sigma)\sin^2\theta d\varphi^2 \\
& - (4aMr\sin^2\theta/\Sigma)d\varphi dt, \qquad (1)
\end{aligned}
$$

where M and a are, correspondingly, the BH's mass and specific angular momentum, $\Sigma \equiv r^2+a^2\cos^2\theta$, and

$$\Delta \equiv r^2 - 2Mr + a^2. \qquad (2)$$

(Throughout this paper we use relativistic units, with $c=G=1$.) We shall consider in this paper only a BH solution with $|a|<M$: the extremal case, $|a|=M$, requires a separate treatment, as we later briefly explain. The event and inner horizons of the (non-extremal) KBH are the two null 3-surfaces $r=r_+$ and $r=r_-$, respectively, where

$$r_{\pm} = M \pm \sqrt{M^2-a^2} \qquad (3)$$

are the two roots of the ''horizons function'' $\Delta(r)$ defined in Eq. (2).

To discuss perturbations of the Kerr spacetime via the Newman-Penrose (NP) formalism [30], introduce Kinnersley's null tetrad basis [31] $[l^\mu, n^\mu, m^\mu, m^{*\mu}]$ (where an asterisk denotes complex conjugation). In BL coordinates, the "legs" of this tetrad are given by

$$l^\mu = \Delta^{-1}[r^2+a^2, \Delta, 0, a]$$

$$n^\mu = (2\Sigma)^{-1}[r^2+a^2, -\Delta, 0, a]$$

$$m^\mu = (2^{1/2}\bar{\rho})^{-1}[ia\sin\theta, 0, 1, i/\sin\theta] \tag{4}$$

(with the fourth tetrad vector obtained from m^μ by complex conjugation), where $\bar{\rho} \equiv r + ia\cos\theta$. In the framework of the NP formalism, the gravitational field in vacuum is completely described by five complex scalars $\Psi_0, \ldots, \Psi_4$, constructed from the Weyl tensor $C_{\alpha\beta\gamma\delta}$ by projecting it on the above tetrad basis. Likewise, the electromagnetic field is completely characterized by the three complex scalars $\varphi_0, \varphi_1, \varphi_2$, constructed by similarly projecting the Maxwell tensor $F_{\mu\nu}$. In particular,

$$\Psi_0 = -C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta \quad \text{and}$$
$$\Psi_4 = -C_{\alpha\beta\gamma\delta} n^\alpha m^{*\beta} n^\gamma m^{*\delta} \tag{5}$$

represent, respectively, the ingoing and outgoing radiative parts of the Weyl tensor, and

$$\varphi_0 = F_{\mu\nu} l^\mu m^\nu \quad \text{and} \quad \varphi_2 = F_{\mu\nu} m^{*\mu} n^\nu \tag{6}$$

represent the ingoing and outgoing radiative parts of the electromagnetic field.

In the (unperturbed) Kerr background all Weyl scalars but Ψ_2 vanish (as directly implied by the Goldberg-Sachs theorem in view of Kerr spacetime being of Petrov type D; see Secs. 9b, 9c in [32]). In the framework of a linear perturbation analysis, the symbols $\Psi_0, \Psi_1, \delta\Psi_2, \Psi_3, \Psi_4$ and $\varphi_0, \delta\varphi_1, \varphi_2$ are thus used to represent first-order perturbations of the corresponding fields (with $\delta\Psi_2 \equiv \Psi_2 - \Psi_2^{\text{background}}$, etc.). One can show (see Sec. 29b in [32]) that Ψ_0 and Ψ_4, and also φ_0 and φ_2, are *invariant* under gauge transformations (namely, under infinitesimal rotations of the null basis and infinitesimal coordinate transformations). The scalars Ψ_1 and Ψ_3 are not gauge invariant, and may be nullified by a suitable rotation of the null frame. The entities $\delta\Psi_2$ and $\delta\varphi_1$ represent perturbations of the "Coulomb-like," non-radiative, part of the fields (in fact, one can also nullify $\delta\Psi_2$ by a suitable infinitesimal coordinate transformation). It is therefore only the scalars defined in Eqs. (5) and (6) which carry significant information about the radiative part of the fields. (Note, however, that gauge invariance of the radiative fields is guaranteed only within the framework of linear perturbation theory.)

Teukolsky [33] first obtained a single master perturbation equation governing linear perturbations of scalar, electromagnetic, and gravitational fields. In vacuum, this master perturbation equation reads

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\Psi^s_{,tt} - \Delta^{-s}(\Delta^{s+1}\Psi^s_{,r})_{,r} + \frac{4Mar}{\Delta}\Psi^s_{,t\varphi}$$
$$+\left(\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right)\Psi^s_{,\varphi\varphi} - \frac{1}{\sin\theta}(\Psi^s_{,\theta}\sin\theta)_{,\theta}$$
$$-2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\Psi^s_{,\varphi} - 2s\left[\frac{M(r^2-a^2)}{\Delta} - r\right.$$
$$\left. - ia\cos\theta\right]\Psi^s_{,t} + (s^2\cot^2\theta - s)\Psi^s = 0. \tag{7}$$

Here, $\Psi^s(t,r,\theta,\varphi)$ represents the various radiative fields according to the following list:

$$\Phi = \Psi^{s=0} \quad (\text{scalar field}),$$

$$\varphi_0 = \Psi^{s=+1},$$

$$\varphi_2 = (\bar{\rho}^*)^{-2}\Psi^{s=-1},$$

$$\Psi_0 = \Psi^{s=+2},$$

$$\Psi_4 = (\bar{\rho}^*)^{-4}\Psi^{s=-2}. \tag{8}$$

The master equation (7) is fully separable only in the frequency domain, by means of the *spin-weighted spheroidal harmonic* functions $S^{slm}(-a^2\omega^2, \cos\theta)$ [33], where ω is the temporal frequency (the separated equations are referred to as "Teukolsky's equation"). Because the functions $S^{slm}(\theta)$ are ω dependent, separation of the θ dependence is not possible in the time domain (namely, without first decomposing the field into its Fourier components).

B. Reduction of the master field equation in the time domain

The target of the present work is to explore the behavior of the fields Ψ^s at late time. In principle, the analysis can be carried out in the frequency domain, as in Refs. [23–25]. In this technique, the requested temporal behavior is finally to be extracted by an inverse Fourier transform. Our analysis is based on a different approach, motivated by the following argument: In the late time, stationary, limit $(t\to\infty)$, one expects the very low frequency $(\omega\to 0)$ Fourier modes to dominate the behavior. For such waves, the functions $e^{im\varphi}S^{slm}(\theta)$ reduce to the *spin-weighted spherical harmonic* functions $Y^{slm}(\theta,\varphi)$ [34]. This may motivate one to try and extract the angular dependence of the fields Ψ^s by using the functions Y^{lms}. As a result of the lack of spherical symmetry, the resulting (time-domain) field equations will possess coupling between the various multipole modes l; however, one should expect this coupling to be "small," in a sense, at late time. In the sequel we show how this coupling can be treated in an iterative manner, in both the frameworks of the *iterative expansion* (Sec. III) and the *late time expansion* (Sec. IV).

Led by the above argument, we expand the fields Ψ^s as

$$\Psi^s(t,r,\theta,\varphi)=(r^2+a^2)^{-1/2}\Delta^{-s/2}\times\sum_{l=|s|}^{\infty}\sum_{m=-l}^{l}Y^{slm}(\theta,\varphi)\psi^{slm}(t,r), \quad (9)$$

where the radial factor in front of the summation symbols is introduced for convenience [as it eliminates the term $\propto\psi_{,r_*}$ from Eq. (23) below]. Note that the summation over modes l excludes the $l<|s|$ modes, which are nonradiative (for a discussion regarding the nonradiatable modes, see Sec. III D 4 in Ref. [2]).

Inserting the expansion (9) into the master field equation (7), we obtain

$$\sum_{lm}Y^{slm}(\theta,\varphi)[\tilde{D}(t,r)\psi^{slm}-a^2\sin^2\theta(\psi^{slm})_{,tt}+2ias\cos\theta(\psi^{slm})_{,t}]=0, \quad (10)$$

where $\tilde{D}(t,r)$ is a certain differential operator independent of θ,φ. Note in Eq. (10) how the two last terms in the squared brackets (proportional to a) avoid a full separation of variables.

Now, the product $\cos\theta\cdot Y^{slm}$ can be re-expanded in terms of the functions Y^{slm} (which form a complete set of functions on the unit 2-sphere for each s). The ''matrix elements'' of the function $\cos\theta$ with respect to the Y^{slm} basis are given by [35]

$$\langle sl'm|\cos\theta|slm\rangle\equiv\oint d\Omega(Y^{slm})^*\cos\theta(Y^{slm})=\left(\frac{2l+1}{2l'+1}\right)^{1/2}\langle l1m0|l'm\rangle\times\langle l1-s0|l'-s\rangle, \quad (11)$$

where $\langle j_1j_2m_1m_2|jm\rangle$ are the standard Clebsch-Gordan coefficients [36]. We find that

$$\cos\theta\cdot Y^l=c_-^{l+1}Y^{l+1}+c_0^lY^l+c_+^{l-1}Y^{l-1}, \quad (12)$$

where

$$c_-^l=\left[\frac{(l^2-s^2)(l^2-m^2)}{l^2(2l-1)(2l+1)}\right]^{1/2},\quad c_0^l=-\frac{ms}{l(l+1)},\quad c_+^l=c_-^{l+1}. \quad (13)$$

(Here, as we shall often do below, we omit the indices s,m for the sake of brevity.) This also easily leads to

$$-\sin^2\theta\cdot Y^l=C_{--}^{l+2}Y^{l+2}+C_-^{l+1}Y^{l+1}+C_0^lY^l+C_+^{l-1}Y^{l-1}+C_{++}^{l-2}Y^{l-2}, \quad (14)$$

where

$$C_{++}^l=c_+^{l+1}c_+^l,\quad C_+^l=c_+^l(c_0^{l+1}+c_0^l),\quad C_0^l=(c_-^l)^2+(c_+^l)^2+(c_0^l)^2-1,\quad C_-^l=c_-^l(c_0^l+c_0^{l-1}),\quad C_{--}^l=c_-^{l-1}c_-^l. \quad (15)$$

All constant coefficients c^l and C^l are nonvanishing, with only the following exceptions:

(i) C_-^l and c_-^l vanish for $l=|m|$ or $l=|s|$.

(ii) C_{--}^l vanishes for $|m|\leqslant l\leqslant|m|+1$ or $|s|\leqslant l\leqslant|s|+1$.

(iii) c_0^l, C_+^l, and C_-^l vanish for $m=0$ or $s=0$.

(It can be verified that C_0^l is always negative definite.)

Substituting Eqs. (12) and (14) in Eq. (10) we obtain, by the orthogonality of the functions Y^{slm},

$$\bar{D}(t,r)\psi^l+\mathcal{I}(\psi^{l\pm1},\psi^{l\pm2})=0 \quad (16)$$

for each l,m,s satisfying $l\geqslant|m|$ and $l\geqslant|s|$, where $\bar{D}$ is yet another differential operator, and $\mathcal{I}$ is a functional describing *coupling* between the l mode and the $l\pm1$ and $l\pm2$ modes:

$$\mathcal{I}(\psi^{l\pm1},\psi^{l\pm2})=a^2(C_{++}^l\psi^{l+2}+C_+^l\psi^{l+1}+C_-^l\psi^{l-1}+C_{--}^l\psi^{l-2})_{,tt}+2ias(c_+^l\psi^{l+1}+c_-^l\psi^{l-1})_{,t}. \quad (17)$$

Note that, obviously, modes of different m do not interact with each other, as the Kerr geometry is axially symmetric. We also point out that the scalar field case is special, in that for this case the l mode does not interact with the $l\pm1$ modes, but only with the $l\pm2$ modes (recall that the interaction coefficients $C_\pm^l$ vanish for $s=0$).

To write Eq. (16) explicitly in a convenient form, we introduce the advanced and retarded time coordinates, defined, respectively, by

$$v\equiv t+r_*\quad\text{and}\quad u\equiv t-r_* \quad (18)$$

(which are nonetheless *not* null coordinates in Kerr spacetime). Here, the ''tortoise'' radial coordinate r_* is defined by

$$r_*=r-r_++(2\kappa_+)^{-1}\ln z_+-(2\kappa_-)^{-1}\ln z_-, \quad (19)$$

with $\kappa_\pm$ being the horizons' ''surface gravity'' parameters,

$$\kappa_\pm=\frac{r_+-r_-}{4Mr_\pm}, \quad (20)$$

and where the dimensionless radial variables $z_\pm$ are given by

$$z_\pm\equiv\frac{r-r_\pm}{r_+-r_-}. \quad (21)$$

[Note the relation $\Delta=z_+z_-(r_+-r_-)^2$. Also, recall that we are dealing in this paper only with non-extremal black holes, for which $r_+>r_-$ and, consequently, $z_\pm$ are well defined.]

The coordinate r_*, satisfying $dr_*/dr=(a^2+r^2)/\Delta$, increases monotonically with r from $-\infty$ (at the EH) to $+\infty$ (at spacelike infinity). Later we shall find useful the asymptotic relations

$$e^{2\kappa_+ r_*}\simeq\frac{\Delta}{(r_+-r_-)^2}\simeq z_+ \quad (\text{for } r_*\ll -M, r\to r_+),$$
$$r_*\simeq r \quad (\text{for } r_*\gg M, r\to\infty). \tag{22}$$

The explicit form of the mode-coupled field equation (16) is now[1]

$$\psi^l_{,uv}+V^l(r)\psi^l+R^l(r)\psi^l_{,t}+K(r)[a^2C^l_0\psi^l_{,tt}+\mathcal{I}(\psi^{l\pm1},\psi^{l\pm2})]=0, \tag{23}$$

where $V^l(r)$, $R^l(r)$, and $K(r)$ are radial functions given by

$$4V^l(r)=(r^2+a^2)^{-2}[(l-s)(l+s+1)\Delta-m^2a^2-2isma(r-M)]-(r^2+a^2)^{-3/2}\times\Delta^{-s/2+1}\frac{d}{dr}\left\{\Delta^{s+1}\frac{d}{dr}[(r^2+a^2)^{-1/2}\Delta^{-s/2}]\right\}, \tag{24a}$$

$$2R^l(r)=(r^2+a^2)^{-2}[2imMar-sM(r^2-a^2)+s\Delta(r+iac^l_0)], \tag{24b}$$

$$4K(r)=(r^2+a^2)^{-2}\Delta. \tag{24c}$$

Of importance will be the large-r asymptotic forms of these functions, which in terms of the $1/r_*$ expansion reads

$$V^l(r)=\frac{l(l+1)}{4r_*^2}+\frac{M-imsa+l(l+1)M\left[2\ln\left(\frac{r_*}{r_+-r_-}\right)-1\right]}{2r_*^3}+O\left[\frac{(\ln r_*)^2}{r_*^4}\right], \tag{25a}$$

$$R^l(r)=\frac{s}{2r_*}+\frac{s\left[iac^l_0-3M+2M\ln\left(\frac{r_*}{r_+-r_-}\right)\right]}{2r_*^2}+O\left[\frac{(\ln r_*)^2}{r_*^3}\right], \tag{25b}$$

$$K(r)=\frac{1}{4r_*^2}-\frac{M\left[1+2\ln\left(\frac{r_*}{r_+-r_-}\right)\right]}{2r_*^3}+O\left[\frac{(\ln r_*)^2}{r_*^4}\right]. \tag{25c}$$

Equation (23) already provides a qualitative picture of one aspect of the fields' evolution: Unlike in spherically symmetric spacetimes, multipole modes of different l interact with each other while propagating on the Kerr background.[2] For example, if a physical ($s\neq0$) perturbation is initially composed of a pure l,m mode, then, in general, we may expect all possible modes l',m—namely, all modes with $l'\geq\max(|m|,|s|)$—to be generated while the perturbation evolves in time. [The unrealistic case of a scalar field ($s=0$) is special, as for this field only those of the above modes with even $l-l'$ are expected to be excited.] Note that all interaction terms in Eq. (23) involve derivatives of the field with respect to t. It is this feature which, by means of the LTE scheme, allows one to effectively decouple Eq. (23), as we show in Sec. IV.

C. Initial setup

We shall consider an initial perturbation in the form of a compact outgoing pulse of radiation, which is relatively short, yet arbitrarily shaped. We take this pulse to be emitted at $u=u_0$, $v=v_0$, and without limiting the generality we take, for simplicity, $v_0=0$. We further assume that the initial pulse has a rather generic angular shape, so that it is composed of all multipole modes l,m [and in particular, for each m it contains the lowest possible mode, $l=\max(|m|,|s|)$]. Formulated mathematically, this initial setup takes the form[3]

$$\psi^{lm}=\begin{cases}\Gamma^{lm}(u) & \text{at } v=0,\\ 0 & \text{at } u=u_0,\end{cases} \tag{26}$$

where, for each l and m, $\Gamma^{lm}(u)$ is an arbitrary (but nonvanishing) function with a compact support between $u=u_0$ and (say) $u=u_1>u_0$, with $u_1-u_0\ll|u_0|$. This type of initial data corresponds to the physical scenario in which no ingoing radiation is coming from past null infinity.

It will be assumed in the following that the initial pulse is emitted far away from the BH; namely, we take $-u_0\gg M$. This assumption greatly simplifies our analysis (as we explain in Sec. III; cf. [7,8]); yet, it seems reasonable to expect the late time behavior in this case to remain characteristic of the general situation.

[1] In Eq. (23), the derivative ∂_u is taken with fixed v, ∂_v is taken with fixed u, and ∂_t with fixed r.

[2] See, however, our remark in the concluding section, with regard to the definition of "multipole moments" in the Kerr geometry being somewhat ambiguous.

[3] It should be noted that, strictly speaking, these initial data [supplemented to the coupled field equation (23)] do *not* form a well posed characteristic initial-value problem, as $u=$const and $v=$const are *not* characteristic hypersurfaces of the Kerr geometry (these hypersurfaces are timelike rather than null). We further comment on this issue below.

III. LATE TIME BEHAVIOR AT NULL INFINITY: THE ITERATIVE SCHEME

In this section we derive the form of the late time decay at future null infinity, that is for $v\to\infty$ at finite $u\gg M$. This has two main motivations: First, the results at null infinity will appear to serve in the framework of the late time expansion scheme as necessary ‘‘boundary conditions’’ for the global late time evolution problem (as we discuss in Sec. IV). The results at null infinity also have their own physical significance, for the following reason: Consider a static (fixed-r) observer located at very large distance. Let Δu and Δt represent, respectively, the retarded and the static observer's time elapsed since this observer gets the first signal from the perturbing field. With respect to this observer, the relevant information about the decay at finite $\Delta u=\Delta t$ is the one calculated at the null infinity domain (which we may call, in this context, the ‘‘astrophysical zone’’ of the waves). Only when the time lapse becomes infinitely large (while r remains finite) does this observer enters the ‘‘future timelike infinity’’ zone, $t\gg r$ (the late time behavior in this domain will be discussed in Sec. VII).

For this part of the analysis (namely, for the derivation of the late time tails at null infinity) we apply the *iterative scheme*, first developed and tested for scalar waves on the Schwarzschild background in Refs. [7,8]. Since the technical details of the calculations involved are often very similar to those in the above references, we mainly describe here the new results (for spin-s fields in Kerr), and direct the reader to Refs. [7,8] for further details.

A. Formulation of the iterative scheme

We define

$$V_0^l(r)\equiv\frac{l(l+1)}{4r_*^2},$$

$$R_0(r)\equiv\frac{s}{2r_*}\tag{27}$$

and

$$\delta V^l(r)\equiv V^l(r)-V_0^l(r),$$

$$\delta R^l(r)\equiv R^l(r)-R_0(r).\tag{28}$$

The functions $V_0^l(r)$ and $R_0(r)$ extract at large r the (flat space) asymptotic behavior of the functions $V^l(r)$ and $R^l(r)$ appearing in Eq. (23). The ‘‘curvature induced’’ residual part of these functions is represented by $\delta V^l(r)$ and $\delta R^l(r)$.

We now decompose each of the functions ψ^{slm} (for each l,m) as

$$\psi^l=\psi_0^l+\psi_1^l+\psi_2^l+\cdots,\tag{29}$$

such that each of the functions ψ_n^l satisfies the field equation

$$\psi_{n,uv}^l+V_0^l(r)\psi_n^l+R_0(r)\psi_{n,t}^l=S_n^l,\tag{30}$$

with $S_{n=0}^l\equiv 0$ and with

$$S_{n>0}^l\equiv-\delta V^l\psi_{n-1}^l-\delta R^l(\psi_{n-1}^l)_{,t}-K(r)[a^2C_0^l(\psi_{n-1}^l)_{,tt}+\mathcal{I}(\psi_{n-1}^{l\pm1},\psi_{n-1}^{l\pm2})].\tag{31}$$

We take the initial conditions for the various functions ψ_n^l to be $\psi_{n=0}^l=\psi^l$ and $\psi_{n>0}^l=0$ on the initial surfaces $u=u_0$ and $v=0$.

Equation (30), supplemented by the above initial data, constitutes a hierarchy of characteristic initial data problems for the various functions ψ_n^l. Formal summation over n recovers the original evolution problem for ψ^l. For each $n>0$ (and for all l), the function ψ_n^l admits an inhomogeneous field equation, with a source term depending only on the functions $\psi_{n'<n}$ preceding it in the hierarchy, and with the function $\psi_{n=0}^l$ satisfying a closed homogeneous equation. This structure allows one, in principle, to solve for all functions ψ_n^l in an iterative manner: first for all modes l of $\psi_{n=0}^l$, which then serve as sources to $\psi_{n=1}^l$, etc. In general, each function ψ_n^l shall have sources coming from the modes l, $l\pm1$, and $l\pm2$ of ψ_{n-1}.

Of course, the effectiveness of the proposed iteration scheme crucially depends on its convergence properties. In that respect, the scheme as formulated above may seem problematic, because, while the ‘‘zeroth order’’ $(n=0)$ field equation well approximates the actual field equation (23) at large distance, it fails to do so at the highly curved small-r region (actually, the functions V_0 and R_0, as defined above, diverge at $r_*=0$).

It is possible (by redefining V_0 and R_0 at small r) to construct a more sophisticated iteration scheme that would account for the small-r region of spacetime as well — as was done in Ref. [8] for the case of a scalar field in the Schwarzschild spacetime. In that case, it was demonstrated [8] that the small-r details of the background geometry have merely a negligible effect on the late time behavior at null infinity. In the Schwarzschild case, we were able to greatly simplify the analysis by considering a toy model of a thin spherically symmetric shell of matter having a flat interior. In that model, the late time behavior of the scalar field at null infinity turned out to well approximate the actual behavior in a ‘‘complete’’ Schwarzschild model (see [7,8] for details).

The above results all indicate that the late time decay of the scalar field is predominantly governed by the large-r structure of the Schwarzschild spacetime. (This conclusion stemmed already from several previous works, as we mentioned in the Introduction.) We shall assume in this paper that the same is also valid for all fields Ψ^s propagating on the Kerr background. To be concrete, we will consider in this section, only for the sake of the calculation at null infinity, a model in which the Kerr interior geometry is replaced by a flat (Minkowski) manifold. There is no way of smoothly attaching a flat interior to a Kerr exterior through a thin spherically symmetric material shell (as was done in [7] in the Schwarzschild case); alternatively, we take this attachment to be made through a material layer of a finite width. The external ‘‘radius’’ of this layer should be of order of a few M. Based on our experience with the Schwarzschild case, we expect the details of our model at small distances (and, in particular, the internal structure of the material layer) to merely have a negligible affect on the behavior of the waves at null infinity, at $u \gg M$.

We emphasize that the above simplified model is adopted only for the analysis at null infinity, where it greatly reduces the amount of technical details one should deal with. For the global analysis carried out in the rest of this paper, the ''complete'' KBH geometry shall be considered.

B. Zeroth order iteration term

By definition, the function $\psi^l_{n=0}$ obeys (for all l) the homogeneous equation

$$\psi^l_{0,uv} + V^l_0(r)\psi^l_0 + R_0(r)\psi^l_{0,t} = 0, \tag{32}$$

with the initial conditions $\psi^l_{n=0}(v=0) = \Gamma^l(u)$ and $\psi^l_{n=0}(u=u_0)=0$. Equation (32) simply describes the free propagation of the field $\psi^{sl}_{n=0}$ in Minkowski spacetime, provided that r_* is replaced with the radial Minkowski coordinate r.

The general solution to this equation reads[4]

$$\psi^{\mathrm{general}}_{n=0} = \sum_{j=0}^{l-s} A^{sl}_j \frac{g_0^{(j)}(u)}{(v-u)^{l-j}} + \sum_{j=0}^{l+s} A^{-sl}_j \frac{h_0^{(j)}(v)}{(u-v)^{l-j}}, \tag{33}$$

in which $g_0(u)$ and $h_0(v)$ are arbitrary functions (with their parenthetical superindices indicating the number of differentiations), and where A^{ls}_j are constant coefficients given by

$$A^{sl}_j = \frac{(2l-j)!}{j!(l-j-s)!}. \tag{34}$$

The above initial data for $\psi^l_{n=0}$ uniquely determine a specific solution for this function [but not for either of the functions $g_0(u)$ and $h_0(v)$ in separate]. For the outgoing pulse initial setup it is possible to write this specific solution in a convenient form by taking $h_0(v) \equiv 0$, in which case we have

$$\psi^l_{n=0} = \sum_{j=0}^{l-s} A^{sl}_j \frac{g_0^{(j)}(u)}{(v-u)^{l-j}}, \tag{35}$$

with the function $g_0(u)$ given by[5]

$$g_0^{l>s}(u) = \theta(-u)\frac{l-s}{(l+s)!}\int_{u_0}^{u} (u/u')^{l+s+1}(u-u')^{l-s-1} \times(-u')^s \Gamma(u')\,du', \tag{36a}$$

$$g_0^{l=s}(u) = \frac{(-u)^s}{(2s)!}\Gamma(u). \tag{36b}$$

[4]Note that the homogeneous equation (32) is invariant upon simultaneously transforming $u \rightleftharpoons v$ and $s \to -s$. The two sums in Eq. (33) constitute two independent homogeneous solutions, which are obtained from each other by this transformation.

[5]The calculation leading to Eq. (36) is a straightforward generalization of the one described in detail in Ref. [7] for $s=0$.

Here, $\theta(x)$ is the standard step function, taking the value 1 or 0 according to whether x is positive or negative, respectively. Equations (35) and (36) describe how the function $\psi^l_{n=0}$ can be constructed given any form $\Gamma(u)$ for the compact initial outgoing pulse.

Equation (36) implies that the wave $\psi^l_{n=0}$ is sharply ''cut off'' at retarded time $u=0$. This effect is due to ingoing and outgoint waves destructively interfering with each other at the origin of coordinates ($r=0$) inside the flat region internal to the material layer, at retarded times $u>0$. [For a scalar field in the ''complete'' Schwarzschild model we found in Ref. [8] an exponential decay of the waves at $u>0$ rather than a sharp cutoff — thus the main support of $\psi^l_{n=0}$ remains compact even in this more sophisticated (and much more complicated) model.]

Obviously, since $\psi^l_{n=0}$ is strictly compact in retarded time (it is supported only in the range $u_0<u<0$), it does not contribute to the overall late time (large u) radiation at null infinity. Rather, the function $\psi^l_{n=0}$ will serve [via Eq. (30)] as a source to higher-order ($n \geq 1$) terms of the iteration scheme, which will form the late time tail of decay, as we show below.

C. Green's function of the iteration scheme

Using the Green's function method, we formally have, for each of the functions $\psi^l_{n>0}$,

$$\psi^l_n(v,u) = \int_0^v dv' \int_{u_0}^u du'\, G^l(v,u;v',u') S^l_n(v',u'), \tag{37}$$

which allows one, in principle, to calculate these functions one by one, in an inductive manner. Here, $G(v,u;v',u')$ is the time domain Green's function in Minkowski spacetime, defined as satisfying the equation

$$G^l_{,uv} + V^l_0(r)G^l + R_0(r)G^l_{,t} = \delta(u-u')\,\delta(v-v'), \tag{38}$$

with the causality condition

$$G^l(u<u') = G^l(v<v') = 0. \tag{39}$$

To solve for G, we use a straightforward generalization of the method used in Ref. [7]. First one shows that for Eq. (38) to be consistent with the causality conditions (39), one must have $G(v=v') = [(v'-u)/(v'-u')]^s$ and $G(u=u') = [(v'-u')/(v-u')]^s$. This establishes a characteristic initial-value problem for the Green's function at $u>u'$ and $v>v'$. Then, with the help of Eq. (36), one can obtain (see [7] for more details)

$$G^l(v,u;v',u') = \sum_{j=0}^{l-s} A^{sl}_j \frac{g^{(j)}(u;v',u')}{(v-u)^{l-j}}\theta(u-u') \times\theta(v-v')\,\theta(v'-u), \tag{40}$$

where the function $g(u;v',u')$ is given by

$$g(u;v',u') = \frac{1}{(l+s)!} \frac{(v'-u)^{l+s}(u-u')^{l-s}}{(v'-u')^{l}}, \quad (41)$$

and where the j differentiations of this function are with respect to u. The factor $\theta(v'-u)$ in Eq. (40) is related, again, to the presence of the origin of coordinates inside the flat internal region.

Finally, a general comment should be made about the application of the iterative scheme to the Kerr spacetime: Here (unlike in the Schwarzschild case) the characteristic surfaces of the iterative scheme (i.e. the surfaces of constant u or constant v, which are not null but rather timelike) do *not* coincide with the actual characteristics of the Kerr geometry. Thus, strictly speaking, the ''causality condition'' stated in Eq. (39) does not hold for the actual Green's function in the Kerr spacetime. However, the surfaces $u=\text{const}$ and $v=\text{const}$ do approach the actual null characteristics of the Kerr background at large distances. Recalling that the form of the late time radiation at null infinity is shaped mainly during the propagation at large distances, it is reasonable to expect that the above problem will not crucially affect the validity of our results.

D. First-order iteration term

The first contribution to the late time (large-u) tail at null infinity comes from the function $\psi^{l}_{n=1}$. This contribution also turns out to be the most dominant one, with those of the functions $\psi^{l}_{n>1}$ smaller by one or more factors of $M/|u_0|$ (recall that in our model we have $|u_0| \gg M$). It is therefore of special importance to analyze in detail the behavior of $\psi^{l}_{n=1}$, and derive its late time form at null infinity, as we shall do now.

The function $\psi^{l}_{n=1}$ is calculated from Eq. (37), with $n=1$. Since we only look for the behavior at null infinity, we take the limit $v\to\infty$ of this equation. At that limit, the Green's function appearing in the integrand is dominated by merely the $j=l-s$ term of the sum in Eq. (40):

$$G^{l}(v\to\infty) \cong \frac{(l+s)!}{(l-s)!} v^{-s} [g(u;u',v')]^{(l-s)} \times \theta(u-u')\,\theta(v'-u) \quad (42)$$

(where the derivatives of g are with respect to u).

With Eq. (41) we now obtain, for $\psi^{l}_{n=1}$ at null infinity (for each l),

$$\psi^{l}_{n=1}(u,v\to\infty) = \frac{v^{-s}}{(l-s)!} \int_{u_0}^{0} du' \int_{u}^{\infty} dv' \times \frac{[(v'-u)^{l+s}(u-u')^{l-s}]^{(l-s)}}{(v'-u')^{l}} S^{l}_{n=1}(u',v') \quad (43)$$

(where, again, the $l-s$ derivatives are with respect to u). In this expression, the lower limit of the integration over v' was set to $v'=u$ due to the factor $\theta(v'-u)$ appearing in the Green's function, and the upper limit of the u' integration was set to $u'=0$ in view of the compactness of $\psi^{l}_{n=0}$ (implying the compactness of the source $S^{l}_{n=1}$ as well).

The source function $S^{l}_{n=1}$ is calculated from Eq. (31), with $n=1$. It contains, in general, contributions from the modes $l'=l, l\pm1, l\pm2$ of $\psi^{l'}_{n=0}$. These contributions to $\psi^{l}_{n=1}$ [via Eq. (43)] are additive, and may be calculated one by one. The details of this calculation are given in the Appendix. In brief, it contains two steps for each of the above contributions: First, the definite integration over v' is carried out explicitly. The integrand of the remaining u' integration then becomes a finite sum over derivatives of $g^{l'}_{0}(u')$, each multiplied by a power of $(u-u')$. In the second and final step we use successive integrations by parts to eliminate the derivatives of $g^{l'}_{0}(u')$, with all resulting surface terms vanishing in virtue of the compactness of this function. (This procedure is clarified in the Appendix.)

The following is a description of the outcome from the above calculation. Let us denote by $\psi^{l'\to l}_{n=1}$ the contribution to the mode l of $\psi_{n=1}$ at null infinity from all terms in $S^{l}_{n=1}$ associated with the mode l' of $\psi_{n=0}$. We then find at $u \gg -u_0$, to leading order in M/u and in u_0/u (see the Appendix for details),

$$\begin{aligned}
\psi^{l-2\to l}_{n=1} &\cong \alpha^{l}_{-2} I^{l-2}_{0} v^{-s} u^{-(l-s+2)}, \\
\psi^{l-1\to l}_{n=1} &\cong \alpha^{l}_{-1} I^{l-1}_{0} v^{-s} u^{-(l-s+2)} \times \left[1+\beta^{l}_{-1}\ln\left(\frac{u}{r_+-r_-}\right)\right], \\
\psi^{l\to l}_{n=1} &\cong \alpha^{l}_{0} I^{l}_{0} v^{-s} u^{-(l-s+2)}, \\
\psi^{l+1\to l}_{n=1} &\cong \alpha^{l}_{+1} I^{l+1}_{0} v^{-s} u^{-(l-s+4)} \times \left[1+\beta^{l}_{+1}\ln\left(\frac{u}{r_+-r_-}\right)\right], \\
\psi^{l+2\to l}_{n=1} &\cong \alpha^{l}_{+2} I^{l+2}_{0} v^{-s} u^{-(l-s+6)},
\end{aligned} \quad (44)$$

where the α's and β's are constant coefficients (depending on s,l,m), and where I^{l}_{0} is, for each l, a simple functional of g^{l}_{0}:

$$I^{l}_{0} \equiv \int_{u_0}^{0} g^{l}_{0}(u')\,du'. \quad (45)$$

For a scalar field, $s=0$, the contributions $\psi^{l\pm1\to l}_{n=1}$ vanish (as the coefficients $\alpha^{l}_{\pm1}$ are proportional to s). Also, there is no contribution $\psi^{l-1\to l}_{n=1}$ for $l<l_0+1$ and no contribution $\psi^{l-2\to l}_{n=1}$ for $l<l_0+2$, where l_0 is the lowest radiatable multipole mode for given s and m:

$$l_0 = \max(|s|,|m|). \quad (46)$$

Equations (44) implies that for any given s and m, the most dominant mode of $\psi_{n=1}$ at null infinity, at large u, is l_0. It also tells us that the dominant contribution to this mode comes solely from this mode itself (namely, it is $\psi_{n=1}^{l_0\to l_0}$), since in this case there are no contributions from lower modes, and the ones from higher modes are negligible at large u. Hence, the decay of the dominant mode of $\psi_{n=1}$ at null infinity (for any given s and m) is described at large retarded time u by

$$\psi_{n=1}^{l_0}(u,v\to\infty)=\alpha_0^{l_0}I_0^{l_0}v^{-s}u^{-(l_0-s+2)}, \tag{47}$$

to leading order in M/u and in u_0/u. Recall that the function $g_0^{l_0}(u)$ and, hence, the functional $I_0^{l_0}$ are nonvanishing,[6] as long as we make the assumption that the initial data have a generic angular form (so that it particularly contains the lowest multipole mode l_0 for any given s and l). The case where the lowest modes are initially missing will be discussed in brief below.

E. Late time tail at null infinity

Equation (37) provides a formal means for calculating the higher iteration terms, $\psi_{n\geqslant 2}^{l}$. However, exact analytic calculations become very tedious already for the $n=2$ term. In the case of a scalar field in Schwarzschild spacetime [7] we have explicitly derived $\psi_{n=2}^{l}$ at null infinity, and showed that it exhibits the same power-law decay as $\psi_{n=1}^{l}$ at large u, yet with an amplitude reduced by a factor proportional to $(M/|u_0|)\ll 1$. For this case, analytic considerations suggested that a similar reduction of the amplitude by a factor $\propto M/u_0$ occurs also for $n>2$, whenever n is increased by 1. This conclusion was verified numerically for the first few iterative terms [7]. Our numerical calculations also indicated that the sum of iterative terms ψ_n seems to converge rather fast at null infinity for large $|u_0|/M$ (say, in the order of 100).

We now proceed under the assumption that the same considerations also apply in our case, of general-s fields on the Kerr background. That is, we assume that (for any given s,l,m) the functions $\psi_{n\geqslant 2}^{l}$ decay at null infinity at large u with the same tail as $\psi_{n=1}^{l}$; yet the amplitude of these functions is smaller by at least one factor of order $O(M/u_0)$. This assumption seems plausible, because the above property of the iterative scheme [namely, the scaling of ψ_n as $\propto(M/u_0)^n$] seems to stem from the basic structure of the iteration procedure, rather than from the details of the source function S_n, which distinguishes the Schwarzschild, $s=0$, case from the more complicated case studied in the current paper.

Adopting the above assumption, we conclude that, for large $|u_0|/M$, the ''overall'' function ψ^l is well approximated at null infinity at large u by merely the term $\psi_{n=1}^{l}$. In particular, l_0 is the dominant mode of ψ there, for any given s and m. By virtue of Eqs. (9) and (47) we then finally obtain, for the Newman-Penrose field Ψ^{sm} (for any given s,m),

$$\Psi^{sm}\cong\alpha_0^{l_0}I_0^{l_0}Y^{sl_0m}(\theta,\varphi)v^{-2s-1}u^{-(l_0-s+2)}$$
$$(\text{dominant mode at scri+, large } u). \tag{48}$$

This result is accurate to leading order in M/u, in u_0/u, and in M/u_0. In the generic case where the initial pulse includes all values of m, we find that the behavior is dominated by the modes with $0\leqslant|m|\leqslant|s|$ and $l=|s|$, which decay at null infinity with the late time tail u^{-2} for $s\geqslant 0$ or u^{2s-2} for $s\leqslant 0$.

One may also ask about the behavior of the other, faster decaying, modes at null infinity. From Eq. (44) we find that (for any given s and m) the modes $l>l_0$ of ψ_1 are also ''fed'' by strong contributions coming from modes of smaller l: In general, the function $\psi_{n=1}^{l>l_0}$ has leading-order contributions from the modes $l-1$ and $l-2$ of $\psi_{n=0}$. In the exceptional scalar field case ($s=0$) this contribution comes only from the $l-2$ mode (provided $l\geqslant l_0+2$), and provides the same tail as the contribution from the mode l itself, namely $u^{-(l-s+2)}$. We thus have, for all $l\geqslant l_0$ in the scalar field case, $\psi_{n=1}^{l}\propto u^{-(l-s+2)}$ at null infinity, large u. Under the above assumption that the ''overall'' field ψ^l is well approximated there by ψ_1^l (for all l), we conclude that *in the scalar field case*, the decay of *any* of the modes $l\geqslant l_0$ at null infinity is given at large u by

$$\Psi^{lm}\propto Y^{lm}(\theta,\varphi)v^{-1}u^{-l-2} \quad (\text{any mode of a scalar field}), \tag{49}$$

where Y^{lm} are the spherical harmonics.

As to the non-dominating modes of the $s\neq 0$ fields: Equation (44) suggests that these modes would exhibit not a strict power-law tail but rather a tail of the form $\propto u^{-(l-s+2)}\times\ln[u/(r_+-r_-)]$. We feel, however, that this result cannot be taken as conclusive, and needs a further support (e.g. from numerical analysis). (We comment that such logarithmic dependence does not arise from the frequency-domain analysis in Ref. [24].) We emphasize our conclusion, Eq. (48), that the leading-order tail at null infinity, belonging to the most dominant mode, decays with a strict power-law.

IV. LATE TIME EXPANSION

The target of this work is to explore the behavior of the fields Ψ^s at late time *anywhere* outside the KBH (and along its EH). To that end we shall apply the late time expansion scheme, a version of which was used in Ref. [8] to analyze a scalar field in the Schwarzschild case.

We assume that at late time, the fields Ψ^s admit an expansion of the form

$$\Psi^s(v,r,\theta,\varphi)=\sum_{k=0}^{\infty}\left[\sum_{l=|s|}^{\infty}\sum_{m=-l}^{l}Y^{slm}(\theta,\varphi)F_k^{slm}(r)\right]v^{-k_0-k}, \tag{50}$$

[6]This is true unless the initial data are very finely tuned such as to make the integral in Eq. (45) vanish.

to which we shall refer as the *late time expansion*. Here, k_0 is a constant parameter which we later determine.[7] As we show in this paper, the LTE is consistent with the field equations, with the tail form at null infinity, and with regularity requirements at the EH. We adopt an expansion in $1/v$, rather than in $1/t$, because it appears to be more adequate for analyzing the behavior near and along the EH (as the coordinate v, unlike t, is regular through the EH).

Inserting the form (50) into the master field equation (7), and collecting terms of common v powers and of common multipole numbers l [with the aid of Eqs. (12) and (14)], yields an ordinary equation for each of the unknown functions $F_k^{slm}(r)$:

$$D^{slm}[F_k^{slm}(r)]=Z_k^{slm}, \tag{51}$$

in which D^{slm} is a differential operator given by

$$D^l\equiv\Delta\frac{d^2}{dr^2}+2(s+1)(r-M)\frac{d}{dr}+\left[\frac{a^2m^2+2isma(r-M)}{\Delta}-(l-s)(l+s+1)\right], \tag{52}$$

and the source term Z_k^l reads

$$Z_k^l=2(k_0+k-1)\left\{(r^2+a^2)\frac{dF_{k-1}^l}{dr}+\left[\frac{2M[s(r^2-a^2)-imar]}{\Delta}+r-iasc_0^l\right]F_{k-1}^l-ias(c_+^lF_{k-1}^{l+1}+c_-^lF_{k-1}^{l-1})+\frac{a^2}{2}(k_0+k-2)(C_0^lF_{k-2}^l+C_{++}^lF_{k-2}^{l+2}+C_+^lF_{k-2}^{l+1}+C_-^lF_{k-2}^{l-1}+C_{--}^lF_{k-2}^{l-2})\right\}, \tag{53}$$

with $F_{k<0}^l\equiv0$, and with the various coefficients c^l and C^l given in Eqs. (13) and (15).

An essential feature of Eq. (51) is the fact that it actually constitutes a hierarchy of effectively *decoupled* equations, as each of the functions $F_{k>0}$ satisfies an inhomogeneous equation whose source depends only on the functions $F_{k'<k}$ preceding it in the hierarchy. The first function, $F_{k=0}^l$, obeys a closed homogeneous equation. Thus, in principle, we may solve for all modes of all functions F_k one by one, starting with $k=0$. For each k, one should be able to solve for all modes l of F_k, and then carry on to $k+1$.

Now, Eq. (51) is a second-order differential equation for each of the various functions F_k^l. In principle, to determine these functions, proper boundary conditions should be specified at the EH (which is, mathematically speaking, a ''regular singular point'' of the equation) and at space-like infinity. (These two boundary conditions should determine the two arbitrary parameters which occur in the general solution for each of the function F_k^l.) The behavior of Ψ^s at infinity is known from the previous section, and in Sec. VII below we discuss the matching to this asymptotic region. At the EH, the only obvious requirement concerns the regularity of the physical fields there. In the rest of the present section we obtain the boundary conditions for the functions F_k^l at the EH, based on local regularity considerations.

One expects ''measurable'' physical quantities to maintain a perfectly regular behavior at the EH, which is a surface of a perfectly regular local geometry. Accordingly, the components of the Weyl and Maxwell tensors should be perfectly smooth through the EH, provided these components are expressed in a coordinate system regular at the EH. To construct boundary conditions for the scalars Ψ^s at the EH, it then remains to relate these scalars to the regular components of the Weyl and Maxwell tensors.

To that end we must first write the tetrad basis (4), used to construct the scalars Ψ^s, in EH-regular coordinates. Recalling that the BL coordinates t and φ go irregular at the EH, we introduce the Kruskal-like coordinates

$$U\equiv-e^{-\kappa_+u}\quad\text{and}\quad V\equiv e^{\kappa_+v}, \tag{54}$$

and the regularized azimuthal coordinate

$$\tilde{\varphi}_+\equiv\varphi-\Omega_+t \tag{55}$$

(see Sec. 58 in Ref. [32]), where

$$\Omega_+=\frac{a}{2Mr_+}. \tag{56}$$

(Ω_+ is the ''angular rate of inertial frame dragging'' at the EH.) In the EH-regular coordinate system $(V,U,\theta,\tilde{\varphi}_+)$, the components of the ingoing and outgoing tetrad legs have the EH-asymptotic forms

$$l^\mu\propto\Delta^{-1}e^{\kappa_+v}[1,0,0,0],$$
$$n^\mu\propto\Delta e^{-\kappa_+v}[0,1,0,0]. \tag{57}$$

Recall that v is regular at the EH, and that $\Delta=0$ there.

Now, the construction of the scalars Ψ^s involves $|s|$ projections of the Weyl and Maxwell tensors on the tetrad legs l^μ (for $s>0$) or n^μ (for $s<0$). Since the components of these tensors must be EH regular in the coordinate system $(V,U,\theta,\tilde{\varphi}_+)$, then by virtue of Eq. (57) we find that $\Delta^s\Psi^s$ must be EH regular as well.

To formulate the regularity condition for the functions $F_k^l(r)$, we note that the functions $Y^{slm}(\theta,\varphi)$ in Eq. (50) are irregular at the EH for $m\neq0$, due to the factor $e^{im\varphi}$: We have

$$e^{im\varphi}=e^{im(\tilde{\varphi}_++\Omega_+t)}=[e^{im\tilde{\varphi}_+}\cdot e^{im\Omega_+v}]e^{-im\Omega_+r_*}, \tag{58}$$

[7]It should be noted that *by definition* the parameter k_0 does not depend on l,m: whereas in the Schwarzschild case [8] a separate parameter k_0^l has been defined for each mode l,m, in the present paper a single parameter k_0 is related with the overall field Ψ^s.

where the factor in square brackets is EH regular, but the following factor oscillates rapidly towards the EH. From Eq. (50) we thus find that it is the quantity $\Delta^s e^{-im\Omega_+ r_*} F_k^l$ which must be perfectly regular at the EH (for all k).

Hence, the regularity condition at the EH can be phrased as follows: Define the ''physical'' variables

$$\hat{\Psi}^s \equiv \Delta^s \Psi^s \quad \text{and} \quad \hat{F}_k^l \equiv \Delta^s e^{-im\Omega_+ r_*} F_k^l. \tag{59}$$

Then, we must have that (for all k)

$$\hat{\Psi}^s \quad \text{and} \quad \hat{F}_k^l \quad \text{are smooth functions at the EH.} \tag{60}$$

Mathematically, we shall require these functions and all their derivatives with respect to an EH-regular coordinate (such as r or U) to be continuous through the EH.

Equation (60) constitutes the required boundary condition at the EH for all functions $F_k^l(r)$.

V. GLOBAL SOLUTION FOR $k=0$

The dominant late time decay at world lines of fixed r is described by the $k=0$ term of the LTE, Eq. (50).[8] In this section we derive an exact analytic expression for this term (namely, for all modes l of the function $F_{k=0}^l$). Since $F_{k=0}^l$ is a solution of a second-order differential equation, it shall contain two arbitrary parameters. One of these parameters will be determined in this section by the regularity condition at the EH. The other parameter will be determined in Sec. VII through matching at infinity.

By definition, $Z_{k=0}^l = 0$, and the function $F_{k=0}^l$ admits the homogeneous equation

$$D^l F_{k=0}^l = 0, \tag{61}$$

with the operator D^l given in Eq. (52). This is nothing but the *static field* equation in Kerr spacetime. The static solutions play an important role in our analysis, for two reasons: (i) As just mentioned, the late time behavior can be approximated by knowing $F_{k=0}^l$, which must be a static solution, and (ii) we shall use a basis of static solutions in constructing the functions $F_{k>0}^l$ using the Wronskian method.

For reasons that will become clear below, we continue by treating separately the cases $s\neq 0$ and $s=0$. Also, for $s\neq 0$ we will consider separately the cases $m\neq 0$ (nonaxially symmetric modes) and $m=0$ (axially symmetric modes).

A. $s\neq 0$ fields: Nonaxially symmetric modes

For $s\neq 0$ and $m\neq 0$, a basis of exact solutions to the homogeneous static field equation (61) is given by

$$\phi_r = (z_+/z_-)^{im\gamma} \Delta^{-s} F(-l-s, l-s+1; 1-s+2im\gamma; -z_+), \tag{62a}$$

$$\phi_{ir} = (z_+/z_-)^{-im\gamma} F(-l+s, l+s+1; 1+s-2im\gamma; -z_+), \tag{62b}$$

where $z_\pm$ are the dimensionless radial variables defined in Eq. (21),

$$\gamma \equiv \frac{a}{r_+ - r_-} = \frac{\Omega_+}{2\kappa_+}, \tag{63}$$

and F denotes the *hypergeometric function* [37]. (We use this notation because, as we discuss below, ϕ_r is physically regular at the EH, whereas ϕ_{ir} is irregular there.)

The hypergeometric function $F(\hat{a},\hat{b};\hat{c};y)$ (where $\hat{a}$, $\hat{b}$, and $\hat{c}$ are complex parameters and y is a complex independent variable) admits the series expansion

$$F(\hat{a},\hat{b};\hat{c};y) = 1 + \sum_{n=1}^{\infty} \frac{(\hat{a})_n (\hat{b})_n}{(\hat{c})_n} \frac{y^n}{n!} \tag{64}$$

(see, e.g., Sec. 2.1.1 in [38]), where

$$(\hat{a})_n \equiv \hat{a}(\hat{a}+1)\cdots(\hat{a}+n-1) = \Gamma(\hat{a}+n)/\Gamma(\hat{a}) \tag{65}$$

is the ''rising factorial.''[9] Two results arising from Eq. (64) are that (i) the hypergeometric function is not defined if $\hat{c}$ is a non-positive integer [as in this case a zero factor occurs in the denominator in Eq. (64)], and that (ii) if either $\hat{a}$ or $\hat{b}$ are non-positive integers, the expansion (64) terminates, and the hypergeometric function becomes a polynomial of order $-\hat{a}$ or $-\hat{b}$, respectively.

Item (i) above implies that for $m\neq 0$ both solutions (62a), (62b) are defined; however, in the case $m=0$ (which we treat separately below) only one of these solutions is defined (ϕ_r for $s<0$, or ϕ_{ir} for $s>0$). We further find, by item (ii) above, that both hypergeometric functions in the solutions (64) are simply *polynomials* of z_+ (and thus of r too).

For $m\neq 0$, the general static solution is constructed from the two basis functions (62). With the help of Eqs. (22), (63), and (64), one finds the asymptotic forms of these functions to be

$$\phi_r \cong \begin{cases} \Delta^{-s} e^{im\Omega_+ r_*} & \text{as } r_* \to -\infty\,(\Delta, z_+ \to 0), \\ B_s^{lm} r^{l-s} & \text{as } r, z_+ \to \infty, \end{cases} \tag{66}$$

and

$$\phi_{ir} \cong \begin{cases} e^{-im\Omega_+ r_*} & \text{as } r_* \to -\infty\,(\Delta, z_+ \to 0), \\ (B_{-s}^{lm})^* r^{l-s} & \text{as } r, z_+ \to \infty, \end{cases} \tag{67}$$

where the coefficient B_s^{lm} is given by

$$B_s^{lm} = \frac{(2l)!\,\Gamma(1-s+2im\gamma)}{(l-s)!\,\Gamma(l+2im\gamma+1)} (r_+ - r_-)^{-l-s}. \tag{68}$$

[8]Actually, in Sec. VI we discuss an exception to this statement: For $m=0$ modes of $s>0$ fields, the behavior along the EH is dominated by the $k=1$ term. However, in this case too, the decay along lines of constant $r>r_+$ is still dominated by the $k=0$ term.

[9]Note that $(\hat{a})_n$ is well defined even when the expression involving the gamma functions is not. In this case, of course, the second equality in Eq. (65) is invalid.

We now use Eq. (59) to construct the ''physical'' fields $\hat{\phi}_r \equiv \Delta^s e^{-im\Omega_+ r_*}\phi_r$ and $\hat{\phi}_{ir} \equiv \Delta^s e^{-im\Omega_+ r_*}\phi_{ir}$ associated with the static solutions ϕ_r and ϕ_{ir}. From the EH-regularity criterion (60) we then learn that the static solution ϕ_r is physically regular at the EH, whereas ϕ_{ir} is irregular there: For $s<0$ the irregularity of $\hat{\phi}_{ir}$ is obvious, as it diverges like $\sim \Delta^{-|s|}$ at the EH. For $s>0$, $\hat{\phi}_{ir}$ is continuous through the EH, yet its sth derivative with respect to U (which is a regular coordinate through the EH) diverges there like $\propto e^{-2im\Omega_+ r_*}$. Higher order U derivatives of $\hat{\phi}_{ir}$ are unbounded in magnitude at the EH.

Obviously, ϕ_r is the only solution of the homogeneous static equation (up to some global constant factor) which is physically-regular at the EH, because any combination $a\phi_r + b\phi_{ir}$ with $b\neq 0$ will be irregular there. In a physical setup where a static source presents outside the BH, the field must behave as $\propto \phi_r$ near and through the EH. In vacuum, ϕ_r is the only global static solution physically regular at the EH. This field does not vanish at infinity (where it behaves like $\propto r^{l-s}$); hence it cannot represent a physical static perturbation. There exists a static solution which dies off fast enough at infinity [this is the solution $(B^{lm}_{-s})^* \phi_r - B^{lm}_s \phi_{ir}$, which dies off as $\propto r^{-l-s-1}$ at large r [37]], yet this solution is physically irregular at the EH. Similar results apply also to the static $m=0$ modes, which we study below. We thus conclude that *there cannot exist physical vacuum static modes* outside the Kerr BH. This, of course, is a manifestation of the ''no hair'' principle.

In the framework of the LTE, each of the functions $F^l_k(r)$ must be subject to the EH-regularity criterion (60). Since $F^l_{k=0}$ must be a static solution, it thus have to be proportional to ϕ^l_r. To conclude the above discussion we therefore take

$$F^l_{k=0}(r) = a^l_0 \phi^l_r(r). \tag{69}$$

The constant a^l_0 is to to be determined in Sec. VII by matching at null infinity.

B. $s\neq 0$ fields: Axially symmetric modes

We remarked earlier that in the case $m=0$ only one of the basis functions (62a), (62b) is defined—the one in which, depending on the sign of s, the third parameter of the hypergeometric function is a positive integer. Denoting this function by $\phi^{m=0}_r$, we have

$$\phi^{m=0}_r = F(-l+s, l+s+1; s+1; -z_+) \equiv \phi^+_r \quad \text{for } s>0 \tag{70a}$$

and

$$\phi^{m=0}_r = \Delta^{-s} F(-l-s, l-s+1; -s+1; -z_+) \equiv \phi^-_r \quad \text{for } s<0. \tag{70b}$$

The asymptotic forms of this solution are

$$\phi^{m=0}_r(r) \cong \begin{cases} \Delta^0 & (\text{for } s>0) \\ \Delta^{-s} & (\text{for } s<0) \end{cases} \quad \text{at the EH} \tag{71}$$

and, for both $s>0$ and $s<0$,

$$\phi^{m=0}_r(r) \cong B^{l,m=0}_{-|s|} r^{l-s} \quad \text{as } r, z_+ \to \infty. \tag{72}$$

Recall that F is simply a polynomial of z_+ (and of r), and thus so is $\phi^{m=0}_r$. From Eq. (70) we find that the ''physical'' field $\hat{\phi}^{m=0}_r \equiv \Delta^s \phi^{m=0}_r$ is also a polynomial, and therefore, clearly, $\phi^{m=0}_r$ is physically regular at the EH.

We still have to construct a second independent basis static solution for the $m=0$ case. (This will allow us to tell whether or not the above solution $\phi^{m=0}_r$ is the only regular one.) Fortunately, at that point we can benefit from the work already done in Ref. [28] for the Schwarzschild case: When expressed in terms of the variable z_+ (rather than r), the static field equation (61) for $m=0$ takes exactly the same form as for the Schwarzschild BH [see Eq. (21) in [28]], where in the latter case we use the variable $z\equiv(r-2M)/(2M)$. Therefore, each static solution in the Schwarzschild spacetime becomes a static axially symmetric ($m=0$) solution in Kerr spactime, upon replacing $z\to z_+$. Moreover, in terms of the variables z (in the SBH case) and z_+ (in the KBH case), the EH-regularity criterion becomes the same for both spacetimes, and thus the classification of regular and irregular solutions at the EH is also conserved.

As a second basis function we then take the static solution given in Eq. (24) of Ref. [28]:

$$\phi^{m=0}_{ir} = \tilde{A}_{ls} z_+^{-s} z_-^{-l-1} F(l-s+1, l+1; 2l+2; z_-^{-1}) \equiv \phi^+_{ir} \quad \text{for } s>0 \tag{73a}$$

and

$$\phi^{m=0}_{ir} = \tilde{A}_{ls} z_-^{-l-s-1} F(l+s+1, l+1; 2l+2; z_-^{-1}) \equiv \phi^-_{ir} \quad \text{for } s<0, \tag{73b}$$

where we have replaced z with z_+ (and thus $z+1$ with z_-). Here, $\tilde{A}_{sl}$ is a normalization factor,

$$\tilde{A}_{sl} = 1/F(l-|s|+1, l+1; 2l+2; 1) = \frac{l!(l+|s|)!}{(2l+1)!(|s|-1)!} \tag{74}$$

[cf. Eq. (46) in Sec. 2.8 of [38]], chosen such that $\phi^{m=0}_{ir}$ takes a simple asymptotic form at the EH (see below). From Ref. [28] we also know that $\phi^{m=0}_{ir}$ admits the following series expansion near the EH:

$$\phi^{m=0}_{ir}(r) = \begin{cases} \Delta^{-s}(1+\tilde{\alpha}^+\Delta+\cdots) + \tilde{\beta}\phi^+_r \ln z_+ & (\text{for } s>0), \\ (1+\tilde{\alpha}^-\Delta+\cdots) + \tilde{\beta}\phi^-_r \ln z_+ & (\text{for } s<0), \end{cases} \tag{75}$$

in which the coefficient $\tilde{\beta}$ is nonvanishing,

$$\tilde{\beta} = \frac{(-1)^{s+1}(l+|s|)!}{(|s|-1)!(|s|)!(l-|s|)!}(r_+-r_-)^{-2|s|}. \tag{76}$$

It is clear (e.g. by comparing the EH-asymptotic forms) that the two solutions $\phi_r^{m=0}$ and $\phi_{ir}^{m=0}$ are independent, and thus form a complete basis of solutions. As already explained in Ref. [28], $\phi_{ir}^{m=0}$ is physically irregular at the EH: For $s<0$ the "physical" field $\hat{\phi}_{ir}^{m=0}$ diverges there as $\Delta^{-|s|}$. For $s>0$ $\hat{\phi}_{ir}^{m=0}$ is continuous through the EH, yet its sth derivative with respect to U diverges there (as $\propto \ln z_+$). Therefore, Eq. (69) applies to $m=0$ as well, where for this case the function ϕ_r is given by Eq. (70).

It is instructive to compare between the asymptotic behavior of the $s\neq 0$ static solutions in the case $m\neq 0$ [Eqs. (66) and (67)] and in the case $m=0$ [Eqs. (71) and (75)]. Focusing on the asymptotic dependence on Δ (and ignoring for this discussion the oscillatory factor presents in the $m\neq 0$ case); we find that for the $s>0$ fields the regular and irregular solutions "switch roles": For $m\neq 0$ modes the regular solution is the one that behaves like Δ^{-s} at the EH and the irregular solution is the one that behaves like $\propto$const there, whereas for $m=0$ modes the opposite is true. Such an interchange of roles does not occur in the case $s<0$. This effect is explored and explained in detail in Ref. [28]

C. Scalar field case $(s=0)$

For $s=0$ we use the new radial variable

$$\overline{z}\equiv z_+ + z_- = \frac{2r-r_+-r_-}{r_+-r_-} \tag{77}$$

(note the relation $\overline{z}=2z_+ +1=2z_- -1$), to write the static field equation (61) in the form

$$(1-\overline{z}^2)F''(\overline{z})-2\overline{z}F'(\overline{z})+[4m^2\gamma^2(1-\overline{z}^2)^{-1} + l(l+1)]F(\overline{z})=0, \tag{78}$$

where a prime denotes $d/d\overline{z}$. This is the familiar Legendre's differential equation (see, for example, Sec. 3.2 in Ref. [38]). Two independent solutions to this equation are [38]

$$\phi_r^{s=0}=(z_+/z_-)^{im\gamma}F(-l,l+1;1+2im\gamma;-z_+), \tag{79a}$$

and

$$\phi_{ir}^{s=0}=\frac{(\overline{z}^2-1)^{im\gamma}}{\overline{z}^{l+2im\gamma+1}}F(l/2+im\gamma+1,1/2+im\gamma + 1/2;l+3/2;\overline{z}^{-2}), \tag{79b}$$

which are (up to a customary normalization) the *associated Legendre functions* of the first and second kinds, $P_l^{\mu}(\overline{z})$ and $Q_l^{\mu}(\overline{z})$, respectively, with $\mu=2im\gamma$.[10]

Using the EH-asymptotic relations $z_+\simeq e^{2\kappa_+ r_*}=e^{\Omega_+ r_*/\gamma}$ and $z_-\simeq 1$, and recalling that the hypergeometric function appearing in Eq. (79a) is a polynomial (of order l) of z_+, we find $\phi_r^{s=0}\simeq e^{im\Omega_+ r_*}$ near the EH. The "physical" field $\hat{\phi}_r^{s=0}\equiv e^{-im\Omega_+ r_*}\phi_r^{s=0}$ associated with this solution is therefore regular at the EH.

It remains to verify that $\phi_r^{s=0}$ represents the *only* physically regular solution for the scalar field (up to a constant factor). For $m\neq 0$, the solution $\phi_r^{s=0}$ and its complex conjugate $(\phi_r^{s=0})^*$ constitute a complete basis of static solutions. The "physical" field $e^{-im\Omega_+ r_*}(\phi_r^{s=0})^*$ becomes indefinite at the EH (where it behaves like $e^{-2im\Omega_+ r_*}$), and we find that for $m\neq 0$, $\phi_r^{s=0}$ is indeed the only physically regular solution. In the case $m=0$, $\phi_r^{s=0}$ becomes real (it is then the Legendre polynomial, up to a normalization), and a complete basis of solutions is given by $(\phi_r^{s=0},\phi_{ir}^{s=0})$. These two basis functions then admit a relation of the form $\phi_{ir}^{s=0}(\overline{z}) \propto \phi_r^{s=0}(\overline{z})\times\ln(z_+/z_-)+$ polynomial in $\overline{z}$ [see Eq. (24) in Sec. 3.6.2 of [38]]. Since $\phi_r^{s=0}$ is physically regular, it is therefore clear that $\phi_{ir}^{s=0}$ is physically irregular.

In conclusion, because the function $F_{k=0}^{s=0}$ must be a static solution physically regular at the EH, it must be proportional to $\phi_r^{s=0}$. Therefore, Eq. (69) applies to the scalar field too, with ϕ_r given in Eq. (79a).

VI. LATE TIME BEHAVIOR AT THE EH

The LTE, Eq. (50), is an expansion in inverse powers of advanced time v, with r-dependent coefficients. Since along the EH itself r is constant and v takes finite values, this expansion seems especially convenient for analyzing the "late time," $v\gg M$, behavior of the fields at the EH. Potentially, this behavior should be described by the $k=0$ of the LTE. However, a possible divergence or vanishing of various "coefficient" functions $F_k(r)$ at $r=r_+$ may alter this simple picture, leading to a different prediction for the late time power-law decay at the EH. Indeed, as it turns out in this section, there is a case (the one of $s>0$, $m=0$) in which the term $k=1$ is found to dominate the term $k=0$ at the EH.

It is therefore important to analyze also the behavior of the $k\geq 1$ terms at the EH. This task is further motivated by our wish to verify that the LTE is fully consistent with regularity at the EH: It will be shown that for each k there exists a solution F_k physically regular at the EH. These EH-regular functions will then construct, via the LTE, a field Ψ^s representing a physical perturbation which is regular along the EH at all v.

With the above motivations in mind, we first derive in this section expressions for the EH-asymptotic behavior of each of the functions $F_{k\geq 1}$.

A. Behavior of the $k>0$ terms at the EH

Each of the functions $F_{k>0}^l$ admits the inhomogeneous equation $D^l F_k^l = Z_k^l$ [Eq. (51)], and is subject to the EH-regularity condition (60). For each $k>0$, the general solution to Eq. (51) has the form

[10]Note that the two independent solutions $\phi_r^{s\neq 0}$ and $\phi_{ir}^{s\neq 0}$, Eqs. (62a) and (62b), degenerate to the single solution (79a) in the scalar field case (i.e., when setting $s=0$).

$$F^l_{k>0}(r)=a_k\phi_r(r)+b_k\phi_{ir}(r)+\phi^{ih}_k(r), \tag{80}$$

where a_k and b_k are (yet) arbitrary coefficients, ϕ_r and ϕ_{ir} are two independent homogeneous solutions (those derived in the previous section), and $\phi^{ih}_k(r)$ is a solution to the inhomogeneous equation.

For each $k>0$, an inhomogeneous solution ϕ^{ih}_k is given by

$$\phi^{ih}_k(r)=\phi_r(r)\int^r\frac{\phi_{ir}(r')Z_k(r')/\Delta(r')}{W(r')}dr' - \phi_{ir}(r)\int^r\frac{\phi_r(r')Z_k(r')/\Delta(r')}{W(r')}dr', \tag{81}$$

in which

$$W=\Delta^{-s-1} \tag{82}$$

is the Wronskian associated with the homogeneous equation $D^lF^l_k=0$. We can now make use of the relation

$$\phi_{ir}=-\phi_r(r)\int^r\phi_r^{-2}(r')W(r')dr' \tag{83}$$

to re-express the above inhomogeneous solution in a more convenient form, as

$$\phi^{ih}_k(r)=\int_{r_1}^r dr'\int_{r_2}^{r'}dr''\frac{\phi_r(r)\phi_r(r'')}{[\phi_r(r')]^2}\frac{W(r')}{W(r'')}\frac{Z_k(r'')}{\Delta(r'')}, \tag{84}$$

where r_1 and r_2 are constant integration limits.[11] This form is obtained from Eq. (81) by first substituting for ϕ_{ir}, using Eq. (83), and then integrating the resulting expression by parts. It is advantageous in that it only involves the homogeneous solution ϕ_r, which is of a more simple form than ϕ_{ir} in all the cases considered in the previous section.

Equation (84) can be used, in principle, to calculate all functions $F^l_k(r)$ in an inductive manner. In general, for each $k>0$ the source function Z^l_k depends on various l modes of the functions F_{k-1} and F_{k-2}, which are to be calculated at previous steps of the induction procedure. As we show below, for each k, the value of one of the coefficients a_k or b_k is dictated by regularity at the EH. The other coefficient is to be specified by matching at large distance, as we explain in Sec. VII.

We now use Eq. (84) to obtain the EH-asymptotic forms of all functions $F^l_{k>0}$. The special case $s>0, m=0$ shall be treated separately from all other cases.

[11]The constant limits r_1 and r_2 will be specified as convenient for each of the various cases analyzed below in separate. Of course, changing these limits amounts to adding a homogeneous solution to F^l_k, which is merely equivalent to re-defining the coefficients a_k or b_k in Eq. (80)

1. $s\leqslant 0$ case and $s>0$ with $m\neq 0$ case

We start with $k=1$. The source $Z^l_{k=1}$ is calculated from Eq. (53), in which, according to the results of the previous section, we set $F^l_{k=0}=a^l_0\phi^l_r$. For $s\leqslant 0$, and also for $s>0$ with $m\neq 0$, we find by Eqs. (62a), (70b), and (79a) that the function $F^l_{k=0}$ has the form $F^l_{k=0}\propto\phi^l_r=\Delta^{-s+im\gamma}\times f_0(r)$, where $f_0(r)$ is a certain function analytic at the EH and nonvanishing there. Substituting this form in Eq. (53) yields

$$Z^l_{k=1}(r)=\Delta^{-s+im\gamma}\times\overline{f_1}(r), \tag{85}$$

where $\overline{f_1}(r)$ is a function analytic at the EH.[12]

With Eqs. (82) and (85), Eq. (84) becomes (for $k=1$)

$$\phi^{ih}_{k=1}=\phi_r(r)\int_{r_+}^r dr'\int_{r_2}^{r'}dr''\frac{[\Delta(r'')]^{-s+2im\gamma}}{[\phi_r(r')]^2[\Delta(r')]^{s+1}}\tilde{f}_1(r''), \tag{86}$$

where $\tilde{f}_1(r'')$ is analytic at the EH, and where we have specified the lower limit of the integration over r' as $r_1=r_+$. Integrating over r'' [recalling $\Delta=(r-r_+)(r-r_-)$], we obtain

$$\phi^{ih}_{k=1}=\phi_r(r)\int_{r_+}^r dr'\frac{[\Delta(r')]^{-s+1+2im\gamma}+\overline{c}}{[\phi_r(r')]^2[\Delta(r')]^{s+1}}\hat{f}_1(r')$$
$$=\phi_r(r)\int_{r_+}^r dr'(\tilde{c}+\overline{c}[\Delta(r')]^{s-1-2im\gamma})\,\overline{\overline{f_1}}(r'), \tag{87}$$

where $\hat{f}_1(r')$ and $\overline{\overline{f_1}}(r')$ are analytic at the EH, $\tilde{c}$ is a certain nonvanishing constant, and $\overline{c}$ is an integration constant, which, of course, depends on the value of the lower integration limit r_2. In the case $s\leqslant 0$, a convenient choice is $r_2=r_+$, which makes $\overline{c}$ vanish. For $s>0$ we take $r_2=2r_+$ (say), as the choice $r_2=r_+$ is forbidden.[13] In that case, the contribution proportional to $\overline{c}$ to the integral in Eq. (87) must coincide (up to a multiplicative constant) with one of the static solutions, ϕ_r or ϕ_{ir}, because changing the integration limit r_2 (thus changing $\overline{c}$) amounts to adding a static solution to Eq. (80). By integrating over r' we find that this contribution has the asymptotic form $\propto\overline{c}\,\Delta^{-im\gamma}\propto\overline{c}e^{-im\Omega_+r_*}$ at the EH; hence it must admit the global form $\propto\overline{c}\phi_{ir}$ [see Eq. (67)]. The term proportional to $\overline{c}$ in $\phi^{ih}_{k=1}$ can therefore be absorbed in the term $b_1\phi_{ir}$ of Eq. (80), by re-defining the coefficient b_1. One is left with the contribution proportional

[12]In deriving Eq. (85) one should notice that when $F^l_{k=0}$ is substituted in Eq. (53), the term containing $dF^l_{k=0}/dr$ and the one containing $F^l_{k=0}/\Delta$ cancel out at the leading order in Δ. As a consequence, leading-order contributions to $Z^l_{k=1}$ arise from all terms in (53), including the interaction terms.

[13]In the case $s=+2$ we can make $\overline{c}$ vanish by taking $r_2=\infty$. However, for $s=+1$ no choice of r_2 nullifies $\overline{c}$.

to $\tilde{c}$, which, after integrating over r', reads $\phi_r \cdot \Delta f_1(r)$, where $f_1(r)$ is a function analytic at the EH.

From Eq. (80) we now find that $F^l_{k=1} = a_1\phi_r + b_1\phi_{ir} + \phi_r \Delta f_1(r)$. The EH-regularity criterion (60) then dictates $b_1 = 0$, finally leading to

$$F^l_{k=1} = \phi_r [a_1 + \Delta \cdot f_1(r)]. \tag{88}$$

Recall (i) that $f_1(r)$ is analytic at the EH, and (ii) that the regular solution is physically regular there in the sense discussed in previous sections. This implies that the function $F^l_{k=1}$ of Eq. (88) is physically regular at the EH (namely, the ''physical'' function $\hat{F}^l_{k=1}$ associated with $F^l_{k=1}$ is mathematically regular there). Note also that, as far as the leading order term in Δ is concerned, $F^l_{k=1}$ has the same asymptotic behavior as $F^l_{k=0}$ at the EH, which is that of the regular static solution: $\propto \Delta^{-s} e^{im\Omega_+ r_*}$.

We can now carry on in an inductive manner, and analyze the terms $k \geq 2$. In general, let us assume that for a given $k' \geq 2$ we have, for all $k < k'$,

$$F^l_k = \phi_r [a_k + \Delta \cdot f_k(r)], \tag{89}$$

where $f_k(r)$ are functions analytic at the EH. (This form was already verified above for $k=0$ and $k=1$.) By substituting in Eq. (53) it is straightforward to show that

$$Z^l_{k'}(r) = \Delta^{-s+im\gamma} \overline{f_{k'}}(r), \tag{90}$$

where $\overline{f_{k'}}(r)$ is analytic at the EH. This, following the same calculation as for $F^l_{k=1}$, leads to $F^l_{k'} = a_{k'}\phi_r + b_{k'}\phi_{ir} + \phi_r \Delta f_{k'}(r)$. The EH-regularity condition (60) then dictates $b_{k'} = 0$, and one finds that Eq. (89) is also valid for $k = k'$. Thus, by induction, Eq. (89) is verified for all $k \geq 0$.

In conclusion, we have constructed EH-regular solutions for each of the functions $F^l_k(r)$. It was found that for all modes l, all functions $F^l_k(r)$ behave near the EH like a regular static solution, $\propto \phi_r$; namely, they all admit the asymptotic form

$$F^l_k(r) \cong a_k \Delta^{-s} e^{im\Omega_+ r_*} \quad \text{near the EH, for all } k \geq 0. \tag{91}$$

2. Case s>0, m=0

In this case, the function $F^l_{k=0} = a_0 \phi^+_r$ is simply a polynomial, admitting the EH-asymptotic form $F^{l,m=0}_{k=0} \cong a_0 \Delta^0$ [see Eq. (71)]. To obtain the source $Z^l_{k=1}$, insert $F^l_{k=0}$ into Eq. (53). This yields

$$Z^{s>0,l,m=0}_{k=1} = 4Msk_0 a_0 (r_+^2 - a^2) \Delta^{-1} + \overline{h}(r), \tag{92}$$

where $\overline{h}(r)$ is a function analytic at the EH. Note that now, since $F^l_{k=0}$ is a polynomial, the asymptotic form of $S^l_{k=1}$ at the EH is dominated by merely the term $\propto \Delta^{-1} F^l_{k=0}$ in Eq. (53), while the other terms (including the derivative term and the interaction terms) contribute only to higher orders in Δ.

To calculate $\phi^{ih}_{k=1}$, we substitute $S^l_{k=1}$ in Eq. (84). Recalling that in the present case the function ϕ_r is a polynomial, one finds

$$\phi^{ih}_{k=1} = 4Msk_0 a_0 (r_+^2 - a^2) \phi^+_r(r) \int_{r_1}^{r} dr' \int_{r_+}^{r'} dr'' \times \frac{[\Delta(r'')]^{s-1} + [\Delta(r'')]^s \tilde{h}(r'')}{[\phi^+_r(r')]^2 [\Delta(r')]^{s+1}}, \tag{93}$$

where $\tilde{h}(r'')$ is analytic at the EH. (Here, we have specified one of the integration limits, $r_2 = r_+$.) Carrying the integration we arrive at

$$\phi^{ih}_{k=1} = 4Mk_0 a_0 \frac{r_+^2 - a^2}{r_+ - r_-} \phi^+_r(r) \int_{r_1}^{r} dr' [\Delta^{-1}(r') + \hat{h}_1(r')] = \tilde{\gamma}_1 \phi^+_r \ln z_+ + \overline{\overline{h}}_1(r), \tag{94}$$

in which $\hat{h}_1(r')$ and $\overline{\overline{h}}_1(r)$ are analytic at the EH, and where the coefficient $\tilde{\gamma}_1$ is given by

$$\tilde{\gamma}_1 = 4Mk_0 a_0 \frac{r_+^2 - a^2}{(r_+ - r_-)^2} \neq 0. \tag{95}$$

By virtue of Eq. (75) we finally obtain, for $F^l_{k=1}$ (in the case $s>0$, $m=0$),

$$\begin{aligned} F^l_{k=1}(r) &= a^+_1 \phi^+_r + b^+_1 \phi^+_{ir} + \phi^{ih}_{k=1} \\ &= a^+_1 \phi^+_r + b^+_1 [\Delta^{-s}(1 + \tilde{\alpha}^+ \Delta + \cdots) \\ &\quad + \tilde{\beta} \phi^+_r \ln z_+] + \tilde{\gamma}_1 \phi^+_r \ln z_+ + \overline{\overline{h}}_1(r). \end{aligned} \tag{96}$$

Now, the EH-regularity criterion (60) forces the ''physical'' function $\hat{F}^l_{k=1} \equiv \Delta^s F^l_{k=1}$ to be perfectly smooth at the EH (where $z_+ = 0$). This implies that $F^l_{k=1}$ must contain no logarithmic terms of the form $\propto \ln z_+$: If such a logarithmic term is present, $\hat{F}^l_{k=1}$ would indeed be continuous at the EH, yet its sth derivative with respect to r or U (which are regular coordinates at the EH) would diverge there. We therefore find that the regularity condition dictates the value of the coefficient b^+_1:

$$b^+_1 = -\tilde{\gamma}_1 / \tilde{\beta} \neq 0. \tag{97}$$

Hence, from Eq. (96) we obtain the form

$$F^l_{k=1}(r) = \Delta^{-s} h_1(r), \tag{98}$$

where $h_1(r)$ is a function analytic at the EH, satisfying

$$h_1(r = r_+) = b^+_1 \neq 0. \tag{99}$$

[The analytic function $h_1(r)$ contains also the terms $a^+_1 \phi^+_r$ and $\overline{\overline{h}}_1(r)$ appearing in Eq. (96), multiplied by Δ^s. Note that the polynomial homogeneous solution does not affect the

TABLE I. The asymptotic behavior of the ''physically regular'' functions $F_k^l(r)$ at the EH. Presented are the leading order in Δ forms of these functions, for the various cases studied in the text. For axially symmetric $(m=0)$ modes the asymptotic behavior depends on whether $s>0$ or $s\leqslant 0$, as discussed in the text. Note that these asymptotic forms are in all cases independent of the multipole number l of the modes under consideration.

Case	$F_{k=0}$	$F_{k=1}$	$F_{k\geqslant 2}$
$am=0, s>0$	Δ^0	Δ^{-s}	$\leqslant \Delta^{-s}$
All other cases	$\Delta^{-s}e^{im\Omega_+ r_*}$ for all $k\geqslant 0$		

leading order term of $F_{k=1}^l$ at the EH.] It is important that $h_1(r)$ does not vanish at $r=r_+$: It implies that at the EH itself (in the case of $s>0$ with $m=0$) the term $F_{k=1}^l$ dominates the term $F_{k=0}^l$, which is only proportional to Δ^0 there. The application of this result to the late time tail will be discussed below.

We now turn to the terms $k\geqslant 2$, and show by mathematical induction that for all $k\geqslant 1$ there exists a solution admitting the form

$$F_{k\geqslant 1}^l(r)=\Delta^{-s}h_k(r) \tag{100}$$

[with $h_k(r)$ being functions analytic at the EH], and thus satisfying the EH-regularity condition (60). This form was already verified in the case $k=1$, for which we also showed that $h_1(r_+)\neq 0$.

Let us assume that Eq. (100) applies for all $1\leqslant k<k'$ where $k'>1$ is arbitrary, and show that it is also valid for $k=k'$. Substituting the form (100) into Eq. (53) we find $Z_{k'}^l=\Delta^{-s}\overline{h}_{k'}(r)$ where $\overline{h}_{k'}(r)$ is analytic at the EH. We then have

$$\begin{aligned}\phi_{k'}^{ih}&=\int_{r_1}^{r}dr'\int_{r_+}^{r'}dr''\frac{\phi_r^+(r)\phi_r^+(r'')}{[\phi_r^+(r')]^2}\frac{\overline{h}_{k'}(r'')}{[\Delta(r')]^{s+1}}\\&=\phi_r^+(r)\int_{r_1}^{r}dr'\Delta^{-s}(r')\widetilde{h}_{k'}(r')\\&=\widetilde{\gamma}_{k'}\phi_r^+(r)\ln z_+ +\Delta^{-s+1}\hat{h}_{k'}(r),\end{aligned} \tag{101}$$

where $\overline{h}_{k'}$, $\widetilde{h}_{k'}$, and $\hat{h}_{k'}$ are analytic at the EH, and $\widetilde{\gamma}_{k'}$ are constant coefficients. (Here we have taken $r_2=r_+$.) For $F_{k'}^l$, we thus obtain

$$\begin{aligned}F_{k'}^l(r)=&a_{k'}^+\phi_r^+ + b_{k'}^+[\Delta^{-s}(1+\widetilde{\alpha}^+\Delta+\cdots)+\widetilde{\beta}\ln(z)\phi_r^+]\\&+\widetilde{\gamma}_{k'}\ln z_+\phi_r^+ +\Delta^{-s+1}\hat{h}_{k'}(r).\end{aligned} \tag{102}$$

This result is analogous to Eq. (96), only here we have not ruled out the possibility that some of the coefficients $\widetilde{\gamma}_{k'}$ may vanish (for $k'>1$) .

For the regularity condition to be met, we must now have $b_{k'}^+=-\widetilde{\gamma}_{k'}/\widetilde{\beta}$. Consequently, Eq. (100) is recovered for $k=k'$ as well. By induction, then, we conclude that Eq. (100) applies to all $k\geqslant 1$.

Recall that it is possible for some of the coefficients $\widetilde{\gamma}_k$ (with $k\geqslant 2$) to vanish. If, for a certain k, $\widetilde{\gamma}_k$ happens to vanish, then, to maintain EH regularity, one must have $b_k^+=0$. In that case Eq. (102) yields for F_k^l a divergence rate slower than Δ^{-s} at the EH (for this specific k). This, of course, does not contradict Eq. (100), which should be regarded as merely setting an upper bound to the divergence rate of the functions $F_{k\geqslant 2}^l$. It is only for $F_{k=1}^l$ that we verified the actual asymptotic behavior $\propto\Delta^{-s}$, by showing $h_1(r_+)\neq 0$. This information, however, would be sufficient for the late time analysis at the EH.

The above results, concerning the behavior of the physically regular functions $F_k^l(r)$ at the EH, are arranged in Table I. The table shows the leading-order forms of these functions for the various cases studied above. Below we use these results to discuss the late time behavior of the fields Ψ^s along the EH.

B. Late time tail along the EH

When discussing the field behavior along the EH, it is most natural to refer to the ''physical field'' $\hat{\Psi}^s=\Delta^s\Psi^s$, which, by construction, is a linear combination (with regular coefficients) of the regular Weyl or Maxwell components. Expressing the LTE in terms of $\hat{\Psi}^s$, and using Eq. (58), we find

$$\begin{aligned}&\hat{\Psi}^s(v,r,\theta,\widetilde{\varphi}_+)\\&\quad=\sum_{k=0}^{\infty}\sum_{l,m}Y^{slm}(\theta,\widetilde{\varphi}_+)e^{im\Omega_+ v}\\&\quad\quad\times[\Delta^s e^{-im\Omega_+ r_*}F_k^{slm}(r)]v^{-k_0-k}.\end{aligned} \tag{103}$$

Here, the factor in the square brackets is the function $\hat{F}_k^l(r)$ which, by the above construction, is regular at the EH (for all k). Recall also that the angular dependence here is EH regular, and that the v-dependent factors take finite values at the EH. Thus, each of the terms in the sum over k in Eq. (103) is indeed physically regular at the EH.

Now, at large v, the field Ψ^s should, potentially, be dominated by the $k=0$ term in Eq. (103). For the $s\leqslant 0$ fields and for $m\neq 0$ modes of the $s>0$ fields, we find from Table I that at the EH itself the factor in the square brackets in Eq. (103) admits $[\ ]_k\propto$const for all $k\geqslant 0$. Therefore, in these cases, the late time decay of Ψ^s along the EH is indeed dominated by the $k=0$ term, with other terms smaller by factors of $1/v$. Let us denote by Ψ^{sm} the part of Ψ^s which includes all multipole modes l of a given m. To leading order in $1/v$, we then find for all modes m of the $s\leqslant 0$ fields, and for non-axially symmetric $(m\neq 0)$ modes of the $s>0$ fields,

$$\hat{\Psi}^{sm}(r=r_+)=\sum_l a_0^l Y^{slm}(\theta,\widetilde{\varphi}_+)e^{im\Omega_+ v}v^{-k_0}. \tag{104}$$

The situation is different in the case of axially symmetric ($m=0$) modes of the $s>0$ fields. Here, the factor in the square brackets in Eq. (103) vanishes at the EH like Δ^s for $k=0$, whereas for $k=1$ it is finite. Hence, in this case, the $k=1$ term dominates the $k=0$ term. For each of the $k\geqslant 2$ terms the above factor is at most finite (and may even vanish for some k); hence these terms are negligible with respect to the $k=1$ term at large v. We conclude that for axially symmetric modes of the $s>0$ fields the late time behavior along the EH is dominated by the $k=1$ term:

$$\hat{\Psi}^{s>0,m=0}(r=r_+)=\sum_l b_1^+ Y^{sl,m=0}(\theta)v^{-k_0+1}, \quad (105)$$

to leading order in $1/v$ (here, the coefficient b_1^+ is also l dependent).

At this stage, we still do not know the mode composition of the above leading-order tails; neither can we tell what the power index k_0 and the amplitude coefficients are. These pieces of information will be obtained below by matching the LTE to the form of the late time field at null infinity.

However, one feature of the behavior along the EH is already manifested in Eq. (104) above: Non-axially symmetric ($m\neq 0$) modes of the fields do not exhibit a strict power-law decay along the EH; rather, the amplitude of the power-law tail *oscillates* along the null generators of the EH, with an (advanced time) frequency $m\Omega_+$. This phenomenon was first observed by Ori [22], and was further analyzed in Refs. [21,23,24].

We comment that the above oscillations are not manifested when using the Kerr coordinate $\tilde{\varphi}$, defined by $d\tilde{\varphi}=d\varphi+(a/\Delta)dr$, instead of $\tilde{\varphi}_+$ (cf. Ref. [18], which adopts the coordinate $\tilde{\varphi}$). Both coordinates are regular at the EH; however, the horizon's null generators are lines of constant $\tilde{\varphi}_+$ but varying $\tilde{\varphi}$.[14] Note that the oscillation of the scalar field along the horizon's null generators is a coordinate-independent phenomenon.

VII. LATE TIME BEHAVIOR AT FIXED $r>r_+$

In this section we obtain the global late time behavior of any of the modes of the field Ψ^s at any fixed value of $r>r_+$. This task is to be accomplished in two steps: First, we consider the mode $l=l_0$ (for each given s and m), which in Sec. III we found to be the (single) dominant mode at null infinity at late time. By evaluating the form of this mode of the LTE at null infinity, and comparing it to the form obtained independently in Sec. III, we derive the unknown power index k_0, as well as the amplitude coefficient of the late time tail at fixed r. Provided with the value of k_0, we can then carry on and, in the second step, obtain the tail form of all other modes at fixed $r>r_+$. In particular, we then find that the single mode l_0 dominates the behavior of the field Ψ^{sm} also at fixed r.

A. Mode $l=l_0$

Substituting $v=t+r_*$ in Eq. (50), we find that along any $r=\text{const}>r_+$ world line, at the late time limit $t\gg|r_*|$, the behavior of the mode Ψ^{sl_0m} (for each given s and m) is described by

$$\Psi^{sl_0m}=Y^{sl_0m}(\theta,\varphi)F_{k=0}^{sl_0m}(r)t^{-k_0}$$
$$=a_0^{l_0}Y^{sl_0m}(\theta,\varphi)\phi_r^{l_0}(r)t^{-k_0} \quad (106)$$

to leading order in $|r_*|/t$, where the second equality is due to Eq. (69). To obtain the unknown power index k_0 and the amplitude coefficient $a_0^{l_0}$, we now match the LTE to the form of the field at null infinity, as derived in Sec. III.

In order for Eqs. (48) and (50) to agree at null infinity for the mode $l=l_0$, we must have

$$\left[\sum_{k=0}^{\infty} F_k^{sl_0m}(r)v^{-k_0-k}\right]_{\text{at scri}+}\equiv\psi_{\text{LTE}}^{l_0}$$
$$=\alpha_0^{l_0}I_0^{l_0}v^{-2s-1}u^{-(l_0-s+2)}. \quad (107)$$

Here, $\psi_{\text{LTE}}^{l_0}$ denotes the time-radial part of the mode l_0 at null infinity, as calculated from the LTE, whereas the expression on the right-hand side (RHS) is the one derived in Sec. III using the iterative expansion scheme. As it turns out (see below), all terms k of the sum on the LHS of this equation contribute in the same order of magnitude at null infinity, and thus should be all summed up when evaluating $\psi_{\text{LTE}}^{l_0}$. To that end, we first need to obtain the large-r asymptotic form of all functions $F_k^{l_0}(r)$.

Starting with $k=0$, we have, from Eq. (69), $F_0^{sl_0m}=a_0^{l_0}\phi_r^{l_0}$. Using Eqs. (66) and (72) we find the large-r asymptotic form

$$F_{k=0}^{sl_0m}(r\gg r_+)\cong\gamma_0^{l_0}r^{l_0-s}, \quad (108)$$

where the constant coefficient $\gamma_0^{l_0}$ is given by

$$\gamma_0^{l_0}=a_0^{l_0}\times\begin{cases}B_{-s}^{l_0,m=0} & \text{for } s>0 \text{ with } m=0,\\ B_s^{l_0m} & \text{in all other cases}\end{cases} \quad (109)$$

[with the coefficients $B_s^{l_0m}$ given in Eq. (68)].

To analyze the functions $F_{k\geqslant 1}^{l_0}$ we use Eq. (80), with the coefficients b_k taken to be the ones determined above by EH-regularity considerations (e.g. $b_k=0$ for $m\neq 0$). We

[14] To see that, we point out that, in the $(v,r,\theta,\tilde{\varphi})$ system, the null generators of the EH are lines of $\theta=\text{const}$, $r=r_+$, $\tilde{\varphi}=2a\lambda$, and $v=2(r_+^2+a^2)\lambda$, where λ is an affine parameter along the generators—see Sec. 33.6 in Ref. [39]. Thus, along the null generators we find $\tilde{\varphi}=\Omega_+v$. Now, at the EH the two coordinates $\tilde{\varphi}$ and $\tilde{\varphi}_+$ are related by $d\tilde{\varphi}=d\tilde{\varphi}_++\Omega_+dv$, from which we conclude that $d\tilde{\varphi}_+=0$ along the null generators of the EH.

now show by mathematical induction that for all k, these functions admit the large-r asymptotic form

$$F_k^{l_0}(r \gg r_+) \cong \gamma_k^{l_0} r^{l_0-s+k} \tag{110}$$

(to leading order in r), where $\gamma_k^{l_0}$ are constant coefficients. To that end, we assume that Eq. (110) applies to all $k<k'$ (where k' is an arbitrary integer greater than zero), and verify its validity to $k=k'$.

First, we must calculate the large-r asymptotic form of the source function $Z_{k'}^{l_0}$. From Eq. (53) we obtain, to leading order in r,

$$Z_{k'}^{l_0} \cong 2(k_0+k'-1)[r^2(dF_{k'-1}^{l_0}/dr) + rF_{k'-1}^{l_0}] \cong \hat{\gamma}_{k'}^{l_0} r^{l_0-s+k'}, \tag{111}$$

where $\hat{\gamma}_{k'}^{l_0} = 2(k_0+k'-1)(l_0-s+k')\gamma_{k'-1}^{l_0}$ are constants.[15]

Substituting this leading-order form in Eq. (84) and performing the double integration, we obtain, to leading order in r, $\phi_{k'}^{ih} \cong \gamma_{k'}^{l_0} r^{l_0-s+k'}$, with

$$\gamma_{k'}^{l_0} = \frac{\hat{\gamma}_{k'}^{l_0}}{k'(2l_0+k'+1)} = \frac{2(k_0+k'-1)(l_0-s+k')}{k'(2l_0+k'+1)}\gamma_{k'-1}^{l_0}.$$

Thus, for all $k'>0$, the contribution from $\phi_{k'}^{ih}$ to $F_{k'}^{l_0}$ at large r [via Eq. (80)] dominates the contribution from the homogeneous solutions, which is at most $\sim r^{l_0-s}$. We therefore have $F_{k'}^{l_0} \cong \phi_{k'}^{ih}$ at large r; hence Eq. (110) is satisfied for $k=k'$ as well. We also know by Eq. (108) that Eq. (110) is valid for $k=0$. This, by mathematical induction, verifies Eq. (110) for all $k \geq 0$, with the coefficients $\gamma_k^{l_0}$ given by

$$\gamma_k^{l_0} = \frac{(2l_0+1)!\,\gamma_0^{l_0}}{(k_0-1)!(l_0-s)!}\left[\frac{2^k(k_0+k-1)!(l_0-s+k)!}{k!(2l_0+k+1)!}\right]. \tag{112}$$

We are now in position to evaluate the sum over k on the LHS of Eq. (107) at null infinity. Substituting for $F_k(r)$ [using Eq. (110)], and recalling $r \cong r_* = (v-u)/2$ at large r, we obtain

$$\psi_{\rm LTE}^{l_0} \cong v^{l_0-s-k_0} \sum_{k=0}^{\infty} \gamma_k^{l_0}\left[\frac{1}{2}\left(1-\frac{u}{v}\right)\right]^{l_0-s+k}. \tag{113}$$

To evaluate the sum of this power series at null infinity, we make use of the auxiliary identity, valid for $|q|<1$,

$$\sum_{k=0}^{\infty} \frac{(k_0+k-1)!(l-s+k)!}{k!(2l+k+1)!} q^k = q^{-2l-1}\left[q^{k_0-1}\left(\frac{q^{l-s}}{1-q}\right)^{(l-s)}\right]^{(k_0-2l-2)}, \tag{114}$$

where the derivatives on the RHS are with respect to q. [To prove this identity, insert the power expansion $q^{l-s}/(1-q) = q^{l-s}\sum_{k=0}^{\infty} q^k$ into the RHS.] We now make the substitutions $q \to (1-u/v)$ and $l \to l_0$. At $v \gg u$ (recall that at null infinity $v \to \infty$ whereas u takes finite values), the expression on the RHS of Eq. (114) is then dominated by the term resulting from $k_0-l-s-2$ differentiations of the factor $(1-q)^{-1}$ — which yields $(k_0-l_0-s-2)!(u/v)^{-(k_0-l_0-s-1)}$. Substituting this result in Eq. (113), we obtain

$$\psi_{\rm LTE}^{l_0} \cong \frac{2^{-l_0+s}(2l_0+1)!(k_0-l_0-s-2)!\,\gamma_0^{l_0}}{(k_0-1)!(l_0-s)!} \times v^{-2s-1} u^{-(k_0-l_0-s-1)}. \tag{115}$$

Comparison of Eqs. (107) and (115) finally yields

$$k_0 = 2l_0+3, \tag{116}$$

and also, with the help of Eq. (109),

$$a_0^{l_0} = \alpha_0^{l_0} I_0^{l_0} \times \frac{2^{l_0-s+1}(l_0+1)}{l_0-s+1} \times \begin{cases} 1/B_{-s}^{l_0,m=0} & \text{for } s>0 \text{ with } m=0, \\ 1/B_s^{l_0 m} & \text{in all other cases.} \end{cases} \tag{117}$$

The parameter k_0 derived above is, in view of Eq. (106), the power index of the l_0 mode's tail at fixed $r>r_+$ for each given s and m [recall $l_0=\max(|s|,|m|)$]. The coefficient $a_0^{l_0}$ describes the amplitude of this tail, with Eq. (117) relating it to the amplitude of the l_0-mode's tail at null infinity (which is $\alpha_0^{l_0} I_0^{l_0}$).

B. Modes $l>l_0$

We now turn to analyze the behavior of the other modes, $l>l_0$, at fixed r. Here, the coupling between modes in the Kerr case shall appear to have a crucial effect on the form of the late time tail. To discuss this effect, it is most instructive to first consider a situation without coupling. Thus, at first, we shall ‘‘turn off’’ the interactions between modes by ignoring for a while all terms in Z_k^l [Eq. (53)] which couples the mode l to other modes. (This will qualitatively describe the situation in the Schwarzschild case.) Then, in the second part of the following discussion, we restore the coupling (by taking into account all terms in Z_k^l), and discuss its important effect on the late time tail of decay.

---

[15]Actually, the source $Z_{k'}^{l_0}$ contains also contributions from other modes ($l=l_0+1, l_0+2$). However, as we show later in this section, such contributions are negligible at large r, and do not affect the asymptotic form (111).

1. Case with no coupling between modes

Considering a mode $l>l_0$ and ignoring its interaction with other modes, we can follow the same calculation as for the l_0 mode, and obtain at null infinity $\psi^l_{\rm LTE} \propto a^l_0 v^{-2s-1}u^{-(k_0-l-s-1)}$, in a full analogy with Eq. (115).[16] However, since for any $l>l_0$ we have $k_0-l-s-1=2l_0-l-s+2<l-s+2$, this result cannot match the known u power at null infinity, $u^{-(l-s+2)}$, unless $a^l_0=0$. The ''boundary condition'' at null infinity thus dictates the vanishing of a^l_0, and thus of F^l_0, for all $l>l_0$. This, of course, means that the modes $l>l_0$ will decay faster than t^{-k_0} at fixed r.

This argument demonstrates that in order to determine the form of the late time tail at fixed r, it is necessary to find out, for each given mode $l>l_0$, what is the smallest k value for which F^l_k does not vanish. Denoting this value by $\tilde{k}(l)$, we find that the tail of any mode l is given by $t^{-k_0-\tilde{k}(l)}=t^{-2l_0-3-\tilde{k}(l)}$.

What is $\tilde{k}(l)$ then? We shall answer this question by matching the LTE at null infinity for each mode $l>l_0$. First, we must obtain the asymptotic form of the functions F^l_k at large r. By definition of $\tilde{k}$ we have $F^l_{k<\tilde{k}}=0$. If no interaction occurs between various modes, then the function $F^l_{k=\tilde{k}}$ must be a static solution (the regular one): $F^l_{k=\tilde{k}}=a^l_{\tilde{k}}\phi_r \cong a^l_{\tilde{k}} r^{l-s}$ at large r, where the constant coefficient $a^l_{\tilde{k}}$ does not vanish by definition of $\tilde{k}$. Following the same calculation as for the l_0 mode (again, with the interaction between modes ignored for a while), we obtain for each $k\geqslant\tilde{k}$ the large-r asymptotic form

$$F^l_{k\geqslant\tilde{k}} \cong \gamma^l_k r^{l-s+(k-\tilde{k})}, \tag{118}$$

where γ^l_k are constant coefficients. Hence, for each mode $l>l_0$ we find, to leading order in u/v,

$$\psi^l_{\rm LTE} \cong \sum_{k=\tilde{k}}^{\infty} \gamma^l_k r_*^{l-s+k-\tilde{k}} v^{-k_0-k} = v^{-k_0+l-s-\tilde{k}} f(u/v), \tag{119}$$

where $f\equiv\sum_{k=\tilde{k}}^{\infty}\gamma^l_k[(1-u/v)/2]^{l-s+k-\tilde{k}}$ is a function of u/v only.[17]

Now, we know independently from Sec. III that any l mode decays at null infinity with a tail of the form $\propto v^{-2s-1}u^{-(l-s+2)}$. By comparing the v power in Eq. (119) to this form, we find (setting $k_0=2l_0+3$) that the function $f(u/v)$ must admit the asymptotic form $f(u/v) \cong (u/v)^{-2l_0-\tilde{k}+l+s-2}$. Then, by comparing the power of u, we finally obtain

$$\tilde{k}(l)=2(l-l_0) \quad \text{(no coupling; SBH case)}. \tag{120}$$

We conclude that for each mode $l>l_0$, the form of the field at null infinity (acting as a boundary condition) dictates the vanishing of all term of the LTE with $k<2(l-l_0)$. The first nonvanishing term, the one with $k=2(l-l_0)$, exhibits the late time tail $t^{-k_0-2(l-l_0)}=t^{-2l-3}$. Other terms, the ones with $k>2(l-l)$, decay faster at late time. Therefore, substituting $v=t-r_*$ in Eq. (50), we find, for each of the modes $l\geqslant l_0$,

$$\Psi^{slm}(t\gg|r_*|) \propto Y^{slm}(\theta,\varphi)\phi^l_r(r)t^{-2l-3} \quad \text{(no coupling; SBH case)} \tag{121}$$

to leading order in $|r_*|/t$.

2. Effect of coupling between modes

Let us now ''turn back on'' the interactions between modes, and consider their effect on the late time tail. For each of the functions F^l_k, the inhomogeneous part ϕ^l_{ih} in Eq. (80) now contains contributions not only from the functions F^l_{k-1} and F^l_{k-2} but also from $F^{l\pm2}_{k-2}$, and (for $s\neq0$) from $F^{l\pm1}_{k-1}$ and $F^{l\pm1}_{k-2}$. For example, the function $F^{l_0+1}_{k=1}$ admits (for $s\neq0$) a nonvanishing source $\propto aF^{l_0}_{k=0}=aa^{l_0}_0\phi^{l_0}_r$. Since we have $a^{l_0}_0\neq0$ (by definition of k_0), then, necessarily, the function $F^{l_0+1}_{k=1}$ is nonvanishing. Thus, for the mode $l=l_0+1$ we find $\tilde{k}=1$, implying a late time tail of the form $t^{-k_0-\tilde{k}}=t^{-2l_0-4}$. This is different than in the ''no-coupling'' situation, in which for the $l=l_0+1$ mode we had $\tilde{k}=2$ [Eq. (120)], leading to the tail t^{-2l_0-5} [Eq. (121)]. We may summarize the result in this example by saying that the interaction ''excites'' the mode l_0+1 already at $k=1$, whereas the boundary condition at null infinity ''excites'' this mode only at $k=2$. We arrive at the conclusion that for this mode the effect of interaction dominates the late time behavior.

The generalization of this result to all $l>l_0$ is straightforward: For each $l\geqslant l_0$, the function $F^l_{\tilde{k}}$ [recall that $\tilde{k}(l)$ is, for a given mode l, the smallest k for which F^l_k is nonvanishing] serves as a source, via $Z^l_{\tilde{k}}$, to $F^{l+2}_{\tilde{k}+2}$ and (for $s\neq0$) to $F^{l+1}_{\tilde{k}+1}$ and $F^{l+1}_{\tilde{k}+2}$. These three functions are then necessarily nonvanishing. Since we have $\tilde{k}(l=l_0)=0$, we obtain in the case $s\neq0$, $\tilde{k}(l)=l-l_0$ for all $l\geqslant l_0$. In the scalar field case ($s=0$) the interaction couples only between next-to-nearest modes. In this case, the mode $l=l_0+1$ (for example) is not excited by interaction, but rather by the boundary condition at null infinity, yielding $\tilde{k}(l=l_0+1)=2$ [see Eq. (120)]. In general, for $s=0$ we thus find $\tilde{k}=l-l_0$ for even $l-l_0$, and $\tilde{k}=l-l_0+1$ for odd $l-l_0$. We can express the results in all the above cases by writing

[16] Recall that in obtaining Eq. (115) for the l_0 mode, the interaction between modes was not taken into account (this will be justified below). Thus, the analogy with this case is straightforward: one only needs to replace l_0 with l in Eq. (115).

[17] Here we do not sum over k in an explicit manner, as we did for the l_0 mode. Rather, we use a simpler argument, which is yet somewhat less rigorous.

$$\tilde{k}(l)=l-l_0+q \quad \text{(with coupling)}, \tag{122}$$

where

$$q=\begin{cases}1 & \text{for } s=0 \text{ with odd } l-l_0,\\ 0 & \text{otherwise.}\end{cases} \tag{123}$$

By comparing Eqs. (120) and (122) we find that interaction excites any of the modes $l>l_0$ at *smaller* k than do the boundary conditions at null infinity. In other words, it is the *interaction* (rather than the boundary conditions at null infinity) which first excites any of these modes, and thus determines the form of the leading order tail at late time. This tail shall admit the form $t^{-k_0-\tilde{k}}=t^{-l-l_0-3-q}$.

More precisely, for each given s and m, and for each of the modes $l\geqslant l_0$, we find from Eq. (50), to leading order in $|r_*|/t$,

$$\Psi^{slm}(t\gg|r_*|) \propto Y^{slm}(\theta,\varphi)F^l_{k=l-l_0+q}(r)t^{-(l+l_0+3+q)} \quad \text{(Kerr case)}. \tag{124}$$

Note that for each $l>l_0$ (for the scalar field—for each $l>l_0+1$) the late time tail decays *slower* than the corresponding tail of the same mode in the Schwarzschild case, Eq. (121). These slowly decaying tails are produced by the interaction, as discussed above. On the other hand, the mode l_0 (for each s and m), whose form is not shaped by interaction, exhibits the same decay rate in both the SBH and the KBH cases: t^{-2l_0-3}.

There now remains an important question to deal with: In what way does the matching of the LTE at null infinity change under the effect of interaction between modes? Does the LTE remain consistent with the boundary condition there? In particular, we would like to show that the calculation made above for the l_0 mode is still valid even when the interaction is taken into account.

To answer these questions we must first examine the large-r asymptotic form of the various functions F^l_k, this time taking into account also the effect of interaction between modes. As we recall from Eq. (80), each function F^l_k contains a ''homogeneous'' part, $a^l_k\phi^l_r+b^l_k\phi^l_{ir}$, and an ''inhomogeneous'' part, $[\phi^l_k]^{ih}$. The homogeneous part vanishes identically for all $k<2(l-l_0)$ [by virtue of Eq. (120)], and behaves as r^{l-s} at large r for $k\geqslant 2(l-l_0)$. The inhomogeneous part vanishes identically for all $k<l-l_0+q$ [by virtue of Eq. (122)]. Its large-r form can be calculated for each k and l using Eqs. (53) and (84), given the large-r form of all functions $F^{l'}_{k'}$ which serve as sources to F^l_k.

Using Eqs. (53) and (84), one may formulate two practical calculation rules:

(i) If the function F^l_{k-1} admits the large-r form $\sim r^p$ (where p is some positive power index, and a ''$\sim$'' symbol represents the asymptotic form to leading order in r), then the contribution to F^l_k due to the term in Z^l_k involving F^l_{k-1} would be of order $\sim r^{p+1}$. For example, the source function $F^{l_0}_{k=0}$ (which is an EH-regular homogeneous solution and thus admits $\sim r^{l_0-s}$) induces on $F^{l_0}_{k=1}$ (through $Z^{l=l_0}_{k=1}$) a nonvanishing contribution of order $\sim r^{l_0-s+1}$.

(ii) The contribution to a function F^l_k due to interaction with a source function $F^{l'\neq l}_{k'}$ admits the same large-r form as the source function itself. This is the situation with all interaction sources of F^l_k and, also, with the source function F^l_{k-2}. For example, the source $F^{l_0}_{k=0}\sim r^{l_0-s}$ induces on $F^{l_0+2}_{k=2}$ (through $Z^{l=l_0+2}_{k=2}$) a nonvanishing contribution of order $\sim r^{l_0-s}$. It also, for example, contributes in order $\sim r^{l_0-s}$ to $F^{l_0}_{k=2}$ (through $Z^{l=l_0}_{k=2}$).

With the above two ''rules of thumb'' one can now inductively construct expressions for the large-r form of each of the functions F^l_k, starting with $F^{l_0}_{k=0}\sim r^{l_0-s}$. This easily yields $F^l_k\sim r^{(l_0-s)+k-(l-l_0)}$, namely,

$$F^l_k(r\gg r+)\sim r^{2l_0-l-s+k} \tag{125}$$

for each $l\geqslant l_0$ and $k\geqslant\tilde{k}(l)=l-l_0+q$.[18] This is the same asymptotic form as obtained for the functions F^l_k in the absence of interaction, for $k\geqslant 2(l-l_0)$ — see Eq. (118) with $\tilde{k}=2(l-l_0)$. [The important difference is that without an interaction the terms $l-l_0+q\leqslant k<2(l-l_0)$ all vanish identically.]

We conclude the following:

(I) For any mode l, the ''inhomogeneous'' part ($\sim r^{2l_0-l-s+k}$) of F^l_k dominates its ''homogeneous part'' ($\sim r^{l-s}$) at large r for all k except $k=2(l-l_0)$, where both parts contribute to order $\sim r^{l-s}$ [recall that the homogeneous contribution vanishes identically for $k<2(l-l_0)$]. Thus, for each mode l, only one arbitrary parameter [the one belonging to the homogeneous solution at $k=2(l-l_0)$] is involved in the leading order form at large r—as we may expect. (For each l, this parameter is to be determined, in principle, by matching at null infinity.) Note also that the first nonvanishing function of each l (namely $F^l_{\tilde{k}}$), is always proportional to the parameter $a^{l_0}_0$, which originates from the mode $l=l_0$ at $k=0$, and ''propagates'' through interaction to higher modes. [This parameter was determined above, Eq. (117), by matching the mode l_0 at null infinity.]

(II) Whether the interactions are taken into account or not, one obtains the large-r form $F^l_k\propto r^{2l_0-l-s+k}$ [for all $k\geqslant 2(l-l_0)$], though with different proportion coefficients. That difference only affects the amplitude of the function ψ^l_{LTE}

[18] In fact, this result, Eq. (125), is not completely accurate, as the functions $F^{l>l_0}_{k>\tilde{k}}$ turn out to involve logarithmic factors which complicate the situation. This logarithmic dependence will be discussed at the end of this section; meanwhile we shall ignore it to make the discussion more clear, and refer only to the power-law dependence of the functions F^l_k (which is not affected by the presence of the logarithmic factors).

[e.g. in Eq. (119)], yet the matching of the modes $l>l_0$ at null infinity, discussed above, remains qualitatively the same.

(III) As to the effect of the interaction on the mode l_0: By Eq. (125) we have for all k, at the leading order in r, $F_k^{l_0} \sim r^{l_0-s+k}$. This function has, in general, interaction-induced contributions coming from F_{k-1}^{l+1}, F_{k-2}^{l+1}, and F_{k-2}^{l+2}, which by Eq. (125) and the above rule (ii) are of order $\sim r^{l_0-s+k-2}$, $\sim r^{l_0-s+k-3}$, and $\sim r^{l_0-s+k-4}$, respectively. Thus, the large-r leading order behavior of the functions $F_k^{l_0}$ is not affected by interaction with other modes. This result is valid only for the l_0 mode (and, for $s=0$, also for l_0+1) which only interacts with modes of larger l. It justifies ignoring the interactions when evaluating the behavior of the mode l_0 at null infinity, as we did above. Particularly, the values derived above for the parameter k_0 [Eq. (116)] and for the coefficient $a_0^{l_0}$ [Eq. (117)] remain valid also with interaction between modes taken into account.

Finally, it should be commented on that in the above discussion (regarding the $l>l_0$ modes) we have ignored a certain complication for the sake of clarity: Actually, the integration in Eq. (84) produces, for certain values of k and l, a logarithmic dependence on r. This leads to an asymptotic form of F_k^l which is not strictly a power law [as in Eq. (125)], but, in fact, having the form $F_k^l \sim r^{2l_0-l-s+k}(\ln r)^L$. It can be shown that for each l and k, the logarithmic power L may take only integer values between 0 and $l-l_0$. Particularly, we find no logarithmic dependence in all functions F_k belonging to the mode l_0, and thus no modification is required in the above analysis for this mode. Also, we can show that $L=0$ for all modes $F_{\tilde{k}}^l$ (i.e. for the first nonvanishing function F_k of each mode l), and therefore the functions F_k^l in Eq. (124) exhibit no logarithmic dependence at large r. Finally, the matching of the $l>l_0$ modes at null infinity, discussed above, was based merely on the power-law dependence of the functions $F_k^l(r)$, which is not affected by the presence of the logarithmic factors. However, it is not clear to us whether the logarithmic factors themselves properly match at null infinity [recall that in Sec. III the logarithmic dependence (in u) of the modes $l>l_0$ at null infinity has not been fully investigated]. This question remains open.

VIII. SUMMARY AND DISCUSSION

In this paper we have explored analytically the late time decay of the Newman-Penrose scalars Ψ^s (representing scalar, electromagnetic, and gravitational perturbations) in the background of a realistic Kerr black hole. Our analytic method can be summarized as follows: We assume that at late time each of the fields Ψ^s admits the *late time expansion*, Eq. (50). This reduces the master perturbation equation to a hierarchy of ordinary differential equations for the radial functions $F_k^l(r)$. The homogeneous part of each of these equations is just the static field equation in Kerr spacetime, to which there exists an analytic basis of exact solutions. In addition, for each l and $k>0$, each of these equations possesses an inhomogeneous part depending on functions $F_{k'<k}$ (including functions which belong to other modes). Using the Wronskian method we can then explore, in an inductive manner, the general solution for each of the functions F_k^l. Each of these solutions contains, in advance, two unknown parameters. One of these parameters is determined by regularity requirements at the EH. The other parameter is determined by the form of the field at null infinity (serving as a boundary condition). To obtain the behavior of the fields at null infinity, we apply the *iterative scheme*, as described in Sec. III.

Following is a summary of our main results. These results are valid in the most realistic initial setup of a compact pulse composed of all multipole modes (and in particular, the lowest radiatable mode l_0 for each value of m; below we also briefly discuss the more special case where this mode is missing).

A. Tail form at fixed $r>r_+$

Along any world line of fixed r outside the KBH, each specific mode l,m decays at late time with the tail

$$\Psi^{slm}(t\gg|r_*|)\propto t^{-(l+l_0+3+q)} \quad \text{for each } l\geqslant l_0=\max(|s|,|m|), \tag{126}$$

to leading order in $|r_*|/t$ [see Eq. (124)]. Recall that $q=0$, except for $s=0$ with odd $l-l_0$ in which case $q=1$.

The most dominant modes of the *overall* field Ψ^s are those with $l=|s|$ and $-|s|\leqslant m\leqslant|s|$. From Eq. (106) we find, to leading order in $|r_*|/t$,

$$\Psi^s(t\gg|r_*|)=\sum_{m=-|s|}^{|s|} a_0^{l=|s|}Y^{s,l=|s|,m}(\theta,\varphi) \times\phi_r^{l=|s|}(r)t^{-(2|s|+3)} \quad \text{(overall behavior)}. \tag{127}$$

Here, the function $\phi_r^l(r)$ is the physically regular static solution, whose exact analytic form is given in Eqs. (62a), (70), and (79a), corresponding, respectively, to the case $s\neq 0$ with $m\neq 0$, the case $s\neq 0$ with $m=0$, and the case $s=0$. The constant coefficient a_0 (which is also m dependent) is related in Eq. (117) to the amplitude of the leading-order tail at null infinity, which, in turn, is expressed as a functional of the initial data function—see Eq. (45) and the Appendix. Note that Eq. (127) constitutes an exact analytic expression (accurate to leading order in $|r_*|/t$) for the late time behavior of the fields Ψ^s, valid *anywhere* at fixed $r>r_+$.

The power-law indices predicted in Eqs. (126) and (127) agree with those obtained by Hod [24] at fixed $r\gg M$ (the result by Hod refers only to this asymptotic domain). The result in Eq. (127) has support from numerical simulations (in 2+1 dimensions) by Krivan *et al.*— see Ref. [18] for $s=0$ and Ref. [19] for $s=-2$. Also, the validity of our prediction, Eq. (126), has recently demonstrated numerically by Krivan [27].

B. Tail form along the EH

It is most natural to express the results at the EH in terms of the ''physical'' fields $\hat{\Psi}^s \equiv \Delta^s \Psi^s$, which are related through an EH-regular transformation to the components of the Maxwell and Weyl tensors (see the discussion in Sec. IV). By virtue of Eqs. (104), (105), and (122), we find each specific mode l,m to decay along the EH with the tail

$$\hat{\Psi}^{slm}(v \gg |r_*|) \propto \begin{cases} v^{-(l+l_0+4+q)} & \text{for } s>0, m=0, \\ e^{im\Omega_+ v} v^{-(l+l_0+3+q)} & \text{in all other cases} \end{cases} \tag{128}$$

(to leading order in $|r_*|/v$), with Ω_+ defined in Eq. (56). Note that for the $s>0$ fields the axially symmetric $(m=0)$ mode decays faster than other modes. Consequently, the late time behavior of the *overall* field $\hat{\Psi}^{s>0}$ is dominated by the non-spherically symmetric, $m\neq 0$, modes. These modes oscillate along the null generators of the EH with (advanced time) frequencies $m\Omega_+$. We find, to leading order in $|r_*|/v$,

$$\hat{\Psi}^s(v \gg |r_*|) = \begin{cases} \displaystyle\sum_{|m|=1}^{|s|} a_0^{l=|s|} Y^{s,l=|s|,m}(\theta,\tilde{\varphi}_+) e^{im\Omega_+ v} v^{-(2|s|+3)}, & \text{overall } s>0 \text{ field}, \\ \displaystyle\sum_{|m|=0}^{|s|} a_0^{l=|s|} Y^{s,l=|s|,m}(\theta,\tilde{\varphi}_+) e^{im\Omega_+ v} v^{-(2|s|+3)}, & \text{overall } s\leq 0 \text{ field}, \end{cases} \tag{129}$$

where the regularized azimuthal coordinate $\tilde{\varphi}_+$ is the one defined in Eq. (55). Thus, the late time behavior of the field $\hat{\Psi}^{s>0}$ along the EH is *characteristically oscillatory*. On the other hand, the behavior of the scalar field $(s=0)$ is characteristically *non-oscillatory*, whereas the field $\hat{\Psi}^{s<0}$ involves both oscillatory and non-oscillatory modes.

As recently discussed by Ori [29], the characteristics of the late time decay along the EH — both the value of the power index and the oscillatory nature of the waves — have important implications to the structure of the infinite blueshift singularity at the inner horizon of the KBH. This singularity is related with the behavior of the ingoing component $\Psi_0 = \Psi^{s=2}$ of the Weyl perturbation. As it turns out [29], this singularity is generically oscillatory.

C. Tail form at null infinity

For each specific mode l,m, the analysis in Sec. III predicts at null infinity a late time tail of the form

$$\Psi^{slm}(u \gg M) \propto Y^{slm}(\theta,\varphi) v^{-2s-1} u^{-(l-s+2)} \times [\text{l.d.}] \tag{130}$$

to leading order in u_0/u and in M/u_0. (We assume here that the initial pulse is emitted at large distance, so that $-u_0 \gg M$.) In this expression, ''[l.d.]'' represents a possible logarithmic dependence of the form $\ln^L[u/(r_+ - r_-)]$, where the power L is some positive integer. Such a logarithmic factor does not occur for the dominant modes l_0 of each m, and also for all modes of a scalar field $(s=0)$. Our analysis indicates that logarithmic factors do occur for the less dominant modes $(l>l_0)$ of $s\neq 0$ field; however, this point was not studied by us in full detail. In any case, for each given s and m, the dominant late time decay at null infinity is described by Eq. (48), in which no logarithmic factors occur.

The *overall* field Ψ^s is dominated at null infinity by the modes with $l=|s|$ and $-|s|\leq m \leq |s|$:

$$\Psi^s(u \gg M) = \sum_{m=-|s|}^{|s|} \alpha_0^{l=|s|} I_0^{l=|s|} Y^{s,l=|s|,m}(\theta,\varphi) \times v^{-2s-1} u^{-(|s|-s+2)} \quad \text{(overall behavior)} \tag{131}$$

to leading order in u_0/u and in M/u_0. Here, $I_0^{l=|s|}$ is a functional whose construction from the initial data is described by Eqs. (45) and (36).

The power-law indices given in Eqs. (130) and (131) are in agreement with Hod's results [23,24], though Hod indicates no logarithmic dependence for any on the modes.

D. Non-generic initial data

We now briefly discuss the case where the initial pulse is of a non-generic mode composition, such that (for given s and m) it does not contain the mode $l=l_0$. For example, what can we say about a case in which, for a $s=\pm 2$ field, the angular dependence of the initial pulse is that of a pure mode $m=0$, $l=4$?

The calculation scheme presented in Sec. VII, based on the LTE, allows one to obtain the power-index of the tail at fixed r regardless of the initial setup, provided only that the power index at null infinity is known. Suppose that, for a specific initial setup and for a certain m, the most dominant mode, $l_0=\max(|s|,|m|)$, falls off at null infinity with a tail of the form $\Psi^{sm} \propto u^{-w}$. Then, from Eq. (115) (whose derivation does not involve any reference to the details of the initial data) we must have $w=k_0-l_0-s-1$, where k_0 is the power index of the tail at fixed r. Therefore, for *any* initial mode composition, there exists a simple relation between the power-law indices at null infinity and at fixed r. Symbolically, we may write (for each m)

$$\Psi^{sm}_{\mathrm{scri}+}\propto u^{-w}\Rightarrow\Psi^{sm}_{r=\mathrm{const}>r_+}\propto t^{-(w+l_0+s+1)}, \qquad (132)$$

which is valid for any initial mode composition.

The main challenge, then, is to obtain w, the power index at null infinity. Our iterative scheme, presented in Sec. III, provides a formal way for accomplishing this task; however, in the case of a non-generic initial mode composition this technique becomes less practical, for the following reason: Consider, for example, an initial pulse of scalar radiation, composed of only the mode $m=0$, $l=4$. Then, the function $\psi_{n=1}$ (namely, the first-order iteration term; we use here the notation of Sec. III) would contain only the three modes $l=2,4,6$. The most dominant mode of the overall field, that is $l=0$, would be excited only at $n=2$ (and, as we suspect, will gain its typical power-law form only at $n=3$). Thus, in this example, it would require us to go through at least three successive iteration stages in order to recover the tail form of the dominant mode. (Recall that in this paper we only discussed the first iteration; the second iteration already becomes very complicated for analytic treatment.) In general, as larger is the difference between the initial mode and l_0, as greater becomes the number of iterations required to extract the exact tail form of the dominant mode. It is only for the generic case discussed in this paper that a single iteration suffices for this goal.

The case of non-generic initial data (specifically, the case of any pure initial mode) was studied by Hod in Refs. [23] and [24]. However, recent numerical experiments by Krivan [26] show disagreement with Hod's results in this case. Further work is needed, both analytic and numerical, to clarify this point.

E. Final remarks

We recall that in this paper we have considered only non-extremal, $|a|<M$, Kerr BHs. Clearly, the extremal case needs to be analyzed separately [note, for example, that Eqs. (19)–(21) cease to be valid in the case $|a|=M$]. A basic property of the effective potential in both the SBH and the non-extremal KBH spacetimes — its exponential decay towards the EH (with respect to r_*) — is no longer valid in the extremal Kerr case: rather than $V(r)\propto e^{2\kappa_+ r_*}$ for $|a|<M$ near the EH, one finds $V(r)\propto r_*^{-2}$ for $|a|=M$ [where in both cases the tortoise variable r_* is defined through the differential relation $dr_*/dr=(a^2+r^2)/\Delta$]. Consequently, some basic parts of the analysis presented in this paper may fail to apply in the extremal case. In particular, the crucial assumption made in Sec. III, that the late time tail at null infinity is exclusively dominated by waves scattered at very large distances, need not necessarily hold in this case: Here, the strong contribution to the tail may occur also from back-scattering at *small* distances. To clarify the situation in the extremal case, a separate detailed analysis is thus required.

Finally, we should comment on the limited significance of ''multipole modes'' in the Kerr spacetime: The spin-weighted spherical harmonic functions Y^{slm} are not related here to an underlying symmetry group, as they are in spherically symmetric backgrounds. Consequently, the ''multipoles'' associated with these functions have no invariant meaning, but are rather related to (and defined through) the specific choice of the coordinates. Yet, the functions $Y^{slm}(\theta,\varphi)$ with θ,φ being the Boyer-Lindquist coordinates are signified as the natural basis for our purpose, because, at the late time limit, the field equation becomes separable in terms of these functions. (This separability is manifested in the $k=0$ term of the LTE, which exhibits no coupling between the various modes l,m.) Note also that the late time behavior of the overall field Ψ^s is in all cases governed by a pure mode l.

ACKNOWLEDGMENTS

I wish to thank Professor Amos Ori for his guidance throughout this research and for many helpful discussions.

APPENDIX: CALCULATION OF $\psi^l_{n=1}$ AT NULL INFINITY

In this appendix we calculate the ''1st-order'' iteration term $\psi^l_{n=1}$ at null infinity, at large retarded time (that is, for $v\to\infty$ with fixed $u\gg M$). In the case $s\geqslant 0$ we shall give full details of the calculation. For brevity, the case $s<0$ (which turns out slightly more complicated to analyze for technical reasons—see below) will be discussed in less detail.

The starting point for the calculation would be Eq. (43), in which the source function $S^l_{n=1}$ is given by Eq. (31) as a function of various modes of $\psi_{n=0}$. The radial functions $\delta V(r)$, $\delta R(r)$, and $K(r)$ involved in the expression for $S^l_{n=1}$ are implicit functions of $r_*=(v-u)/2$ (because r is an implicit function of r_*). We can expand each of these radial functions in powers of $1/r_*$. By virtue of Eqs. (25) and (28) we then obtain the leading order forms

$$\delta V(r)=\frac{M-imsa+l(l+1)M\left[2\ln\left(\frac{r_*}{r_+-r_-}\right)-1\right]}{2r_*^3}+O\left[\frac{(\ln r_*)^2}{r_*^4}\right] \qquad \text{(A1a)}$$

and

$$\delta R(r)=\frac{s\left[iac_0^l-3M+2M\ln\left(\frac{r_*}{r_+-r_-}\right)\right]}{2r_*^2}+O\left[\frac{(\ln r_*)^2}{r_*^3}\right], \qquad \text{(A1b)}$$

with the asymptotic form of $K(r)$ given in Eq. (25c).

The various terms in the source $S^l_{n=1}$ contribute additively to $\psi^l_{n=1}$ [via Eq. (43)]. The analysis below implies that the dominant contribution to $\psi^l_{n=1}$ at null infinity at large u comes only from the leading-order form (in $1/r_*$) of each of these source terms: Roughly speaking, each additional $1/r_*$ factor in the source leads to an additional factor of $1/u$ in the contribution to $\psi^l_{n=1}$ at null infinity. Hence, to calculate

$\psi^l_{n=1}$ to leading order in $1/u$, one may replace the actual functions δV, δR, and $K(r)$, with their above asymptotic forms.

Let us now consider the contribution to $\psi^l_{n=1}$ due to a source term of the form

$$S^l_{n=1}(u',v';l'Pd)\equiv(2r_*)^{-P}[\alpha+\beta\ln(2\tilde{r}_*)]\frac{d^d(\psi^{l'}_0)}{dt^d}, \tag{A2}$$

where a tilde symbol over a quantity shall represent the ratio of that quantity and (r_+-r_-) [so that $\tilde{r}_*\equiv r_*/(r_+-r_-)$]. This is the general form (to leading order in $1/r^*$) of all contributing terms in $S^l_{n=1}$, with the integer numbers P, l', and d admitting the possible values $P=2,3$, $l'=l,l\pm1,\pm2$ and $d=0,1,2$, and where α and β are constant coefficients.

Let us denote the contribution of $S^l_{n=1}(u',v';l'Pd)$ to $\psi^l_{n=1}$ at null infinity by $\psi^{ll'Pd}_{n=1}$. Then, from Eqs. (43) and (35) we find

$$\psi^{ll'Pd}_{n=1}=\frac{v^{-s}}{(l-s)!}\sum_{j=0}^{l'-s}A^{l's}_j \times\int_{u_0}^{0}du'[g^{l'}_0(u')]^{(j+d)} \times\int_u^\infty dv'\frac{d^{l-s}}{du^{l-s}}\left[\frac{(v'-u)^{l+s}(u-u')^{l-s}}{(v'-u')^{l+l'-j+P}}\right] \times[\alpha+\beta\ln(\tilde{v}'-\tilde{u}')]. \tag{A3}$$

Note that for $s\geqslant0$, all derivatives with respect to u can be ''taken out'' of the v' integration, due to the factor $(v'-u)^{l+s}$ appearing in the integrand. This manipulation (which is not possible for $s<0$) much simplifies the calculation in the $s\geqslant0$ case. For brevity, we therefore continue from this point by concentrating on the case $s\geqslant0$. Our calculations for $s<0$, whose details shall not be presented here, yield the same qualitative results as those obtained below for $s\geqslant0$, yet they are slightly more tedious (as they involve more complicated combinatorial expressions). Unfortunately, we could not figure out a way for treating the $s<0$ case in a simple manner as the $s\geqslant0$ case, though we think this should be possible.

After ''taking the u derivatives out'' of the integration over v', we can now easily extract the u dependence of $\psi^{ll'Pd}_{n=1}$ by transforming in Eq. (A3) to the new integration variable, $x(v')=(u-u')/(v'-u')$. This yields

$$\psi^{ll'Pd}_{n=1}=\frac{v^{-s}}{(l-s)!}\sum_{j=0}^{l'-s}A^{l's}_j\int_{u_0}^{0}du'[g^{l'}_0(u')]^{(j+d)}\frac{d^{l-s}}{du^{l-s}} \times((u-u')^{l-l'-P+1+j}\{\overline{\alpha_j}[\alpha+\beta\ln(\tilde{u}-\tilde{u}')] -\overline{\beta_j}\beta\}), \tag{A4}$$

where $\overline{\alpha_j}$ and $\overline{\beta_j}$ are constant coefficients given by

$$\overline{\alpha_j}\equiv\int_0^1dx(1-x)^{l+s}x^{l'-s-j+P-2} =\frac{(l'-s+P-2-j)!(l+s)!}{(l+l'+P-1-j)!},$$
$$\overline{\beta_j}\equiv\int_0^1dx(1-x)^{l+s}x^{l'-s-j+P-2}\ln x. \tag{A5}$$

[The coefficient $\overline{\alpha_j}$ is just the standard beta function, $B(l'-s+P-1-j,l+s+1)$.]

Next, in Eq. (A4) we integrate by parts $j+d$ successive times with respect to u'. All resulting surface terms vanish due to the compactness of $g^{l'}_0(u')$, and one is left with

$$\psi^{ll'Pd}_{n=1}=\frac{v^{-s}}{(l-s)!}\sum_{j=0}^{l'-s}A^{l's}_j\int_{u_0}^{0}du'g^{l'}_0(u')\frac{d^{l-s+j+d}}{du^{l-s+j+d}} \times((u-u')^{l-l'-P+1+j}\{\overline{\alpha_j}[\alpha+\beta\ln(\tilde{u}-\tilde{u}')] -\overline{\beta_j}\beta\}). \tag{A6}$$

Here, we have used the fact that the u' derivatives operate on functions of $(u-u')$ only, to make the replacement $\partial_{u'}\to-\partial_u$. Evaluated at late retarded time, $u\gg-u_0$, the last expression takes the form (accurate to leading order in u/u_0)

$$\psi^{ll'Pd}_{n=1}\cong I^{l'}_0\frac{v^{-s}}{(l-s)!}\sum_{j=0}^{l'-s}A^{l's}_j\frac{d^{l-s+j+d}}{du^{l-s+j+d}}\{u^{l-l'-P+1+j} \times[\overline{\alpha_j}(\alpha+\beta\ln\tilde{u})-\overline{\beta_j}\beta]\}, \tag{A7}$$

where $I^{l'}_0$ is the functional constructed from the function $g^{l'}_0(u)$ according to Eq. (45).

Finally, performing the multiple differentiation in Eq. (A7), we find

$$\psi^{ll'Pd}_{n=1}\cong I^{l'}_0[\lambda^{ll'Pd}(\alpha+\beta\ln\tilde{u})+\beta\eta^{ll'Pd}] \times v^{-s}u^{-(l'-s+P-1+d)}, \tag{A8}$$

in which $\lambda^{ll'Pd}$ and $\eta^{ll'Pd}$ are constant coefficients. Here, the term proportional to $\lambda^{ll'Pd}$ contains only contributions which arise from all $l-s+j+d$ derivatives in Eq. (A7) acting on the power $u^{l-l'+j-P+1}$, with nonacting on $\ln\tilde{u}$. The order of differentiation, $l-s+j+d$, is in all relevant cases greater than the power index $l-l'+j-P+1$. Hence, the only contributions to the coefficient $\lambda^{ll'Pd}$ arise when the power $l-l'+j-P+1$ is *negative*, i.e. from the terms with $j\leqslant l'-l+P-2$. Since the index j takes no negative values, we find that there would be no contribution to $\lambda^{ll'Pd}$ unless $P\geqslant l-l'+2$. Namely,

$$\lambda^{P<l-l'+2}=0. \tag{A9}$$

For $P \geqslant l-l'+2$, the coefficient $\lambda^{ll'Pd}$ is given by

$$\lambda^{P\geqslant l-l'+2}=\frac{(l+s)!}{(l-s)!}\sum_{j=0}^{\bar{j}} A_j^{l's}(-1)^{l'-s+j+d} \times\frac{(l'-s+P-2-j)!(l'-s+P+d-2)!}{(l+l'+P-1-j)!(l'-l+P-2-j)!}, \tag{A10}$$

where $\bar{j}=\min(l'-s,l'-l+P-2)$ (in accordance with the above discussion). An expression for the coefficient $\eta^{ll'Pd}$ can also be written down explicitly, using Eq. (A7). It can be verified that this coefficient is non-vanishing for all relevant values of l,l',P,d.

Equation (A8) suggests that, potentially, one may expect logarithmic dependence to occur at the leading order tail at null infinity. In the following we show, however, that due to vanishing of the coefficient $\lambda^{ll'Pd}$ in certain cases, this logarithmic dependence is avoided as far as the most dominant mode of the overall field is concerned: this dominant mode shall appear to die off with a pure power-law tail.

Note also that Eq. (A8) confirms our above assertion, that to leading order in $1/u$, the contribution from each given term in $S_{k=1}^{l'}$ (with given l' and d) comes merely from the leading order in $1/r'_*$: For higher-order terms in the $1/r'_*$ expansion of the source there correspond larger values of P, leading to a faster decay in Eq. (A8).[19]

Using Eq. (A8) we can now analyze the contribution to $\psi_{n=1}^{l}$ at null infinity at late retarded time, belonging to each of the various source modes l':

(a) *Contribution of the mode* $l'=l-2$. For this source mode we have $d=2$, and, at the leading order in $1/r'_*$, $P=2$ and $\beta=0$ [see Eqs. (31) and (25c)]. Since for this order $P<l-l'+2=4$, the corresponding λ coefficient vanishes in Eq. (A8). Thus, we find no contribution at all to $\Psi_{n=1}^{l}$ from the order $O[(r'_*)^{-2}]$ of this source mode. Turning next to the following order, with $P=3$, we find again that λ vanishes (as $P<4$). However, at this order the logarithmic coefficient β of the source does not vanish [we have $\beta=8Ma^2C_{--}^{l}$—see Eqs. (31) and (25c)], and from Eq. (A8) we find the nonvanishing, *nonlogarithmic* contribution

$$\psi_{n=1}^{l-2\rightarrow l}\cong 8Ma^2C_{--}^{l}\,\eta_{P=3,d=2}^{l'=l-2}I_0^{l-2}v^{-s}u^{-(l-s+2)}, \tag{A11}$$

where we adopt the notation $\psi_{n=1}^{l'\rightarrow l}$ to represent the late time contribution to $\psi_{n=1}^{l}$ at null infinity due to the mode l'. Here, the symbol "$\cong$" stands for "leading order in M/u and in u_0/u."

(b) *Contribution of the mode* $l'=l-1$. For this source mode there are two contributions: one with $d=1$ [the one proportional to c_-^l — see Eq. (17)] and one with $d=2$ (proportional to C_-^l). Both contributions have $P=2$ and $\beta=0$ at the leading order in $1/r'_*$. Clearly, in view of Eq. (A8), the term with $d=1$ dominates the contribution from this mode. Now, for the leading order, $O[(r'_*)^{-2}]$, we have $P<l-l'+2=3$; thus the corresponding λ coefficient vanishes. Since for this order we also have $\beta=0$, one finds no contribution from this order to $\psi_{n=1}^{l}$. The dominant contribution would come from the next order (with $P=3$), for which both coefficients λ and $\beta\cdot\eta$ are nonvanishing. Therefore, this contribution will contain a logarithmic dependence:

$$\psi_{n=1}^{l-1\rightarrow l}\cong 8iasM\,c_-^l I_0^{l-1}[\lambda_{P=3,d=1}^{l'=l-1}(1/2+\ln\tilde{u}) + \eta_{P=3,d=1}^{l'=l-1}]v^{-s}u^{-(l-s+2)}. \tag{A12}$$

(c) *Contribution of the mode* $l'=l$. There are three terms in the source $S_{n=1}^{l}$ which are not due to interaction with other modes [see Eq. (31)]. These are (i) the term proportional to δV, for which $P=3$ and $d=0$; (ii) the term proportional to δR, for which $P=2$ and $d=1$; and (iii) the term proportional to C_0^l, with $P=2$ and $d=2$. Clearly, the δV term and the δR term contribute to $\psi_{n=1}^{l}$ at the same order of $1/u$, whereas the contribution from the third term in smaller by a factor of $1/u$. We thus concentrate on the first two terms, both of which have $P\geqslant l-l'+2=2$ already at the leading order in $1/r'_*$. Hence, the dominant contribution to $\psi_{n=1}^{l}$ would come from this leading order. From Eq. (A8), using Eqs. (A1a) and (A1b), we obtain, for the contributions of these two terms,

$$\psi_{n=1}^{l\rightarrow l}(\text{due to }\delta V)\cong -4M\,I_0^l v^{-s}u^{-(l-s+2)}\{\lambda_{P=3,d=0}^{l'=l} \times[1-imsa/M-l(l+1) +2l(l+1)\ln\tilde{u}]+2l(l+1)\,\eta_{P=3,d=0}^{l'=l}\}, \tag{A13a}$$

$$\psi_{n=1}^{l\rightarrow l}(\text{due to }\delta R)\cong -2Ms\,I_0^l v^{-s}u^{-(l-s+2)} \times[\lambda_{P=2,d=1}^{l'=l}(ic_0^l a/M-3+2\ln\tilde{u}) +2\,\eta_{P=2,d=1}^{l'=l-1}]. \tag{A13b}$$

Now, from Eq. (A10) we find

$$\lambda_{P=3,d=0}^{l'=l}=\frac{s(-1)^{l-s}(l-s+1)(l+s)!}{2l(l+1)(2l+1)},$$

$$\lambda_{P=2,d=1}^{l'=l}=\frac{(-1)^{l-s+1}(l+s)!}{2l+1}. \tag{A14}$$

Substituting these values in Eqs. (A13a) and (A13b), and adding up these two equations to construct the overall contribution from the mode $l'=l$, we find that the two logarithmic terms exactly cancel each other. (In the scalar field case,

[19]Higher-order terms in the $1/r'_*$ expansion of the source would exhibit higher $\ln\tilde{r}'_*$ powers, leading to higher logarithmic powers in Eq. (A8); however, the $u^{-(l'-s+P-1+d)}$ power law would remain the same for the higher order terms as well, with P denoting the power of $1/r'_*$ for each term.

$s=0$, each of the two logarithmic terms vanishes independently.) The leading order overall contribution from the mode $l'=l$ will therefore be *nonlogarithmic*, exhibiting a strict power-law tail of the form

$$\psi_{n=1}^{l\to l} \propto M\, I_0^l\, v^{-s}\, u^{-(l-s+2)}. \tag{A15}$$

(d) *Contribution of the mode* $l'=l+1$. For this mode, there contribute two terms in $S_{n=1}^l$, the ones proportional to c_+^l and to C_+^l. These two terms have, respectively, $d=1$ and $d=2$, and both have (at the leading order in $1/r'_*$) $P=2$ and $\beta=0$. However, both coefficients $\lambda_{P=2,d=1}^{l'=l+1}$ and $\lambda_{P=2,d=2}^{l'=l+1}$ turn out to vanish, resulting in the dominant contribution to $\psi_{n=1}^l$ coming from $P=3$. The contribution from the term proportional to c_+^l (which is $\propto u^{-(l-s+4)}\ln\tilde{u}$) dominates the one from the term proportional to $C_+^l(\propto u^{-(l-s+5)}\ln\tilde{u})$, and one finds, in summary,

$$\psi_{n=1}^{l+1\to l} \cong 8ias M c_+^l I_0^{l+1}[\lambda_{P=3,d=1}^{l'=l+1}(1/2+\ln\tilde{u}) + \eta_{P=3,d=1}^{l'=l-1}]v^{-s}u^{-(l-s+4)}. \tag{A16}$$

(e) *Contribution of the mode* $l'=l+2$. This mode has $d=2$ and (at the leading order in $1/r'_*$) $P=2$ and $\beta=0$. However, the coefficient $\lambda_{P=2,d=2}^{l'=l+2}$, as well as $\lambda_{P=3,d=2}^{l'=l+2}$, turns out to vanish, resulting in the leading order contribution from this mode to $\psi_{n=1}^l$ coming at $P=3$ from the term in Eq. (A8) proportional to $\beta\cdot\eta$. Hence, this contribution would be *nonlogarithmic*:

$$\psi_{n=1}^{l+2\to l} \cong 8Ma^2 C_{++}^l\, \eta_{P=3,d=2}^{l'=l+2} I_0^{l+2}\, v^{-s}\, u^{-(l-s+6)}. \tag{A17}$$

Equations (A11), (A12), (A15), (A16), and (A17) describe the various contributions to the tail of $\psi_{n=1}^l$ at null infinity from all various source modes. These results are summarized in Eq. (44) (in Sec. III D) using a different notation for the amplitude coefficients. We point out that Eq. (A8), as well as all power-law formulas derived in this appendix, is also valid in the case $s<0$, though with different amplitude coefficients. (As we mentioned above, we found these coefficients to be more complicated to calculate for $s<0$; still, the same coefficients λ found to vanish for $s\geqslant 0$ also appear to vanish in the case $s<0$, which finally leads to the same power-law contributions.)

The following are the main conclusions that can be drawn from the analysis in this appendix:

(i) In general, the dominant contribution to the late time tail of a mode l of $\psi_{n=1}$ at null infinity is due to the source modes l, $l-2$, and (for $s\neq 0$) $l-1$. These contributions all have the form $\propto u^{l-s+2}$ (multiplied by a logarithmic factor in the $l'=l-1$ case). Contributions due to the modes $l+1$ and $l+2$ are negligible.

(ii) Consequently, for given s and m, the most dominant mode of the field $\psi_{n=1}^{sm}$ at null infinity is the lowest radiatable one, namely the multipole $l=l_0\equiv\min(|s|,|m|)$.

(iii) This mode (l_0) admits no contributions from lower, $l<l_0$, multipoles, and thus, to leading order in $1/u$, it is not affected by interactions with other modes. Equation (A15) then implies that this mode (and thus also the overall field $\psi_{n=1}$) admits the strict late time power-law tail $\propto u^{-(l_0-s+2)}$, with no logarithmic dependence.

[1] R.H. Price, Phys. Rev. D **5**, 2419 (1972).
[2] R.H. Price, Phys. Rev. D **5**, 2439 (1972).
[3] E. Leaver, J. Math. Phys. **27**, 1238 (1986); Phys. Rev. D **34**, 384 (1986).
[4] C. Gundlach, R.H. Price, and J. Pullin, Phys. Rev. D **49**, 883 (1994).
[5] R. Gómez, J. Winicour, and B.G. Schmidt, Phys. Rev. D **49**, 2828 (1994).
[6] N. Andersson, Phys. Rev. D **55**, 468 (1997).
[7] L. Barack, Phys. Rev. D **59**, 044016 (1999).
[8] L. Barack, Phys. Rev. D **59**, 044017 (1999).
[9] J. Bičák, Gen. Relativ. Gravit. **3**, 331 (1972).
[10] E.S.C. Ching, P.T. Leung, W.M. Suen, and K. Young, Phys. Rev. Lett. **74**, 2414 (1995).
[11] L.M. Burko and A. Ori, Phys. Rev. D **56**, 7820 (1997).
[12] P.R. Brady, C.M. Chambers, W. Krivan, and P. Laguna, Phys. Rev. D **55**, 7538 (1997).
[13] S. Hod, Plup. Rev. D **60**, 104053 (1999).
[14] For a generalization to other types of scalar fields, see S. Hod and T. Piran, Phys. Rev. D **58**, 044018 (1998) and references therein (for a massive scalar field); **58**, 024019 (1998) and references therein (for a charged scalar field).
[15] C. Gundlach, R.H. Price, and J. Pullin, Phys. Rev. D **49**, 890 (1994).
[16] J.M. Bardeen, Nature (London) **226**, 64 (1970); K.S. Thorne, Astrophys. J. **191**, 507 (1974).
[17] For a detailed review of the ''no hair'' theorems by Hawking, Israel, Carter, and Robinson, see B. Carter, in *Les Astres Occlus*, edited by C. DeWitt and B.S. DeWitt (Gordon and Breach, New York, 1973).
[18] W. Krivan, P. Laguna, and P. Papadopoulos, Phys. Rev. D **54**, 4728 (1996).
[19] W. Krivan, P. Laguna, P. Papadopoulos, and N. Andersson, Phys. Rev. D **56**, 3395 (1997).
[20] L. Barack, in *Internal Structure of Black Holes and Spacetime Singularities*, Volume XIII of the Israel Physical Society, edited by L.M. Burko and A. Ori (Institute of Physics, Bristol, 1997).
[21] L. Barack and A. Ori, Phys. Rev. Lett. **82**, 4388 (1999).
[22] A. Ori, Gen. Relativ. Gravit. **29**, 881 (1997).
[23] S. Hod, Phys. Rev. D (to be published), gr-qc/9902072.
[24] S. Hod, gr-qc/9902073. See also S. Hod, Phys. Rev. D **58**, 104022 (1998), which, however, does not correctly handle the coupling between modes.
[25] S. Hod, gr-qc/9907096v2.
[26] W. Krivan, Phys. Rev. D **60**, 101501 (1999).
[27] W. Krivan (private communication).
[28] L. Barack and A. Ori, Phys. Rev. D **60**, 124005 (1999).

[29] A. Ori, Phys. Rev. D **61**, 024001 (2000).
[30] E.T. Newman and R. Penrose, J. Math. Phys. **3**, 566 (1962).
[31] W. Kinnersley, J. Math. Phys. **10**, 1195 (1969).
[32] S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Oxford University Press, New York, 1983).
[33] S.A. Teukolsky, Phys. Rev. Lett. **29**, 1114 (1972).
[34] J.N. Goldberg, A.J. Macfarlane, E.T. Newman, F. Rohrlich, and E.C.G. Sudarshan, J. Math. Phys. **8**, 2155 (1967).
[35] In W.B. Campbell and T. Morgan, Physica (Amsterdam) **53**, 264 (1971), use has been made of the relation between the spin-weighted spherical harmonics and the rotation matrices, to express the product of any three Y^{slm} functions as a sum of functions Y^{slm}, using the standard Clebsch-Gordan coefficients. The special case relevant to our paper [Eq. (11)] is stated in W.H. Press and S.A. Teukolsky, Astrophys. J. **185**, 649 (1973) [Eq. (3.9b) therein].
[36] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I.A. Stegun, U.S. Nat. Bur. Stand. Appl. Math. Ser. No. 55 (U.S. GPO, Washington, D.C., 1968), Table 27.9.2.
[37] For a representation of the general static solution in a different basis, see P.L. Chrzanowski, Phys. Rev. D **11**, 2042 (1975). Note, however, that the expression given in Eq. (5.31) of that paper is not defined when $m=0$ and $s>0$. Yet another basis of solutions is given in Ref. [29], Eqs. (96) and (97).
[38] A. Erdélyi *et al.*, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. 1.
[39] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).