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Improved upper bound to the entropy of a charged system

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Recently, we derived an *improved* universal upper bound to the entropy of a *charged* system $S \leqslant \pi(2Eb - q^2)/\hbar$. There was, however, some uncertainty in the value of the numerical factor which multiplies the q^2 term. In this paper we remove this uncertainty; we rederive this upper bound from an application of the generalized second law of thermodynamics to a gedanken experiment in which an entropy-bearing charged system falls into a Schwarzschild black hole. A crucial step in the analysis is the inclusion of the effect of the spacetime curvature on the electrostatic self-interaction of the charged system.

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According to the thermodynamical analogy in black-hole physics, the entropy of a black hole [1–3] is given by $S_{\mathrm{bh}} = A/4\hbar$, where A is the black-hole surface area. (We use gravitational units in which $G = c = 1$.) Moreover, a system consisting of ordinary matter interacting with a black hole is widely believed to obey the generalized second law of thermodynamics (GSL): ''*The sum of the black-hole entropy and the common (ordinary) entropy in the black-hole exterior never decreases*.'' This general conjecture is one of the corner stones of black-hole physics.

It is well known, however, that the validity of the GSL depends on the (plausible) existence of a universal upper bound to the entropy of a bounded system [4]: Consider a box filled with matter of proper energy E and entropy S which is dropped into a black hole. The energy delivered to the black hole can be arbitrarily redshifted by letting the assimilation point approach the black-hole horizon. If the box is deposited with no radial momentum a proper distance R above the horizon and then allowed to fall in such that

$$R < \hbar S/2\pi E, \tag{1}$$

then the black-hole area increase (or equivalently, the increase in black-hole entropy) is not large enough to compensate for the decrease of S in common (ordinary) entropy. Arguing from the GSL, Bekenstein [4] has proposed the existence of a universal upper bound to the entropy S of any system of total energy E and effective proper radius R:

$$S \leqslant 2\pi RE/\hbar, \tag{2}$$

where R is defined in terms of the area A of the spherical surface which circumscribe the system $R = (A/4\pi)^{1/2}$ [4]. This restriction is necessary for enforcement of the GSL; the box's entropy disappears but an increase in black-hole entropy occurs which ensures that the GSL is respected provided S is bounded as in Eq. (2). Evidently, this universal upper bound is a quantum phenomena (the upper bound goes to infinity as $\hbar \to 0$). This provides a striking illustration of the fact that the GSL is intrinsically a quantum law. The universal upper bound equation (2) has the status of a supplement to the second law; the latter only states that the entropy of a closed system tends to a maximum without saying how large that should be.

Other derivations of the universal upper bound equation (2) which are based on black-hole physics have been given in [5–8]. Few pieces of evidence exist concerning the validity of the bound for self-gravitating systems [5,6,9,10]. However, the universal bound equation (2) is known to be true independently of black-hole physics for a variety of systems in which gravity is negligible [11–15].

We noted [16,17], however, that there is one disturbing feature of the universal bound equation (2): Black holes conform to the bound [4]; however, it is only the Schwarzschild black hole which actually saturates the bound. This uniqueness of the Schwarzschild black hole (in the sense that it is the only black hole which have the maximum entropy allowed by quantum theory and general relativity) is somewhat disturbing. Recently, Hod [16] derived an (improved) upper bound to the entropy of a spinning system and proved that *all* electrically neutral Kerr black holes have the maximum entropy allowed by quantum theory and general relativity. The unity of physics (and of black holes in particular) motivates us to look for an improved upper bound to the entropy of a charged system.

Moreover, the plausible existence of an upper bound stronger than Eq. (2) on the entropy of a charged system has nothing to do with black-hole physics; a part of the energy of the electromagnetic field residing outside the charged system seems to be irrelevant for the system's statistical properties. This reduces the phase space available to the components of a charged system. Evidently, an improved upper bound to the entropy of a charged system must *decrease* with the (absolute) value of the system's charge. However, our simple argument cannot yield the exact dependence of the entropy bound on the system's parameters: its energy, charge, and proper radius.

It is black-hole physics (more precisely, the GSL) which yields a concrete expression for the universal upper bound; recently, we have derived an *improved* universal upper bound to the entropy of a *charged* system $S \leqslant \pi(2Eb - q^2)/\hbar$ [17]. There was, however, some uncertainty in the value of the numerical factor which multiplies the q^2 term. In this paper we remove this uncertainty.

We consider a charged body of rest mass μ and charge q, which is dropped into a Schwarzschild black hole. The equation of motion of a charged body on a Schwarzschild background is a quadratic equation for the conserved energy E (energy-at-infinity) of the body [18]

$$r^4E^2-\Delta(\mu^2r^2+p_\phi^2)-(\Delta p_r)^2=0, \qquad (3)$$

where $\Delta=r^2-2Mr$. The quantities p_ϕ and p_r are the conserved angular momentum of the body and its covariant radial momentum, respectively.

The conserved energy E of a body having a radial turning point at $r=r_++\xi$ [19] (for $\xi\ll r_+$ where $r_+=2M$ is the location of the black-hole horizon) is given by Eq. (3)

$$E=\sqrt{\mu^2+p_\phi^2/r_+^2}(\xi/r_+)^{1/2}[1+O(\xi/r_+)]. \qquad (4)$$

This expression is actually the effective potential (gravitational plus centrifugal) for given values of μ and p_ϕ. It is clear that it can be minimized by taking $p_\phi=0$ (which also minimizes the increase in the black-hole surface area).

However, the well-known analysis of [18] is not complete because it does not take into account the effect of the spacetime curvature on the particle's electrostatic self-interaction. The black-hole gravitational field modifies the electrostatic self-interaction of a charged particle in such a way that the particle experiences a repulsive (i.e., directed away from the black hole) self-force. A variety of techniques have been used to demonstrate this effect [20–24]. The physical origin of this force is the distortion of the charge's long-range Coulomb field by the spacetime curvature. The contribution of this effect to the particle's energy is $Mq^2/2r^2$ [24].

In order to find the change in black-hole surface area caused by an assimilation of the body, one should evaluate E at the point of capture, a proper distance b outside the horizon. The relevant dimension of the body in our gedanken experiment is its shortest length. In other words, the entropy bound is set by the smallest body's dimension (provided $b\gg\hbar/E$ [12]). This conclusion is supported by numerical computations [11] for neutral systems. Thus, we should evaluate E at $r=r_++\delta(b)$, where $\delta(b)$ is determined by

$$\int_{r_+}^{r_++\delta(b)}(1-2M/r)^{-1/2}dr=b. \qquad (5)$$

Integrating Eq. (5) one obtains (for $b\ll r_+$)

$$\delta(b)=b^2/8M, \qquad (6)$$

which implies (to leading order in b/M)

$$E=(2\mu b+q^2)/8M. \qquad (7)$$

An assimilation of the charged body results in a change $\Delta M=E$ in the black-hole mass and a change $\Delta Q=q$ in its charge. The relation $A=4\pi[M+(M^2-Q^2)^{1/2}]^2$ implies that (for $Q=0$) $\Delta A=8\pi[4M\Delta M-(\Delta Q)^2]$ (terms of order $(\Delta M)^2$ are negligible for $b\ll M$ and $|q|\ll M$). Thus, taking cognizance of Eq. (7) we find

$$(\Delta A)_{\min}=4\pi(2\mu b-q^2), \qquad (8)$$

which is the *minimal* black-hole area increase for given values of the body's parameters μ, q, and b. Assuming the validity of the GSL, one can derive an upper bound to the entropy S of an arbitrary system of proper energy E, charge q, and circumscribing radius R (by definition, $R\geq b$):

$$S\leq\pi(2ER-q^2)/\hbar. \qquad (9)$$

It is evident from the minimal black-hole area increase Eq. (8) that in order for the GSL to be satisfied $[(\Delta S)_{\mathrm{tot}}\equiv(\Delta S)_{\mathrm{bh}}-S\geq 0]$, the entropy S of the charged system must be bounded as in Eq. (9). This upper bound is universal in the sense that it depends only on the system's parameters (it is independent of the black-hole mass which was used to derive it).

This improved bound is very appealing from a black-hole physics point of view [17]: consider a charged Reissner-Nordström black hole of charge Q. Let its energy be E; then its surface area is given by $A=4\pi r_+^2=4\pi(2Er_+-Q^2)$. Now since $S_{\mathrm{bh}}=A/4\hbar$, $S_{\mathrm{bh}}=\pi(2Er_+-Q^2)/\hbar$, which is the maximal entropy allowed by the upper bound equation (9). Thus, *all* Reissner-Nordström black holes saturate the bound. This proves that the Schwarzschild black hole is *not* unique from a black-hole entropy point of view, removing the disturbing feature of the entropy bound Eq. (2). This is precisely the kind of universal upper bound we were hoping for.

Evidently, systems with negligible self-gravity (the charged system in our gedanken experiment) and systems with maximal gravitational effects (i.e., charged black holes) both satisfy the upper bound equation (9). Therefore, this bound appears to be of universal validity. One piece of evidence exists concerning the validity of the bound for the specific example of a system composed of a charged black hole in thermal equilibrium with radiation [6].

The intriguing feature of our derivation is that it uses a law whose very meaning stems from gravitation (the GSL, or equivalently the area-entropy relation for black holes) to derive a universal bound which has nothing to do with gravitation (written out fully, the entropy bound would involve $\hbar$ and c, but not G). This provides a striking illustration of the unity of physics.

In summary, an application of the generalized second law of thermodynamics to a gedanken experiment in which an entropy-bearing charged system falls into a Schwarzschild black hole, enables us to derive an *improved universal upper bound* to the entropy of a *charged* system [17]. In doing so, we removed the former uncertainty regarding the precise value of the numerical coefficient which multiplies the q^2 term. A crucial step in the analysis is the inclusion of the influence of the spacetime curvature on the system's electrostatic self-interaction.

Note added. I have learned that recently Bekenstein and Mayo [25] analyzed the same problem, and independently obtained the universal upper bound (which was already derived in [17]).

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