Integrable models and degenerate horizons in two-dimensional gravity

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We analyze an integrable model of two-dimensional gravity which can be reduced to a pair of Liouville fields in the conformal gauge. Its general solution represents a pair of ''mirror'' black holes with the same temperature. The ground state is a degenerate constant dilaton configuration similar to the Nariai solution of the Schwarzschild–de Sitter case. The existence of $\phi=$const solutions and their relation to the solution given by the 2D Birkhoff theorem is then investigated in a more general context. We also point out some interesting features of the semiclassical theory of our model and the similarity with the behavior of AdS_2 black holes.

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I. INTRODUCTION

The existence of exactly solvable models of gravity in two dimensions [1] provides a rich arena for the study of the quantum aspects of black holes. These two-dimensional black holes, in addition to their own interest, can describe particular regimes of higher-dimensional black holes. The Callan-Giddings-Harvey-Strominger (CGHS) model [2] describes low-energy excitations of extremal (magnetic) string black holes in four dimensions. AdS_2 black holes arise in the near-horizon limits of extremal or near-extremal Reissner-Nordström black holes [3,4]. By dimensional reduction, spherically symmetric gravity can also be described in terms of an effective two-dimensional model.

The aim of this paper is to analyze a general family of integrable models [5] which can recover all known solvable models [CGHS [2], Jackiw-Teitelboim [6], and exponential (Liouville) [7] models] in some particular limits. The equations of motion of the models, in conformal gauge, are equivalent to those of a pair of Liouville fields for linear combinations of the conformal factor and the dilaton field. These properties will be briefly reviewed in Sec. II. In Sec. III we investigate the properties of the classical solutions, showing that, in the absence of matter fields, they represent a pair of eternal black holes. In Sec. IV we shall focus on one particular model [with a potential of the form $V(\phi)=2\sinh\beta\phi$] which allows a degenerate solution having a constant value for the dilaton and a two-dimensional de Sitter (or anti–de Sitter, depending on the sign of the constant β) geometry. The situation is similar to that encountered in the Schwarzschild–de Sitter case where the degenerate case of the Nariai metric [8] is also described by a constant dilaton (i.e., the radial coordinate). In Sec. V we shall analyze the existence of such dilaton-constant solutions in a more general setting. We will show in a simple way that these configurations are possible for the zeros of the potential, after removing the kinetic term of the two-dimensional dilaton-gravity theory, and are always accompanied by a constant curvature geometry. Furthemore, they are always connected with the presence of degenerate horizons in the theory. Finally, in Sec. VI we make some comments on the semiclassical behavior of our solutions and show interesting similarities with the behavior of AdS_2 black holes.

II. INTEGRABILITY OF 2D DILATON GRAVITY MODELS

Let us consider the general functional action describing a 2D dilaton-gravity model coupled to N 2D massless and minimal scalar fields:

$$S=\frac{1}{2\pi}\int d^2x\sqrt{-g}\left(R\phi+4\lambda^2V(\phi)-\frac{1}{2}\sum_{i=1}^{N}(\nabla f_i)^2\right), \tag{1}$$

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where $V(\phi)$ is an arbitrary function of the dilaton field and f_i are the scalar matter fields. The above expression represents a generic model because one can get rid of the kinetic term of the dilaton by a conformal reparametrization of the fields and bring the action into the form (1) [9]. In the conformal gauge $ds^2=-e^{2\rho}dx^+dx^-$, the equations of motion derived from the action (1) are

$$2\partial_+\partial_-\rho+\lambda^2V'(\phi)e^{2\rho}=0, \tag{2}$$

$$\partial_+\partial_-\phi+\lambda^2V(\phi)e^{2\rho}=0, \tag{3}$$

$$\partial_+\partial_-f_i=0, \tag{4}$$

$$-\partial_\pm^2\phi+2\partial_\pm\phi\partial_\pm\rho-\frac{1}{2}\sum_{i=1}^{N}(\partial_\pm f_i)^2=0. \tag{5}$$

By introducing an arbitrary parameter β we can rewrite the above equations of motion (2), (3) in the form

$$\partial_+\partial_-(2\rho+\beta\phi)+\lambda^2e^{2\rho}\left(\beta V(\phi)+\frac{dV(\phi)}{d\phi}\right)=0, \tag{6}$$

$$\partial_+\partial_-(2\rho-\beta\phi)-\lambda^2e^{2\rho}\left(\beta V(\phi)-\frac{dV(\phi)}{d\phi}\right)=0. \tag{7}$$

One way to ensure the integrability of the above equations is to reduce them to a pair of Liouville equations [5]. The most general potential satisfying this requirement is

$$V(\phi)=\gamma_+e^{\beta\phi}+\gamma_-e^{-\beta\phi}, \tag{8}$$

so that the corresponding equations of motion are a pair of Liouville equations:

$$\partial_+\partial_-(2\rho\pm\beta\phi)\pm2\gamma_\pm\beta\lambda^2e^{2\rho\pm\beta\phi}=0. \tag{9}$$

This potential includes all known integrable models, that is, for $\gamma_\pm=\frac{1}{2}$ and $\beta\to0$ the CGHS model, for $\gamma_+=-\gamma_-=1/2\beta$ and $\beta\to0$ the Jackiw-Teitelboim theory, and for $\gamma_+=1$, $\gamma_-=0$ the exponential (Liouville) model [10].

The general solution to Eqs. (9) can be written in terms of four arbitrary chiral functions $A_\pm(x^\pm)$, $a_\pm(x^\pm)$,

$$2\rho+\beta\phi=\ln\frac{\partial_+A_+\partial_-A_-}{(1+\gamma_+\beta\lambda^2A_+A_-)^2}, \tag{10}$$

$$2\rho-\beta\phi=\ln\frac{\partial_+a_+\partial_-a_-}{(1-\gamma_-\beta\lambda^2a_+a_-)^2}, \tag{11}$$

and allows us to recover the general solution of the limiting models. The solution for the exponential model is immediately recovered, making $\gamma_+=1$ and $\gamma_-=0$ in Eqs. (10), (11). In the Jackiw-Teitelboim theory $(\gamma_+=-\gamma_-=1/2\beta)$ we have to redefine the functions $a_\pm$, introducing a new pair $\tilde{a}_\pm$, $a_\pm=A_\pm+\beta\tilde{a}_\pm$. Afterwards we realize the $\beta\to0$ limit and then we get

$$\rho=\frac{1}{2}\ln\frac{\partial_+A_+\partial_-A_-}{\left(1+\frac{\lambda^2}{2}A_+A_-\right)^2}, \tag{12}$$

$$\phi=-\frac{1}{2}\left(\frac{\partial_+\tilde{a}_+}{\partial_+A_+}+\frac{\partial_-\tilde{a}_-}{\partial_-A_-}\right)+\frac{\lambda^2}{2}\frac{A_+\tilde{a}_-+A_-\tilde{a}_+}{1+\frac{\lambda^2}{2}A_+A_-}, \tag{13}$$

as it was found in [10]. Finally we can also recover the solution for the CGHS model $(\gamma_\pm=\frac{1}{2})$ in a similar way. Redefining $a_\pm=A_\pm-2\beta\int^{x^\pm}\hat{a}_\pm\partial_\pm A_\pm$ in Eqs. (10), (11) we can perform the $\beta\to0$ limit and we get

$$\rho=\frac{1}{2}\ln\partial_+A_+\partial_-A_-, \tag{14}$$

$$\phi=-\lambda^2A_+A_-+\hat{a}_++\hat{a}_-. \tag{15}$$

The above mechanism provides a very simple picture of the origin of the integrability of these models and suggests a particular analysis of the most general integrable hyperbolic model (8). The hidden reason for this integrability can now be understood as all them are particular cases of a general Liouville integrability of which the hyperbolic model is, in a sense, the maximal one. The hyperbolic model is then the most complicated solvable model that we can study.

III. CLASSICAL THEORY AND ETERNAL BLACK HOLE SOLUTIONS

In this section we shall study the classical theory of the model (8) and look for black hole solutions. The functional action is given by

$$S=\frac{1}{2\pi}\int d^2x\sqrt{-g}\left(R\phi+4\lambda^2(\gamma_+e^{\beta\phi}+\gamma_-e^{-\beta\phi})-\frac{1}{2}\sum_{i=1}^{N}(\nabla f_i)^2\right), \tag{16}$$

and we have to note that, although one of the three parameters λ, γ_+, γ_- is redundant, we shall maintain all of them in order to simplify the equations.

The solutions to the unconstrained equations of motion of the above theory are given by Eqs. (10) and (11). Now, in terms of the $A_\pm$, $a_\pm$ functions the constraint equations (5) become

$$T^f_{\pm\pm}=-\frac{1}{2\beta}(\{A_\pm,x^\pm\}-\{a_\pm,x^\pm\}), \tag{17}$$

where $\{\,,\}$ denotes the Schwartzian derivative.

In the absence of matter fields and in an appropriate Kruskal-type gauge $a_\pm=x^\pm$ the general solution is given by

$$ds^2=\frac{-dx^+dx^-}{\left(\frac{\lambda^2\beta}{C}+\gamma_+Cx^+x^-\right)(1-\gamma_-\lambda^2\beta x^+x^-)}, \tag{18}$$

$$e^{\beta\phi}=\frac{1-\gamma_-\lambda^2\beta x^+x^-}{\frac{\lambda^2\beta}{C}+\gamma_+Cx^+x^-}, \tag{19}$$

where the parameter C is related to the conserved quantity M [proportional to the Arnowitt-Deser-Misner (ADM) mass]:

$$M=\frac{1}{\beta}\left(\frac{C}{\lambda^2\beta}\gamma_+-\frac{\lambda^2\beta}{C}\gamma_--\gamma_++\gamma_-\right). \tag{20}$$

In a ''pure'' two-dimensional context and in order to study the full spacetime structure of the solution we will place no restriction on the range of variation of the field ϕ. Of course, if our starting point were four dimensional, the identification of ϕ with the radius of the two-sphere r would imply that only $\phi>0$ is allowed. The curvature of the solution is

$$R=-4\lambda^2\beta\left(\gamma_+\frac{1-\gamma_-\lambda^2\beta x^+x^-}{\frac{\lambda^2\beta}{C}+\gamma_+Cx^+x^-}-\gamma_-\frac{\frac{\lambda^2\beta}{C}+\gamma_+Cx^+x^-}{1-\gamma_-\lambda^2\beta x^+x^-}\right), \tag{21}$$

and there are two curvature singularities at

$$x^+x^-=\frac{-\lambda^2\beta}{\gamma_+C^2}, \tag{22}$$

$$x^+x^-=\frac{1}{\gamma_-\lambda^2\beta}. \tag{23}$$

In order to avoid timelike singularities we have two possibilities: $\beta<0$, $\gamma_+>0$, $\gamma_-<0$, or $\beta>0$, $\gamma_+<0$, $\gamma_->0$. They are actually the same because the potential (8) is symmetric under the interchange of both cases. The Kruskal diagram is represented in Fig. 1.

The horizons ($\partial_\pm e^{\beta\phi}=0$) are located at $x^\pm=0,\pm\infty$, respectively. The Killing vector $\partial/\partial t$ is timelike in regions I and spacelike in the others.

Choosing $\beta<0$ we can define $\gamma>0$ so that $-\gamma_+/\gamma_-=1/\gamma$ and we are able to redefine the parameter λ in order to absorb the extra parameter. In this way, a hyperbolic model having eternal black hole solutions is given by the following functional action:

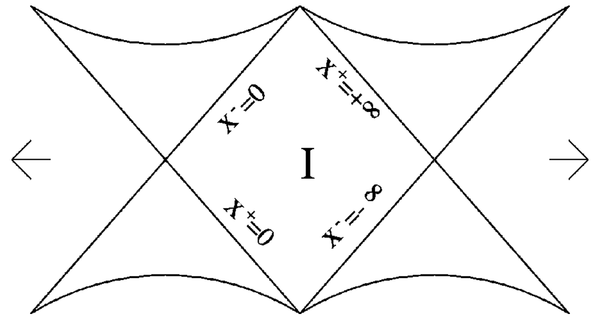

FIG. 1. Kruskal diagram for the hyperbolic model.

$$S=\frac{1}{2\pi}\int d^2x\sqrt{-g}\left(R\phi+4\lambda^2(e^{\beta\phi}-\gamma e^{-\beta\phi})-\frac{1}{2}\sum_{i=1}^{N}(\nabla f_i)^2\right). \tag{24}$$

It is interesting to note that we do not lose generality by restricting ourselves to the case $\gamma=1$. In fact, even if we consider $\gamma\neq1$, the redefinitions $e^{\beta\phi}\to e^{\beta\phi}\sqrt{\gamma}$ and $\lambda^2\sqrt{\gamma}\to\lambda^2$ recast the potential in the $\gamma=1$ form. Moreover, the constant shift in the field ϕ will produce an extra piece in the action proportional to R, but this, being just a boundary term, does not affect the equations of motion. We then consider the following potential:

$$V(\phi)=2\sinh\beta\phi. \tag{25}$$

Its geometry is given by the metric

$$ds^2=\frac{-dx^+dx^-}{\left(\frac{\lambda^2\beta}{C}+Cx^+x^-\right)(1+\lambda^2\beta x^+x^-)}, \tag{26}$$

with dilaton function

$$e^{\beta\phi}=\frac{1+\lambda^2\beta x^+x^-}{\frac{\lambda^2\beta}{C}+Cx^+x^-}, \tag{27}$$

and M and the curvature read

$$M=\frac{1}{\beta}\left(\frac{C}{\lambda^2\beta}+\frac{\lambda^2\beta}{C}-2\right), \tag{28}$$

$$R=-4\lambda^2\beta\left(\frac{1+\lambda^2\beta x^+x^-}{\frac{\lambda^2\beta}{C}+Cx^+x^-}+\frac{\frac{\lambda^2\beta}{C}+Cx^+x^-}{1+\lambda^2\beta x^+x^-}\right). \tag{29}$$

This model is interesting due to the presence of a dilaton-constant solution. The curvature has generically two singularities at points (22), (23) ($\gamma_+=-\gamma_-=1$). However, in the limit $C\to\lambda^2\beta$ it becomes regular and constant everywhere and the dilaton field is constant $e^{\beta\phi}=1$. The similarity of this solution with a known one in Einstein gravity will be explored in the next section.

IV. DEGENERATE HORIZON SOLUTIONS AND COMPARISON WITH THE SCHWARZSCHILD–de SITTER CASE

In this section we shall study the particular $\phi=0$ solution of the model (25) because it has a special similarity with the Nariai solution appearing in the Schwarzschild–de Sitter solution [11,12]. The Schwarzschild–de Sitter metric is the static spherically symmetric solution of the Einstein equations with a cosmological constant Λ. It is

$$ds^2=-\tilde{U}(r)dt^2+\tilde{U}(r)^{-1}dr^2+r^2d\Omega^2, \tag{30}$$

where

$$\tilde{U}(r)=1-\frac{2m}{r}-\frac{\Lambda}{3}r^2. \tag{31}$$

For $0<m<\frac{1}{3}\Lambda^{-1/2}$, $\tilde{U}(r)$ has two positive roots corresponding to the black hole and cosmological horizons. But in the limit $m\to\frac{1}{3}\Lambda^{-1/2}$ the two roots coincide and the horizons apparently merge. In this degenerate case the Schwarzschild coordinates become inappropriate since $\tilde{U}(r)\to 0$ between the two horizons. According to Ginsparg and Perry [12] we can define new coordinates ψ and χ:

$$t=\frac{1}{\epsilon\sqrt{\Lambda}}\psi, \quad r=\frac{1}{\sqrt{\Lambda}}\left[1-\epsilon\cos\chi-\frac{1}{6}\epsilon^2\right], \tag{32}$$

where

$$9m^2\Lambda=1-3\epsilon^2, \tag{33}$$

with the property that the new metric has a well-defined limit in the degenerate case $\epsilon\to 0$,

$$ds^2=-\frac{1}{\Lambda}(\sin^2\chi d\psi^2-d\chi^2)+\frac{1}{\Lambda}d\Omega^2, \tag{34}$$

which turns out to be the Nariai solution.

A similar situation is found in the model (25). To see this feature we consider the static solution (we call this the Schwarzschild gauge) that the 2D Birkhoff theorem [13] provides for a generic model (1). This solution is written as

$$ds^2=-[4J(\phi)-4M]dt^2+[4J(\phi)-4M]^{-1}dr^2, \tag{35}$$

$$\phi=\lambda r, \tag{36}$$

where

$$M=J(\phi)-\frac{1}{4\lambda^2}(\nabla\phi)^2 \tag{37}$$

is a diffeomorphism invariant parameter related to the ADM mass and $J(\phi)=\int_0^\phi d\tilde{\phi}V(\tilde{\phi})$. For the model (25) we get the following metric:

$$ds^2=-U(r)dt^2+U(r)^{-1}dr^2, \tag{38}$$

where

$$U(r)=\frac{8}{\beta}(\cosh\lambda\beta r-1)-4M. \tag{39}$$

If we consider $\beta<0$, $M\leqslant 0$ solutions, there are two horizons $(U(r_\pm)=0)$ located at

$$r_\pm=\pm\frac{1}{\lambda\beta}\operatorname{arcosh}\left(1+\frac{1}{2}\beta M\right), \tag{40}$$

but in the limit $M\to 0$ the horizons become coincident $(r_\pm=0)$ and $U(r)\to 0$ between them. There are two curvature singularities at $r=\pm\infty$ since the curvature is

$$R=-8\lambda^2\beta\cosh\lambda\beta r. \tag{41}$$

We can interpret this solution as two "mirror" black holes located "at infinity" hidden by two horizons $r_\pm$. The spacetime between the horizons admits a timelike Killing vector $[U(r)>0]$, which becomes spacelike behind the horizons $[U(r)<0]$. In the limit $M\to 0$ the horizons coalesce and in this region $U(r)\to 0$, $\phi\to 0$, $R\to -8\lambda^2\beta$. In this limit the (t,r) coordinates become inappropriate and we need to perform a coordinate change.

If we define a parameter C so that M is written as Eq. (28), the $M\to 0$ limit is recovered in the $C\to\lambda^2\beta$ limit. Thus let us try the following transformation:

$$-x^+x^-=\frac{\dfrac{\lambda^2\beta}{C}-e^{-\lambda\beta r}}{C-\lambda^2\beta e^{-\lambda\beta r}}, \tag{42}$$

$$-\frac{x^+}{x^-}=e^{4\lambda(\lambda^2\beta/C-C/\lambda^2\beta)t}, \tag{43}$$

relating both Kruskal and Schwarzschild gauges as it brings Eqs. (38), (39), (36) into Eqs. (26), (27), respectively. This transformation is singular for the degenerate case $C=\lambda^2\beta$ $(M=0)$ as the Ginsparg-Perry one for the 4D Schwarzschild–de Sitter gravity [12]. We can actually see it as a perturbation around the point $r=0$ where both horizons coincide. When $C=\lambda^2\beta$ there are no singularities and the metric (26) turns into

$$ds^2=\frac{-dx^+dx^-}{(1+\lambda^2\beta x^+x^-)^2} \tag{44}$$

and, finally, the new transformation

$$x^\pm=\frac{1}{\lambda\sqrt{-\beta}}(\sinh\psi\pm\cosh\psi)\frac{\sin\chi}{1+\cos\chi} \tag{45}$$

brings it into the 2D-reduced part of the Nariai solution [8] with topology H^2:

$$ds^2=\frac{-1}{4\lambda^2\beta}(-\sin^2\chi d\psi^2+d\chi^2). \tag{46}$$

Note that even though the transformation (42), (43) is singular for the degenerate case, the coordinates $x^\pm$ remain appropriate for this case too and the horizons' radii also remain different. The true reason for which this transformation becomes singular in the limit $C\to\lambda^2\beta$ is due to the fact that both Kruskal gauge (constant dilaton) and Schwarzschild gauge (linear dilaton) solutions are not diffeomorphism connected. They are indeed two different solutions and this motivates a revision of the 2D Birkhoff theorem which will be made in the next section.

To finish this section we shall consider the thermodynamics of this model. Since the static Schwarzschild gauge (38) is not the appropriate one to study the thermodynamics due to the degenerate limit, we look for another one, starting from the conformal-Kruskal gauge (26). This is possible since the model always admits a timelike Killing vector. Thus let us introduce new static coordinates $y^\pm$ given by

$$\pm\omega x^\pm=e^{\pm\omega y^\pm}, \tag{47}$$

where $\omega^2=-C$. In terms of these coordinates the metric becomes

$$ds^2=\frac{-dt^2+dy^2}{\left(1-\frac{\lambda^2\beta}{C}\right)^2+\frac{4\lambda^2\beta}{C}\cosh^2\omega y}. \tag{48}$$

The metric is manifestly static in this form and it is straightforward to find a new Schwarzschild-type gauge by means of the new spacelike coordinates defined by

$$\sigma=\frac{1}{\omega\left(1+\frac{\lambda^2\beta}{C}\right)\left(1-\frac{\lambda^2\beta}{C}\right)}\times\operatorname{arctanh}\left[\frac{\left(1-\frac{\lambda^2\beta}{C}\right)}{\left(1+\frac{\lambda^2\beta}{C}\right)}\tanh\omega y\right]. \tag{49}$$

The new Schwarzschild-type metric is then

$$ds^2=-U(\sigma)dt^2+\frac{d\sigma^2}{U(\sigma)}, \tag{50}$$

where

$$U(\sigma)=\frac{1-\frac{\left(1+\frac{\lambda^2\beta}{C}\right)^2}{\left(1-\frac{\lambda^2\beta}{C}\right)^2}\tanh^2\left[\left(1+\frac{\lambda^2\beta}{C}\right)\left(1-\frac{\lambda^2\beta}{C}\right)\omega\sigma\right]}{\left(1+\frac{\lambda^2\beta}{C}\right)^2\left\{1-\tanh^2\left[\left(1+\frac{\lambda^2\beta}{C}\right)\left(1-\frac{\lambda^2\beta}{C}\right)\omega\sigma\right]\right\}}. \tag{51}$$

The horizons $U(\sigma_\pm)=0$ are

$$\sigma_\pm=\pm\frac{\operatorname{arctanh}\frac{\left(1-\frac{\lambda^2\beta}{C}\right)}{\left(1+\frac{\lambda^2\beta}{C}\right)}}{\left(1+\frac{\lambda^2\beta}{C}\right)\left(1-\frac{\lambda^2\beta}{C}\right)\omega}. \tag{52}$$

In these coordinates we can study the degenerate case $C=\lambda^2\beta$ since they will still be able to "see" the region between the horizons. In this limit the solution becomes

$$U(\sigma)=\frac{1-(4\omega\sigma)^2}{4} \tag{53}$$

and the horizons still remain uncoincident:

$$\sigma_\pm=\pm\frac{1}{4\omega}. \tag{54}$$

To get the horizon temperature we should construct the Euclidean metric, setting $it=\tau$ and identifying τ with an appropriate period in order to remove the singularities. But this is not so in this case because the Killing vector cannot be normalized at infinity as in the standard Schwarzschild case, due to the presence of the singularities. Bousso and Hawking [14] give the correct prescription. We need to find the point σ_g for which the orbit of the Killing vector coincides with the geodesic going through σ_g. In such a point the effects of both black hole attractions balance out exactly and an observer will need no acceleration ($\Gamma^\rho_{\mu\nu}=0$) to stay there, just like an observer at infinity in the standard Schwarzschild case. A straightforward calculation shows that this point is just where both horizons coincide in the degenerate case ($r=0$), that is, $\sigma_g=0$. With an adequate normalization the horizon temperatures are given by [14]

$$T_{\pm}=\frac{1}{2\pi}\frac{1}{2\sqrt{U(\sigma_g)}}\left|\frac{\partial U}{\partial\sigma}\right|_{\sigma_{\pm}}, \qquad (55)$$

and then we get

$$T_{+}=T_{-}=\frac{1}{2\pi}\left(1+\frac{\lambda^2\beta}{C}\right)\sqrt{-C} \qquad (56)$$

and, in the $C\rightarrow\lambda^2\beta$ limit,

$$T_{+}=T_{-}\rightarrow\frac{\sqrt{-\lambda^2\beta}}{\pi}. \qquad (57)$$

Note that the horizon temperatures are always coincident in either the nondegenerate or degenerate case in a different way from the 4D Schwarzschild–de Sitter case. We can then complete the physical picture of this model; the two mirror black holes are at the same temperature. This feature will have some important consequences on the semiclassical theory as we will see later.

Finally we have to note that the transformation (47) is performed in the region between the horizons $x^+x^-<0$. We can realize a new transformation in order to take into account the black hole interiors $x^+x^->0$:

$$\omega x^{\pm}=e^{\pm\omega y^{\pm}}. \qquad (58)$$

In this case the static metric is

$$ds^2=\frac{-dt^2+dy^2}{\left(1-\frac{\lambda^2\beta}{C}\right)^2-\frac{4\lambda^2\beta}{C}\sinh^2\omega y} \qquad (59)$$

and a further transformation

$$\sigma=\frac{1}{\omega\left(1+\frac{\lambda^2\beta}{C}\right)\left(1-\frac{\lambda^2\beta}{C}\right)}\times\text{arctanh}\left[\frac{\left(1-\frac{\lambda^2\beta}{C}\right)}{\left(1+\frac{\lambda^2\beta}{C}\right)}\text{cotanh}\,\omega y\right] \qquad (60)$$

brings the metric into the same geometry (50), (51) so that there is no difference from the last one as expected.

We now wish to comment briefly on the case $\beta>0$ where the physical picture is completely different; i.e., the singularities are timelike and in the region between the horizons the Killing vector is spacelike. The Kruskal diagram is similar to that of a point electric charge in 2+1 dimensions [15]. Formally, the analysis of this section can be repeated step by step for this solution as well. When the two horizons become degenerate there is again a Ginsparg-Perry-type transformation connecting the constant (now negative) curvature, $\phi=0$ solution with Eqs. (38), (39), in much the same the way as it has been done in [16] for the 2D dilaton-Maxwell gravity.

V. 2D BIRKHOFF THEOREM REVISITED

Now we analyze the existence of dilaton-constant solutions in a more general context. This feature leads us to perform a revision of the 2D Birkhoff theorem. Under some assumptions one can ensure that the general solution is given, up to space-time diffemorphisms, by a one-parameter family of static metrics [13]. The parameter, related with the ADM mass, is diffeomorphism invariant and classifies all of them. In particular, there exists a Schwarzschild gauge in which the solution is manifestly static and the dilaton field is linear in the spacelike coordinate.

Considering the gravitational sector of Eq. (1),

$$S=\frac{1}{2\pi}\int d^2x\sqrt{-g}[R\phi+4\lambda^2V(\phi)], \qquad (61)$$

this solution is written as Eqs. (35), (36), where M, given by Eq. (37), is the diffeomorphism-invariant parameter. We shall show that there is also another type of solutions. For certain potentials there is, in fact, another static solution providing a constant curvature space with a constant dilaton field. The equations of motion (2), (3) of the above functional action in a static gauge $\partial\phi/\partial t=0=\partial\rho/\partial t$ (where $x^{\pm}=t\pm x$) are

$$-\frac{d^2\rho}{dx^2}+2\lambda^2e^{2\rho}\frac{dV}{d\phi}=0, \qquad (62)$$

$$-\frac{d^2\phi}{dx^2}+4\lambda^2e^{2\rho}V=0. \qquad (63)$$

If $d\phi/dx\neq0$, Eq. (63) admits a first integral

$$-\frac{d\phi}{dx}+4\lambda^2\int dxe^{2\rho}V(\phi)=4\lambda M, \qquad (64)$$

where M is an integration constant and, using the constraints, Eq. (62) turns into

$$\lambda e^{2\rho}=\frac{d\phi}{dx}. \qquad (65)$$

Equation (64) gives the conformal factor $e^{2\rho}=4J(\phi)-4M$ and in the Schwarzschild gauge, defined by $dr=e^{2\rho}dx$, we get finally the set (35), (36). This is essentially the Birkhoff theorem [13]. Now we are going to consider the $d\phi/dx=0$ case, that is, dilaton-constant solutions.[1] This kind of solution $\phi=\phi_0$ can only exist for certain potentials $V(\phi)$ satisfying

[1]The existence of these kinds of solutions was already noted in [17].

$$V(\phi_0)=0, \quad \left.\frac{dV(\phi)}{d\phi}\right|_{\phi_0} \neq 0, \tag{66}$$

so that Eq. (63) is trivially satisfied and Eq. (62) becomes

$$\frac{d^2\rho}{dx^2}+\frac{R_0}{2}e^{2\rho}=0, \tag{67}$$

where

$$R_0=-4\lambda^2 \left.\frac{dV}{d\phi}\right|_{\phi_0}=\text{const.} \tag{68}$$

Thus these solutions lead to constant curvature spacetimes. Making the coordinate change $dr=e^{2\rho}dx$ into the Schwarzschild gauge Eq. (67) is easily integrated and the solution is written as

$$ds^2=-\left(k-\frac{R_0}{2}r^2\right)dt^2+\left(k-\frac{R_0}{2}r^2\right)^{-1}dr^2, \tag{69}$$

$$\phi=\phi_0=\text{const}, \tag{70}$$

where k is an integration constant.

Obviously both solutions (35), (36) and (69), (70) are not diffeomorphism connected as is manifested by the scalar dilaton function. Note that this last dilaton-constant solution is not available for a generic potential $V(\phi)$ but only for those satisfying the conditions (66). One example is the $\sinh \beta\phi$ potential (25); another one is provided in the Appendix, starting from Einstein-Maxwell gravity in 4D. In the conformal gauge, in the special limit $C\rightarrow\lambda^2\beta$, we obtained the dilaton-constant $(\phi=0)$ solution (44) with $M=0$ and constant curvature $R=R_0=-8\lambda^2\beta$. In a manifestly static gauge, it reads

$$ds^2=-(1+4\lambda^2\beta r^2)dt^2+(1+4\lambda^2\beta r^2)^{-1}dr^2. \tag{71}$$

But $\phi=0$ is just the dilaton-constant solution for the $\sinh\beta\phi$ potential: $V(0)=0$, $dV(\phi)/d\phi|_{\phi_0}\neq 0$ and moreover $J(0)=0$ so that the expression (37) becomes identically zero. The above solution coincides with Eq. (69) (with $k=1$). Now we can complete our understanding of the $C\rightarrow\lambda^2\beta$ limit of the solution (26), (27) in the Kruskal gauge. The $C\neq\lambda^2\beta$ case coincides, up to diffeomorphisms, with the $M\neq 0$ parametrized solution (35), (36) and the $C=\lambda^2\beta$ case with the unparametrized solution (69), (70). These solutions are different and they cannot be diffeomorphism connected. The special case $M=0$ in Eqs. (35), (36), which at first sight we could be tempted to identify with Eqs. (69), (70), is the horizon coincident case and region I of Fig. 1 is reduced to the point $r=0$ where $\phi=0$ and $R=-8\lambda^2\beta$. The transformation (42), (43) connects both gauges in a similar way to the Ginsparg-Perry one and this suggests that there is a deep relation between the existence of constant dilaton solutions and horizon degeneration. In fact this is what happens in general and we shall show this in the remaining part of this section.

Let us consider again the general solution (35), (36) for a general potential $V(\phi)$ and introduce $U(r)=4J(r)-4M$ so that the horizons are the roots of $U(r)$. In order to study models with horizon degeneration we want $U(r)$ to have two or more roots. Although all roots are distinct we can always fit a value M_0 of the parameter M for which two neighboring roots become coincident in, say, r_0 which is then a double root of $U(r)$. The "critical" value of M is $M_0=J(r_0)$ and the dilaton function at this point is $\phi_0=\lambda r_0$. Now, since r_0 is an extremal of $U(r)$, we get

$$0=\left.\frac{dU}{dr}\right|_{r_0}=4\lambda V(\phi_0), \tag{72}$$

$$0\neq \left.\frac{d^2U}{dr^2}\right|_{r_0}=4\lambda^2\left.\frac{dV}{d\phi}\right|_{\phi_0}, \tag{73}$$

which are just the conditions (66), and then $\phi=\phi_0$ gives the constant dilaton solution (69), (70). It is straightforward to check that the opposite is true as well: if ϕ_0 is a constant dilaton solution, $r_0=\phi_0/\lambda$ is a degenerate horizon for $M=M_0=J(\phi_0)$.

Let us now perform a perturbation around the degenerate radius of coincident horizons, as happens in the limit $M\rightarrow M_0$ and $U(r)\rightarrow 0$ between the two horizons. We write

$$M=M_0-\frac{k}{4}\epsilon^2, \tag{74}$$

where $\epsilon\ll 1$ and k is a constant with the same sign as R_0. The degenerate case corresponds to $\epsilon\rightarrow 0$. We introduce a new coordinate pair $(\tilde{t},\tilde{r})$ defined by

$$t=\frac{\tilde{t}}{\epsilon}, \quad r=r_0+\epsilon\tilde{r}. \tag{75}$$

Expanding the function $U(r)$ in powers of $r-r_0$ we get

$$U(r)=\left(k-\frac{R_0}{2}\tilde{r}^2\right)\epsilon^2+O(\epsilon^3), \tag{76}$$

which finally turns Eq. (35) into

$$ds^2=-\left(k-\frac{R_0}{2}\tilde{r}^2+O(\epsilon)\right)d\tilde{t}^2+\left(k-\frac{R_0}{2}\tilde{r}^2+O(\epsilon)\right)^{-1}d\tilde{r}^2. \tag{77}$$

This in the "near-horizon" limit $\epsilon\rightarrow 0$ becomes Eq. (69).

We end by noting that for $R_0<0$—i.e., the solution has constant negative curvature—k is negative and redefining it as $k\equiv -m$ this is nothing but the AdS_2 black hole.

VI. SEMICLASSICAL THEORY AND CONCLUSIONS

We shall now make some semiclassical considerations concerning the $\sinh\beta\phi$ model. The Hawking radiation is determined by the usual expression

$$\langle T^f_{--}\rangle = \frac{N}{12}[\partial_-^2\rho - (\partial_-\rho)^2 - t_-], \tag{78}$$

and we now show why the choice $t_-=0$ in Kruskal coordinates is the most natural one. The privileged point $(r=0)$ in which the Killing vector must be normalized corresponds in Kruskal coordinates with the curve

$$x^+x^- = \frac{1}{C}. \tag{79}$$

If we calculate $\langle T_{--}\rangle$, we get

$$\langle T^f_{--}\rangle = \frac{N(x^+)^2}{48}\left[\frac{1}{1+\lambda^2\beta x^+x^-} - \frac{1}{Cx^+x^- + \dfrac{\lambda^2\beta}{C}}\right]^2. \tag{80}$$

This expression exactly vanishes when evaluated over the points of the curve (79). The interpretation is then that because the two black holes placed at infinity have the same temperature, there is a compensation between the Hawking radiation coming from each black hole, giving no net Hawking flux. The same considerations apply if we interchange $-$ with $+$ in the previous formulas, and we have $t_+=0$ as well. We can also wonder if it makes sense to choose ''evaporating'' boundary conditions $t_-\neq t_+$. At the classical level and by virtue of the Birkhoff theorem the solutions are parametrized by a single constant C, forcing the two black holes to have the same mass and temperature. However, at the semiclassical level the Birkhoff theorem no longer applies and we could try, for instance, to increase the mass of one of the black holes and to see whether or not a new equilibrium state is reached. Moreover, if in view of a higher-dimensional interpretation we restrict ourselves to the case $\phi>0$, then the physical spacetime contains only one black hole and it would seem natural to impose boundary conditions different from the ones used above. These questions and the related semiclassical dynamical evolutions will be studied elsewhere.

It is interesting to comment that in the Jackiw-Teitelboim limit the curvature singularities disappear and we get contant curvature AdS_2 black holes (if $\beta>0$). AdS_2 black holes have been claimed not to emit Hawking radiation [18] (if a nontrivial dilaton is present, however, this might not be true; see [19]), which is exactly what happens in our $\sinh\beta\phi$ model although there the lack of radiation can be understood by the presence of the mirror black hole. Therefore intuitively the AdS_2 black hole inherits the no-radiation property of the more general model they arise in a certain limit. This is not the case of the exponential model in which black holes evaporate [7]. In this model and with the boundary conditions $t_\pm=0$ the solutions represent black holes in equilibrium with a thermal bath. So the role of the mirror black hole is interchanged with the existence of external radiation incoming onto the black hole.

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APPENDIX: $AdS_2\times S^2$ GEOMETRY IN EINSTEIN-MAXWELL THEORY

In this appendix we shall describe a way to generate the Robinson-Bertotti ($AdS_2\times S^2$) geometry in Einstein-Maxwell gravity based on the possibility of constructing constant-dilaton solutions explained in Sec. V. Let us start with the Einstein-Maxwell action

$$I = \frac{1}{16\pi G^{(4)}}\int d^4x\sqrt{-g^{(4)}}[R^{(4)} - (F^{(4)})^2]. \tag{A1}$$

If we impose spherical symmetry on the gauge field and the metric

$$ds^2_{(4)} = g_{\mu\nu}dx^\mu dx^\nu + \frac{\phi^2}{2\lambda^2}d\Omega^2, \tag{A2}$$

where $x^\mu=(t,r)$, $d\Omega^2$ is the metric on the two-sphere, and λ^{-1} is the Planck length ($\lambda^{-2}=G^{(4)}$), the dimensionally reduced action functional is [20]

$$\int d^2x\sqrt{-g}\left[\frac{1}{2}\left(\frac{\phi^2}{4}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \lambda^2\right) - \frac{1}{8}\phi^2F^{\mu\nu}F_{\mu\nu}\right]. \tag{A3}$$

After an appropriate reparametrization

$$\frac{\phi^2}{4}\to\phi, \tag{A4}$$

$$g_{\mu\nu}\to g_{\mu\nu}(2\phi)^{-1/2}, \tag{A5}$$

the two-dimensional action takes the form [21]

$$\int d^2x\sqrt{-g}\left[\frac{1}{2}[\phi R + \lambda^2 V(\phi)] - \frac{1}{4}W(\phi)F^{\mu\nu}F_{\mu\nu}\right], \tag{A6}$$

where

$$V(\phi) = \frac{1}{\sqrt{2\phi}}, \quad W(\phi) = (2\phi)^{3/2}. \tag{A7}$$

The equations of motion imply that [21]

$$F = q\frac{e^{2\rho}}{W(\phi)}, \tag{A8}$$

where $F=2F_{+-}$ and q is a constant. Substituting the above solution for F into the other equations of motion one finds that they are equivalent to those of the model (61) with the replacement

$$V(\phi)\rightarrow V(\phi)-\frac{q^2}{\lambda^2 W(\phi)}=V_{eff} \tag{A9}$$

and so in our case

$$V_{eff}=\frac{1}{\sqrt{2\phi}}-\frac{q^2}{\lambda^2(2\phi)^{3/2}} \tag{A10}$$

and we can apply the arguments of Sec. V. We then have a constant dilaton solution $\phi=\phi_0$ for

$$V_{eff}=0 \tag{A11}$$

and therefore

$$\phi_0=\frac{q^2}{2\lambda^2}, \tag{A12}$$

which turns out to be the radius of the horizon for the extremal Reissner-Nordström solution $r_+=r_-=(1/\lambda)\sqrt{2\phi_0}=q/\lambda^2$. Moreover, the two-dimensional geometry is AdS_2 with curvature

$$R=-\frac{2\lambda^4}{q^2}=-\frac{2}{r_+^2}. \tag{A13}$$

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