Lattice QCD calculation of $\bar{B}\to Dl\bar{\nu}$ decay form factors at zero recoil

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A lattice QCD calculation of the $\bar{B}\to Dl\bar{\nu}$ decay form factors is presented. We obtain the value of the form factor $h_+(w)$ at the zero-recoil limit $w=1$ with high precision by considering a ratio of correlation functions in which the bulk of the uncertainties cancels. The other form factor $h_-(w)$ is calculated, for small recoil momenta, from a similar ratio. In both cases, the heavy quark mass dependence is observed through direct calculations with several combinations of initial and final heavy quark masses. Our results are $h_+(1)=1.007(6)(2)(3)$ and $h_-(1)=-0.107(28)(04)(^{10}_{30})$. For both the first error is statistical; the second stems from the uncertainty in adjusting the heavy quark masses and the last from omitted radiative corrections. Combining these results, we obtain a precise determination of the physical combination $\mathcal{F}_{B\to D}(1)=1.058(^{20}_{17})$, where the mentioned systematic errors are added in quadrature. The dependence on lattice spacing and the effect of quenching are not yet included, but with our method they should be a fraction of $\mathcal{F}_{B\to D}-1$.

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I. INTRODUCTION

The precise determination of the Cabibbo-Kobayashi-Maskawa (CKM) matrix element V_{cb} is a crucial step for B physics to pursue phenomena beyond the standard model. In particular, the precision achieved in determining the apex of the unitarity triangle may be limited by $|V_{cb}|$, even with future high-statistics experiments. The current determination of $|V_{cb}|$ [1] is made through inclusive [2,3] and exclusive [4,5] B decays.

The heavy quark expansion offers a method to evaluate the hadronic transition amplitude in a systematic way. In particular, at the kinematic end point the exclusive $\bar{B}\to D^*$ matrix element is normalized in the infinite heavy quark mass limit, and the correction of order $1/m_Q$ vanishes as a consequence of Luke's theorem [6]. It is thus possible to achieve an accuracy on $|V_{cb}|$ of a few percent. Calculations of the $1/m_Q^2$ (and higher order) deviations from the heavy quark limit have previously been attempted with the nonrelativistic quark model and with QCD sum rules.

Lattice QCD has the potential to calculate exclusive transition matrix elements from first principles. The shapes of the $\bar{B}\to D^{(*)}l\bar{\nu}$ decay form factors have already been calculated successfully with propagating [7–9], static [10–13], and non-relativistic [14] heavy quarks. On the other hand, a precise determination of the absolute normalization of the form factors has not been achieved. This paper fills that gap for the decay $\bar{B}\to Dl\bar{\nu}$.

Previous lattice calculations were unable to obtain the normalization of the form factors for various reasons. First, the statistical precision of the three point function $\langle DV_\mu B^\dagger\rangle$, which is calculated by Monte Carlo integration, has not been enough. Second, perturbative matching between the lattice and the continuum currents has been a large source of uncertainty. Since the local vector current defined on the lattice is not a conserved current at finite lattice spacing a, the matching factor is not normalized even in the limit of degenerate quarks. Although one-loop perturbation theory works significantly better with tadpole improvement [15], the two-loop contribution remains significant ($\alpha_s^2\sim 5\%$). Last, the systematic error associated with the large heavy quark mass must be understood. Previous work with Wilson quarks [7–9], for which the discretization error was as large as $O(am_Q)$, could not address the $1/m_Q$ dependence in a systematic way when $m_Q\gtrsim 1/a$.

In this paper we present a lattice QCD calculation of the $\bar{B}\to Dl\bar{\nu}$ decay form factor. For the heavy quark we use an improved action [16] for Wilson fermions, reinterpreted in a way mindful of heavy-quark symmetry [17]. Discretization errors proportional to powers of am_Q do not exist in this approach. Instead, discretization errors proportional to powers of $a\Lambda_{\rm QCD}$ remain, although they are intertwined with the $1/m_Q$ expansion. The first extensive application of this approach to heavy-light systems was the calculation [18,19] of the heavy-light decay constants, such as f_B and f_D. There the lattice spacing dependence was studied from direct calculations at several lattice spacings, and a very small a dependence was observed. The third difficulty mentioned above is, thus, no longer a problem.

To obtain better precision on the semi-leptonic form factors, we introduce ratios of three-point correlation functions. The bulk of statistical fluctuations from the Monte Carlo integration cancels between numerator and denominator. Furthermore, the ratios are, by construction, identically one in both the degenerate-mass limit and the heavy-quark-symmetry limit. Consequently, statistical and all systematic errors, as well as the signal, are proportional to the deviation from 1. The first and second difficulties given above are, thus, also essentially cured.

The ratio of correlation functions for the calculation of $h_+(1)$ corresponds to the ratio of matrix elements,

$$\frac{\langle D|\bar{c}\gamma_0 b|\bar{B}\rangle\langle\bar{B}|\bar{b}\gamma_0 c|D\rangle}{\langle D|\bar{c}\gamma_0 c|D\rangle\langle\bar{B}|\bar{b}\gamma_0 b|\bar{B}\rangle}=|h_+(1)|^2, \tag{1}$$

in which all external states are at rest. The denominator may be considered as a normalization condition of the heavy-to-heavy vector current, since the vector current $\bar{q}\gamma_\mu q$ with degenerate quark masses is conserved in the continuum limit, and its matrix element is, therefore, normalized. As a result the perturbative matching between the lattice and continuum currents gives only a small correction to $|h_+(1)|$.

For the calculation of $h_-(w)$ we define another ratio, corresponding to matrix elements

$$\frac{\langle D|\bar{c}\gamma_i b|\bar{B}\rangle}{\langle D|\bar{c}\gamma_0 b|\bar{B}\rangle}\frac{\langle D|\bar{c}\gamma_0 c|D\rangle}{\langle D|\bar{c}\gamma_i c|D\rangle}=1-\frac{h_-(w)}{h_+(w)}, \tag{2}$$

where equality holds when the final-state D meson has small spatial momentum. By construction, the ratio produces a value of h_- that vanishes when the b quark has the same mass as the c quark, as required by current conservation.

This method does not work as it stands for the $\bar{B}\to D^* l\bar{\nu}$ decay form factors. The axial vector current mediates this decay, and it is neither conserved nor normalized. We will deal separately with this case in another paper.

This paper is organized as follows. Section II contains a general discussion of form factors for the exclusive decay $\bar{B}\to D l\bar{\nu}$. Sections III and IV discuss heavy quark effective theory and the $1/m_Q$ expansion in the continuum and with the lattice action used here. Section V contains details of the numerical calculations. Sections VI–IX present our results. Sections VI and VII discuss the form factor h_+ and its mass dependence. Sections VIII and IX do likewise for h_-. We compare the results from the fits of the mass dependence to corresponding results from QCD sum rules in Sec. X. The values of $h_+(1)$ and $h_-(1)$ at the physical quark masses are combined in Sec. XI into a result for the form factor $\mathcal{F}_{B\to D}(1)$, which with experimental data determines $|V_{cb}|$. We give our conclusions in Sec. XII.

II. $\bar{B}\to D l\bar{\nu}$ FORM FACTORS

The decay amplitude for $\bar{B}\to D l\bar{\nu}$ is parametrized with two form factors $h_+(w)$ and $h_-(w)$ as

$$\langle D(\mathbf{p}')|\mathcal{V}_\mu|\bar{B}(\mathbf{p})\rangle=\sqrt{m_B m_D}[h_+^{B\to D}(w)(v+v')_\mu + h_-^{B\to D}(w)(v-v')_\mu], \tag{3}$$

where v and v' are the velocities of the B and D mesons, respectively, and $w=v\cdot v'$. The square of the momentum transferred to the leptons is then $q^2=m_B^2+m_D^2-2m_B m_D w$. We denote by the symbol $\mathcal{V}_\mu$ the physical vector current, to distinguish it from currents in heavy quark effective theory (HQET) and in lattice QCD.

The differential decay rate reads

$$\frac{d\Gamma(\bar{B}\to D l\bar{\nu})}{dw}=\frac{G_F^2}{48\pi^3}(m_B+m_D)^2 m_D^3 \times(w^2-1)^{3/2}|V_{cb}|^2|\mathcal{F}_{B\to D}(w)|^2, \tag{4}$$

with

$$\mathcal{F}_{B\to D}(w)=h_+^{B\to D}(w)-\frac{m_B-m_D}{m_B+m_D}h_-^{B\to D}(w). \tag{5}$$

At zero recoil ($v'=v$, so $w=1$) one expects $\mathcal{F}_{B\to D}(1)$ to be close to 1, because of heavy quark symmetry. From Eq. (4) a determination of $|V_{cb}|$ consists of the following three steps: measure $|V_{cb}||\mathcal{F}_{B\to D}(w)|$ in an experiment, extrapolate it to the zero-recoil limit assuming some functional form, and use the theoretical input of $\mathcal{F}_{B\to D}(1)$.

In this paper we report on a new calculation of $\mathcal{F}_{B\to D}(1)$ with lattice QCD, which is model independent, at least in principle.[1] The present calculation includes the leading corrections to the heavy-quark limit: radiative corrections to the static limit of $h_+^{B\to D}(1)$, the $1/m_Q$ contribution of $h_-^{B\to D}(1)$, and the $1/m_Q^2$ contributions of $h_\pm^{B\to D}(1)$. Radiative corrections of order α_s to $h_-(1)$ are not yet available, but these and further corrections, of order α_s^2, α_s/m_Q, etc., could be included in future applications of our numerical technique, once the needed perturbative results become available.

An obvious disadvantage in using the $\bar{B}\to D l\bar{\nu}$ decay mode is that the branching fraction is much smaller than the $\bar{B}\to D^* l\bar{\nu}$ mode. Another, but not less important, shortcoming is that the phase-space suppression factor $(w^2-1)^{3/2}$ makes the extrapolation of the experimental data to $w=1$ more difficult than for $\bar{B}\to D^* l\bar{\nu}$, where the corresponding factor is $(w^2-1)^{1/2}$. Nevertheless, the experimental result of the CLEO Collaboration [20] shows that the above method certainly works, even with current statistics. That means that future improvement of the statistics will allow a much better determination of $|V_{cb}|$, providing an important cross check against other methods.

III. HQET AND THE $1/m_Q$ EXPANSION

Many important theoretical results have been obtained for the form factors with HQET. The Lagrangian of HQET uses fields of infinitely heavy quarks, so that heavy quark symmetries are manifest. The effects of finite quark mass are included through the $1/m_Q$ expansion and through radiative corrections. For example, at zero recoil the form factor h_+ is given by

$$h_+(1)=\eta_V\left[1-c_+^{(2)}\left(\frac{1}{m_c}-\frac{1}{m_b}\right)^2+O(m_Q^{-3})\right], \tag{6}$$

where η_V represents a matching factor relating the vector current in Eq. (3) to the current in HQET [21]. The absence of the $O(1/m_Q)$ term in Eq. (6) is a result of a symmetry under an interchange of initial and final states in Eq. (3), and it is known as a part of Luke's theorem [6]. The same symmetry also restricts the form of the $O(1/m_Q^2)$ terms.

[1] Our calculations are done in the quenched approximation, for example, but this is a removable uncertainty and not a permanent limitation of the method.

The matching factor, defined so that the identity $\mathcal{V}_0 = \eta_V V_0^{\rm HQET}$ holds for matrix elements, is an ultraviolet- and infrared-finite function of m_c/m_b. Through one-loop perturbation theory

$$\eta_{V^{cb}} = 1 + 3C_F \frac{\alpha_s}{4\pi}\left(\frac{m_b+m_c}{m_b-m_c}\ln\frac{m_b}{m_c} - 2\right). \tag{7}$$

The two-loop coefficient is also available [22].

The vector current defined with lattice fermion fields has properties similar to $V_0^{\rm HQET}$. There is a normalization factor Z_{V_0} defined so that $\mathcal{V}_0 = Z_{V_0} V_0^{\rm lat}$ holds for matrix elements. The factor Z_{V_0} depends strongly on the (lattice) quark masses am_c and am_b [17], and its one-loop corrections are large. In the past, such uncertainties in the normalization prevented a calculation of $h_+^{B\to D}(1)$ with the sought-after accuracy. One can, however, capture most of the normalization nonperturbatively by writing, with explicit flavor indices,

$$Z_{V_0^{cb}} = \sqrt{Z_{V_0^{cc}} Z_{V_0^{bb}}}\,\rho_{V_0^{cb}}. \tag{8}$$

In our ratio (1) the flavor-diagonal factors cancel, so our method avoids the major normalization uncertainties.

The remaining radiative correction $\rho_{V_0^{cb}}$ depends on the ratio of quark masses and the lattice spacing. In the continuum limit, $am_c \to 0$ and $am_b \to 0$ with m_c/m_b fixed,

$$\rho_{V_0} \to 1, \tag{9}$$

by construction. In the static limit, $am_c \to \infty$ and $am_b \to \infty$ with a and m_c/m_b fixed,

$$\rho_{V_0} \to \eta_V, \tag{10}$$

because the lattice theory strictly obeys heavy-quark symmetries. In numerical work one is somewhere in between, but the limits imply that ρ_{V_0} is never far from unity. Two of us have computed ρ_{V_0} at one loop in perturbation theory [23], verifying explicitly that the radiative correction is small.

Similarly, the ratio (2) is described by the expansion

$$1 - \frac{h_-(1)}{h_+(1)} = 1 - \beta_V + c_-^{(1)}\left(\frac{1}{m_c} - \frac{1}{m_b}\right) - c_-^{(2)}\left(\frac{1}{m_c^2} - \frac{1}{m_b^2}\right) + O(m_Q^{-3}), \tag{11}$$

where β_V is a coefficient from matching the currents in Eq. (2) to HQET. Like η_V, it is an ultraviolet- and infrared-finite function of m_c/m_b, and

$$\beta_{V^{cb}} = 2C_F \frac{\alpha_s}{4\pi}\left(\frac{2m_b m_c}{(m_b-m_c)^2}\ln\frac{m_b}{m_c} - \frac{m_b+m_c}{m_b-m_c}\right) \tag{12}$$

at leading order.

The ratio (2) again captures nonperturbatively most of the renormalization of the lattice currents, apart from a factor $\rho_{V_i^{cb}}$ to compensate for the difference between the radiative corrections with a fixed lattice cutoff and with no ultraviolet cutoff. In the continuum limit $\rho_{V_i} \to 1$, and in the static limit $\rho_{V_i} \to 1 - \beta_V$. Again, explicit calculation verifies that the one loop contribution remains small between the limits.

In the rest of this paper, we do not write the matching factors ρ_{V_μ} when there is no risk of confusion. In the final result, on the other hand, they are included.

IV. LATTICE QCD AND HEAVY QUARK SYMMETRY

In Ref. [17], it was shown that the usual action for light quarks [16] can be analyzed in terms of the operators of HQET. Therefore, it can be used as the basis of a systematic treatment of heavy quarks on the lattice, even when the quark mass in lattice units, am_Q, is not especially small. The key is to adjust the couplings in the lattice action so that operators are normalized as they are in HQET. When $am_Q < 1$, as is the case for charmed quarks at the smaller lattice spacings in common use, this is essentially automatic, because the higher order terms of the heavy quark expansion come from the Dirac term of the lattice action, as in continuum QCD. When $am_Q > 1$, as is the case for bottom quarks, one can apply the formalism of HQET to the lattice theory to obtain the normalization conditions, as sketched below. In either case, the kinetic energy is normalized nonperturbatively by tuning the quark mass according to some physical condition. Other operators are often normalized perturbatively as an initial approximation but ultimately may be normalized nonperturbatively.

In the numerical calculations presented here, we use an action introduced by Sheikholeslami and Wohlert [16],

$$S = \sum_{x,f} \overline{\psi}_x^f \psi_x^f - \sum_{x,y,f} \kappa_f \overline{\psi}_x^f M_{xy} \psi_y^f + \frac{i}{2} c_{\rm SW} \sum_{x,f} \kappa_f \overline{\psi}_x^f \sigma_{\mu\nu} F_{\mu\nu} \psi_x^f, \tag{13}$$

where the index f runs over heavy and light flavors. The hopping parameter κ_f is related to the bare quark mass,

$$am_{0f} = \frac{1}{2\kappa_f} - \frac{1}{2\kappa_{\rm crit}}, \tag{14}$$

where $\kappa_{\rm crit}$ is the value of κ needed to make a quark massless. The flavor-independent matrix M_{xy} vanishes except when $y = x \pm \hat{\mu} a$, for some spacetime direction μ. The kinetic energy arises from this term. The gluons' field strength $F_{\mu\nu}$ is defined on a set of paths shaped liked a four-leaf clover, so S is often called the ''clover'' action. With $c_{\rm SW} = 0$ one has the Wilson action.

For the light quark the clover coupling $c_{\rm SW}$ can be chosen so that there are lattice artifacts of order $a^2\Lambda_{\rm QCD}^2$. In our numerical work we take an approximation to the optimal value, leaving an artifact of order $\alpha_s a \Lambda_{\rm QCD}$.

For heavy quarks, the clover action (13) has the same heavy-quark spin and flavor symmetries as continuum QCD, even at nonzero lattice spacing. Consequently, we can use the machinery of HQET to characterize the lattice theory. The same operators as in continuum QCD appear, but the

coefficients can differ. Through first order in $1/m_Q$ there are three operators in the heavy quark effective Hamiltonian,

$$H = m_1 \bar{h} h - \frac{\bar{h}\mathbf{D}^2 h}{2m_2} - i\frac{\bar{h}\boldsymbol{\Sigma}\cdot\mathbf{B}h}{2m_B} + \cdots, \tag{15}$$

where h is a heavy quark field, and the coefficients m_1, $1/m_2$, and $1/m_B$ depend on the bare mass and the gauge coupling. Because the lattice breaks relativistic invariance, the three ''masses'' are not necessarily equal, except as $am_0 \to 0$.

At the tree level, the rest mass $am_1 = \log(1+am_0)$, and the (inverse) kinetic mass

$$\frac{1}{am_2} = \frac{2}{am_0(2+am_0)} + \frac{1}{1+am_0}. \tag{16}$$

The first term can be traced to the Dirac term of the lattice action, and the second to the Wilson term. The one-loop corrections to am_1 and am_2 are also available [24]. The chromomagnetic mass m_B is considered below.

In the heavy quark effective theory, the rest mass term $m_1\bar{h}h$ commutes with the rest of the Hamiltonian and, thus, decouples from the dynamics. As with decay constants [25], one can derive the expansions like Eqs. (6) and (11) within the lattice theory, and the rest mass disappears from physical amplitudes [26]. On the other hand, adjusting the bare quark mass so that $m_2 = m_Q$ is the way to normalize the kinetic operator $\bar{h}\mathbf{D}^2 h/2m_2$ correctly. This normalization can be implemented nonperturbatively by demanding that the energy of a hadron have the correct momentum dependence. In our numerical work we use the B and D mesons for this purpose. Furthermore, one can correctly normalize the chromomagnetic operator $\bar{h}\boldsymbol{\Sigma}\cdot\mathbf{B}h/2m_B$ by adjusting the clover coupling $c_{\rm SW}$, as a function of the gauge coupling, so that $m_B = m_2$. For example, at the tree level the desired adjustment is $c_{\rm SW} = 1$. In our numerical work, we choose $c_{\rm SW}$ in a way that sums up tadpole diagrams, which dominate perturbation theory. This amounts to normalizing the chromomagnetic operator perturbatively.

In summary, we adjust the bare mass am_0 and clover coupling $c_{\rm SW}$ so that the leading effects of the heavy-quark expansion are correctly accounted for [17]. Previous work in the literature chose instead to adjust the bare mass until $m_1 = m_Q$, which introduces an unnecessarily large error,[2] proportional to $1 - m_1(m_0)/m_2(m_0)$.

Under renormalization the heavy quark kinetic energy can mix with the rest mass term in a power divergent way. Because the lattice action used here contains both, the rest mass fully absorbs the power divergence. A related problem is the ambiguity owing to renormalons [27], which appears in some quantities in HQET or nonrelativistic QCD (NRQCD). It is irrelevant to our work, because we calculate physical quantities, namely the masses of the B and D mesons and decay amplitude for $\bar{B} \to D l \bar{\nu}$.

To complete the correspondence of the lattice theory to HQET we must consider the vector current. At order $1/m_Q$ of HQET,

$$V_\mu^{cb} = \left(\bar{h^c} + \mathbf{D}\bar{h^c}\cdot\frac{\boldsymbol{\gamma}}{2m_{3c}}\right)\gamma_\mu\left(1 - \frac{\boldsymbol{\gamma}\cdot\mathbf{D}}{2m_{3b}}\right)h^b + \cdots, \tag{17}$$

where the coefficient $1/m_3$ depends on the current employed. The heavy-heavy current on the lattice is constructed by defining a rotated field [17,25],

$$\Psi^f = \sqrt{2\kappa_f}[1 + ad_1^f(am_{0f}, g_0^2)\boldsymbol{\gamma}\cdot\mathbf{D}]\psi^f, \tag{18}$$

where ψ is the quark field in the hopping parameter form of the action (13). Then the lattice vector current

$$V_\mu^{cb} = \bar{\Psi}^c \gamma_\mu \Psi^b \tag{19}$$

and $\mathcal{V}_\mu^{cb} = Z_{V_\mu^{cb}} V_\mu^{cb}$. Both $Z_{V_\mu^{fg}}$ and d_1^f depend on the gauge coupling, the masses, and (at higher orders) on the Dirac matrix in Eq. (17). They are adjusted so that the normalization and momentum dependence of matrix elements matches the continuum, respectively. In particular, at the tree level the coefficient in Eq. (17) is

$$\frac{1}{am_3} = \frac{2(1+am_0)}{am_0(2+am_0)} - 2d_1, \tag{20}$$

and the condition $m_3 = m_2$ prescribes a condition on d_1 [17,25].

From the properties of the operators under heavy-quark symmetry, it follows that the $1/m_2$ and $1/m_B$ terms in Eq. (15) could give a contribution to $h_+(1)$, but not to $h_-(1)$ [6]. On the one hand, these contributions to $h_+(1)$ must be symmetric under interchange of the initial and final states, but, on the other hand, they must vanish when the initial and final quark masses are the same. Consequently, there can be no contributions linear in either $1/m_2$ or $1/m_B$. Our definition of $h_+(1)$ enjoys this property, by construction, because Eq. (1) manifestly respects the interchange symmetry.

Similarly, the $1/m_3$ terms in Eq. (17) give a contribution only to $h_-(1)$. It must be anti-symmetric under interchange of the initial and final states and must vanish when the initial and final quark masses are the same. Our definition of $h_-(1)$, again by construction, ensures that only the combination $1/m_{3c} - 1/m_{3b}$ appears. This feature is taken into account in Sec. IX.

In Eqs. (6) and (11) we seek contributions of order $1/m_Q^2$. These come from double insertions of the $1/m_Q$ terms in Eqs. (15) and (17), and from $1/m_Q^2$ terms implied by the ellipses. Remarkably, the latter cancel when $h_+(1)$ and $h_-(1)$ are defined by the double ratios (1) and (2) [26]. This is easy to understand if one starts with the matrix elements. The $1/m_b^2$ and $1/m_c^2$ corrections to the action arise from the initial or final state only. To this order, one can factorize them. They drop out of the double ratios, because the numerator and

[2]To mitigate this error, these calculations are often carried out at artificially small quark masses. Ensuing extrapolations to larger masses contaminate lower orders in the (physical) $1/m_Q$ expansion with higher orders.

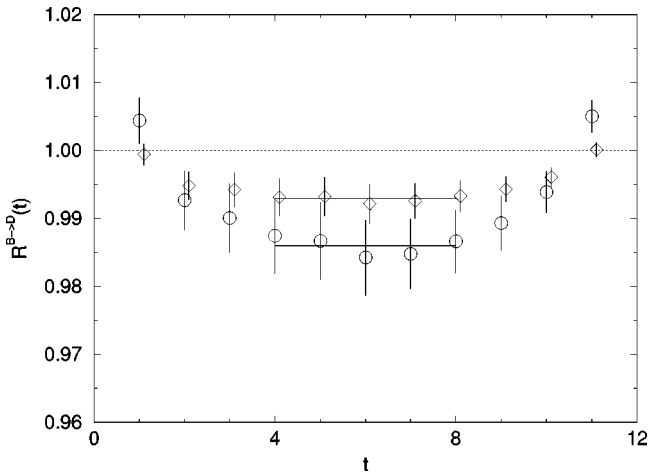

FIG. 1. $R^{B\to D}(t)$ as a function of t. The heavy quark hopping parameters for the initial and final mesons are $(\kappa_b,\kappa_c)=(0.089,0.110)$ (diamonds) and (0.089,0.119) (circles). The light quark corresponds to the strange quark, $\kappa_l=0.1405$. The solid lines represent a constant fit with $4\leqslant t\leqslant 8$.

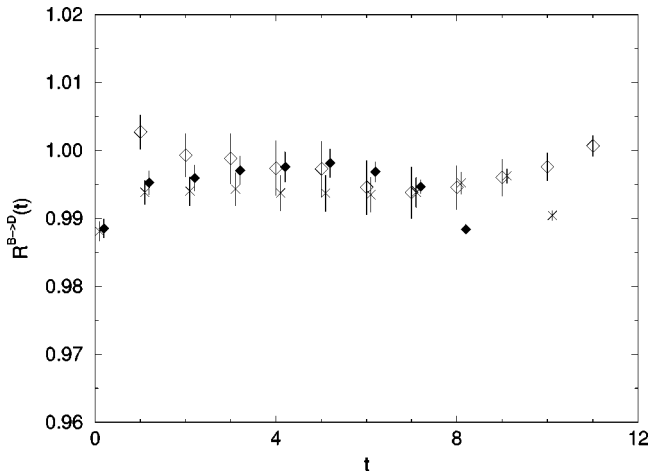

FIG. 2. Check of the plateau in $R^{B\to D}(t)$ by varying the time slice t_B of the B meson interpolating field. Open diamonds, crosses and solid diamonds correspond to the results with $t_B=12$, 10, and 8, respectively. The heavy quark hopping parameters are $(\kappa_b,\kappa_c)=(0.089,0.110)$, and $\kappa_l=0.1405$.

denominator of Eq. (1) or (2) contain the same number of B and D factors. The same applies to $1/m_b^2$ and $1/m_c^2$ corrections to the current. There may be a nonfactorizable correction to the current with coefficient $C(m_c,m_b)/m_c m_b$, where the function C is unknown, except that in perturbation theory it starts at one loop and that $C(m,m)=0$.

In the long run, one would like to pick up terms of order $1/m_Q^3$ and higher. Because the bottom quark is so heavy, these are dominated by the $1/m_c^n$ terms. With the normalization conditions outlined here [17], these come automatically from the Dirac term, as in continuum QCD. In future work, at smaller lattice spacings, the Dirac term will dominate, generating contributions to all orders in $1/m_c$.

V. LATTICE DETAILS

Our numerical data are obtained in the quenched approximation on a $12^3\times 24$ lattice with the plaquette gluon action at $\beta=6/g_0^2=5.7$. We take a mean-field-improved [15] value of the clover coupling, which on this lattice is $c_{\rm SW}=1.57$. Out of 300 configurations generated for our previous work [18], we use 200 configurations. We usually define the inverse lattice spacing through the charmonium $1S$-$1P$ splitting, finding $a^{-1}(1S\text{-}1P)=1.16^{+3}_{-3}$ GeV. For comparison, with the kaon decay constant $a^{-1}(f_K)=1.01^{+2}_{-1}$ GeV, and the difference is thought to be part of the error of quenching. Because the form factors are dimensionless, the lattice spacing affects them only indirectly, through the adjustment of the quark masses.

To investigate the heavy quark mass dependence of the form factors we take $\kappa_h=0.062$, 0.089, 0.100, 0.110, 0.119 and 0.125, and consider several combinations for the heavy quarks in the initial and final states. The mass of the spectator light quark is usually taken to be close to that of the strange quark, for which $\kappa_l=0.1405$. We examine the effect of chiral extrapolation using four κ_l values, 0.1405, 0.1410, 0.1415, and 0.1419, for various combinations of the initial and final heavy quark masses $\kappa_h=0.089$, 0.110, and 0.119. The critical hopping parameter is $\kappa_{\rm crit}=0.14327^{+5}_{-3}$.

For the computation of the matrix element[3] $\langle D(\mathbf{p}')|V_\mu|B(\mathbf{p})\rangle$ we calculate the three point correlation function

$$C^{DV_\mu B}(t,\mathbf{p}',\mathbf{p}) = \sum_{\mathbf{y},\mathbf{x}} e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{y}} e^{-i\mathbf{p}\cdot\mathbf{x}} \langle D(0,\mathbf{0}) V_\mu(t,\mathbf{y}) B^\dagger(T/2,\mathbf{x})\rangle \quad (21)$$

with V_μ from Eq. (19) and $\mathbf{p}=\mathbf{0}$. The light quark propagator is solved with a source at time slice 0, and we place the interpolating field for B at $T/2$, where we use the source method. The interpolating fields B and D are constructed with the $1S$ state smeared source as in Ref. [18]. The spatial momentum $\mathbf{p}'$ carried by the final state is taken to be (0,0,0), (1,0,0), (1,1,0), (1,1,1) and (2,0,0) in units of $2\pi/L$, where L is the physical size of the box; in our case, $L=12a$.

The numerical results presented below are obtained from uncorrelated fits to ratios of these three-point functions. The statistical errors are estimated with the jackknife method. For a subset of the data we have repeated the analysis with correlated fits and the bootstrap method. We find no statistically significant difference.

In much of the numerical work presented in this paper, we set the coefficients d_1 of the rotation (18) to zero. From the discussion following Eq. (17) the dependence on d_1 enters directly through $1/m_3$, and indirectly by changing ρ_{V_μ}. On the scattering matrix elements of the spatial current V_i, this should make a small ($\lesssim 10\%$ or so) effect. On the temporal current V_0, the effect should be tiny. Both expectations are

[3]For simplicity we use ''B'' instead of ''$\bar{B}$'' to indicate the $(b\bar{q})$ meson, and we use ''B'' or ''D'' for any values of the heavy quark masses.

TABLE I. Numerical data for $R^{B\to D}$, which corresponds to $|h_+(1)|^2$, at $\kappa_l=0.1405$. Rows (columns) are labeled by the value of κ_h in the initial (final) state. Combinations without data have not been calculated in this work. The diagonal elements are 1 by construction.

κ_h	0.062	0.089	0.100	0.110	0.119	0.125
0.062	1	0.989(07)	0.979(12)			0.947(24)
0.089	0.989(07)	1	0.998(01)	0.993(02)	0.986(05)	0.983(07)
0.100	0.979(12)	0.998(01)	1			0.992(04)
0.110		0.993(02)		1	0.999(01)	
0.119		0.986(05)		0.999(01)	1	
0.125	0.947(24)	0.983(07)	0.992(04)			1

checked at representative choices of the heavy quark masses, and the uncertainty introduced into the spatial current is propagated to the final result.

VI. CALCULATION OF $|h_+(1)|$

The form factor $|h_+(w)|$ at zero recoil is obtained directly from the three-point correlation functions (21), setting all three momentum to be zero. We define a ratio[4]

$$R^{B\to D}(t)\equiv\frac{C^{DV_0B}(t,\mathbf{0},\mathbf{0})C^{BV_0D}(t,\mathbf{0},\mathbf{0})}{C^{DV_0D}(t,\mathbf{0},\mathbf{0})C^{BV_0B}(t,\mathbf{0},\mathbf{0})}, \tag{22}$$

in which the exponential dependence on t associated with the ground state masses cancels between the numerator and denominator. When the current and two interpolating fields are separated far enough from each other, the contribution of the ground state dominates and

$$R^{B\to D}(t)\to\frac{\langle D(\mathbf{0})|V_0|B(\mathbf{0})\rangle\langle B(\mathbf{0})|V_0|D(\mathbf{0})\rangle}{\langle D(\mathbf{0})|V_0|D(\mathbf{0})\rangle\langle B(\mathbf{0})|V_0|B(\mathbf{0})\rangle}$$
$$=\frac{|h_+^{B\to D}(1)h_+^{D\to B}(1)|}{|h_+^{D\to D}(1)h_+^{B\to B}(1)|}=|h_+^{B\to D}(1)|^2, \tag{23}$$

suppressing radiative corrections. Here we use the definition (3) and the unit normalization of $|h_+(1)|$ in the equal mass case. Thus, we expect $R^{B\to D}$ to be constant as a function of t, and its value represents the form factor squared.

In Fig. 1 we plot the ratio $R^{B\to D}(t)$ for two representative combinations of mass parameters. We observe a nice plateau extending over about five time slices, and our fit over the interval $4\leqslant t\leqslant 8$ is shown by the solid line.

To see if the plateau is stable under the change of the position of the interpolating field, we repeat the calculation changing the time t_B of the B-meson interpolating field. The results with $t_B=10$ and 8 are shown in Fig. 2 together with the one with $t_B=T/2=12$. We observe that the plateau is very stable and conclude that the extraction of the ground state is reliable. In the following analysis we use the result with $t_B=T/2$, and the numerical data for each κ_h are given in Table I.

We examine the chiral limit by computing with four values of the light quark mass (14), roughly in the range $m_s/2\leqslant m_q\leqslant m_s$. Figure 3 shows that the am_q dependence of $|h_+(1)|^2$, for two combinations of (κ_b,κ_c), is very slight. A linear fit in am_q gives a slope consistent with zero, and the value in the chiral limit is still consistent with that at the finite light quark mass. With our present statistics, we cannot study the dependence on the light and heavy quark masses simultaneously. Instead we take from Fig. 3 two lessons: the dependence on the light quark mass is insignificant, but the (statistical) uncertainty increases, by a factor of 2, in the chiral limit.

A small, but non-analytic, dependence on m_π is expected from chiral perturbation theory [28,29]. Such effects may be different in the quenched approximation. If so, the difference should be counted as part of the error of the quenched approximation.

VII. HEAVY QUARK MASS DEPENDENCE OF $|h_+(1)|$

In the heavy quark limit of QCD, the heavy quark mass dependence of $|h_+(1)|$ can be described with a $1/m_Q$ expansion. Using a symmetry of its definition (3) under the ex-

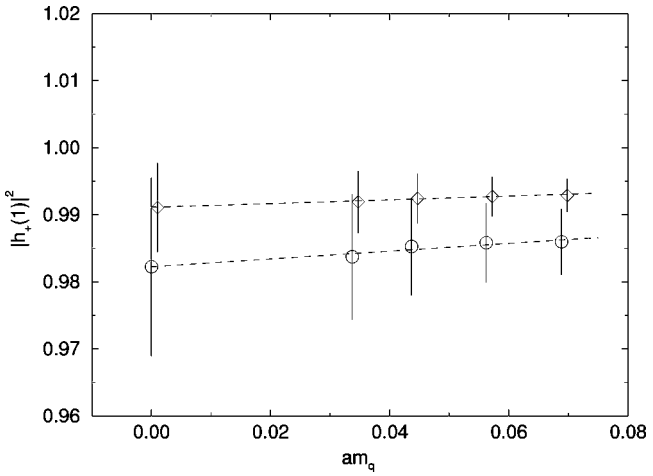

FIG. 3. Chiral extrapolation of $|h_+(1)|^2$. The heavy quark hopping parameters for the initial and final mesons are $(\kappa_b,\kappa_c)=(0.089,0.110)$ (diamonds) and $(0.089,0.119)$ (circles).

[4]Mandula and Ogilvie [10] used a similar ratio, with nonzero velocity transfer, to study the w dependence of the Isgur-Wise function, which is the infinite mass limit of $h_+(w)$.

change of the initial and final states and the normalization in the limit of degenerate heavy quark mass, the form of the $1/m_b$ and $1/m_c$ expansion is restricted to be

$$|h_+(1)| = 1 - c_+^{(2)}\left(\frac{1}{m_c}-\frac{1}{m_b}\right)^2 + c_+^{(3)}\left(\frac{1}{m_c}+\frac{1}{m_b}\right)\left(\frac{1}{m_c}-\frac{1}{m_b}\right)^2 + O(1/m_Q^4), \tag{24}$$

suppressing the radiative correction η_V. The term $O(1/m_Q^4)$ denotes all possible combinations of $1/m_c$ and $1/m_b$ with total mass dimension -4. The absence of terms of order $1/m_Q$ is implied by Luke's theorem [6], but in this particular case it can be understood as a result of the symmetry $1/m_c \leftrightarrow 1/m_b$.

If we take the radiative corrections into account, the data presented in the last section correspond to $|h_+(1)|/\rho_{V_0}$. To use the right-hand side of Eq. (24), on the other hand, we must multiply them by ρ_{V_0}/η_V to obtain $|h_+(1)|/\eta_V$. At $\beta=5.7$ and our choices of quark masses we find, at one loop, that ρ_{V_0}/η_V is very nearly 1, so that we do not need to carry out this conversion.[5]

In the lattice theory, the masses entering Eq. (24) are m_2, m_B, and m_3, as explained in Sec. V. In particular, if one follows Refs. [30,31] to see how higher-dimension tree-level operators affect the matrix elements, one sees that the $1/m_c^2$ and $1/m_b^2$ corrections to the action and current do not affect $R^{B\to D}$ [26].

We study the relation (24) with several combinations of the initial and final heavy quark masses. We require a relation between the hopping parameters, which are inputs to the numerical calculation, and the quark masses. To simplify the analysis, we set $m_3=m_B=m_2$ and estimate the kinetic quark mass by applying tadpole improvement [15] to include the dominant tadpole contribution to the perturbation series. The tadpole-improved kinetic mass is given by substituting $a\tilde{m}_0 = am_0/u_0$ for am_0 on the right-hand side of Eq. (16), with the mean link variable $u_0=1/8\kappa_{\rm crit}$. We do not bother with the one-loop correction to m_2 [24], because it is smaller than the uncertainty from a. This way of parametrizing the quark masses is for interpolating only; when reconstituting the physical result, the hopping parameters κ_c and κ_b are chosen nonperturbatively from the masses of the D and B mesons.

Figure 4 shows the $1/am_c$ dependence of $|h_+(1)|$. The initial heavy quark mass is set to be $1/am_b=0.475$ ($\kappa_b=0.089$), and we vary $1/am_c$ between 0.2 and 2.0. (Here we misuse the meaning of subscript b or c to indicate the initial or final state heavy quark, respectively.) At $1/am_c=1/am_b$ the form factor becomes exactly 1 by construction, and the deviation from unity increases as $1/am_c$ moves away from $1/am_b$. The statistical error grows as the difference of heavy quark masses increases. When one approaches the static limit the signal becomes much noisier, as in many other Monte Carlo calculations with heavy-light mesons. In our case, the statistical error of the point with heaviest am_c is very large.

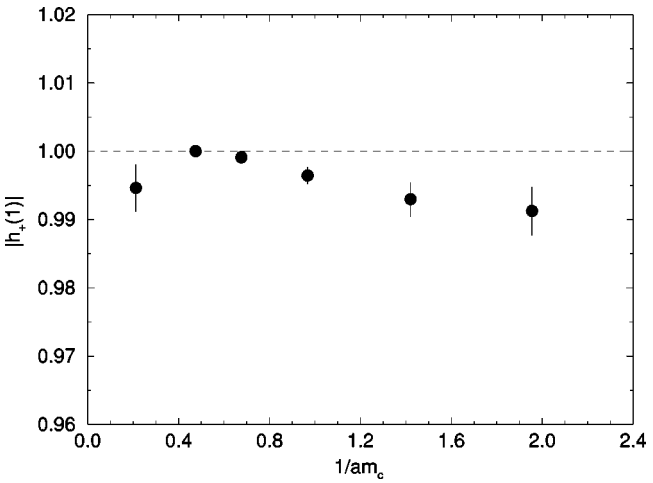

FIG. 4. $1/am_c$ dependence of $|h_+(1)|$. The initial heavy quark mass is fixed at $\kappa_b=0.089$, which corresponds to $1/am_b=0.475$. The light quark corresponds to the strange quark, $\kappa_l=0.1405$.

To see the mass dependence more clearly, we rewrite the relation (24) as

$$\frac{1-|h_+(1)|}{\Delta^2} = c_+^{(2)} - c_+^{(3)}\left(\frac{1}{am_c}+\frac{1}{am_b}\right), \tag{25}$$

where $\Delta = 1/am_c - 1/am_b$. The left-hand side is plotted in Fig. 5. The data exhibit a very good linear dependence on Δ, except in the heavy mass regime, where the error grows rapidly. Fitting all data linearly, we obtain $c_+^{(2)}=0.029(11)$ and $c_+^{(3)}=0.011(4)$. In physical units, and absorbing factors of a into the coefficients, these coefficients have a size typical of the QCD scale: $c_+^{(2)}=[0.20(4)\ \text{GeV}]^2$ and $c_+^{(3)}=[0.26(3)\ \text{GeV}]^3$.

The dotted line marking the physical value of $1/am_c + 1/am_b$ shows that we are, in effect, using Eq. (25) as an ansatz for interpolation. Although the coefficients are inter-

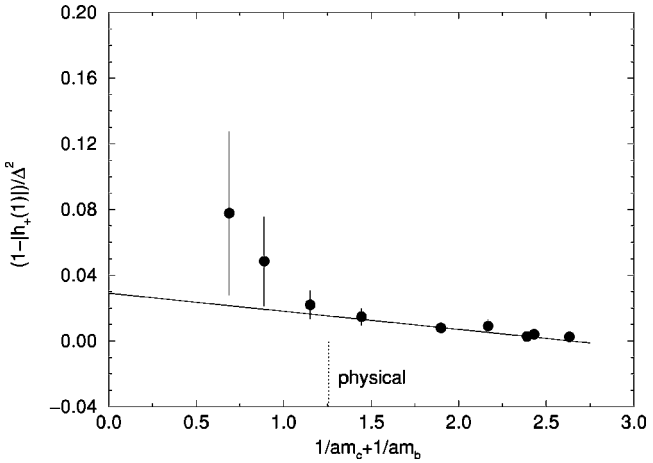

FIG. 5. $[1-|h_+(1)|]/\Delta^2$ vs $1/am_c+1/am_b$. The dotted vertical line indicates the physical value of $1/am_c+1/am_b$. The light quark corresponds to the strange quark, $\kappa_l=0.1405$.

[5]This is an accident at our choice of lattice spacing. For smaller lattice spacings, this would not be so. See Ref. [23] for details.

esting in their own right, we caution the reader that the values extracted from the fit are highly correlated, and we have not made a full analysis of the errors on them. Below we prefer to give $h_+(1)$, evaluated at physical values of the masses, as the principal result of this section.

We have checked the influence of the rotation by repeating the calculations with d_1 set to

$$\tilde{d}_1 = \frac{1}{2+a\tilde{m}_0} - \frac{1}{2(1+a\tilde{m}_0)} \tag{26}$$

at several of the heavy-quark mass combinations. This is the correctly tuned value at the (mean-field improved) tree level [17]. The primary effect of varying d_1 is through the combination $(1/m_{3c} - 1/m_{3b})^2$ [26]. A secondary effect is to modify the radiative corrections factor ρ_{V_0}. After interpolating the masses to the physical point, the change on $h_+(1)$ is $+0.00013$, which is much smaller than several other uncertainties. Owing to this, our central value for $h_+(1)$ can come safely from data with $d_1 = 0$, the only value of d_1 for which ρ_{V_0} is already available.

VIII. CALCULATION OF $h_-(1)$

To obtain $h_-(w)$, it is necessary to consider nonzero recoil momentum. From the definition of the form factors (3), the matrix elements of the spatial and temporal vector current for the nonzero recoil final state $D(\mathbf{p}')$ read

$$\langle D(\mathbf{p}')|\mathcal{V}_i|B(\mathbf{0})\rangle = \sqrt{m_B m_D}[h_+^{B\to D}(w) - h_-^{B\to D}(w)]v_i', \tag{27}$$

$$\langle D(\mathbf{p}')|\mathcal{V}_0|B(\mathbf{0})\rangle = \sqrt{m_B m_D}[h_+^{B\to D}(w)(1+w) + h_-^{B\to D}(w)(1-w)], \tag{28}$$

where $w = v\cdot v' = \sqrt{1+\mathbf{v}'^2}$, and $\mathbf{v}' = \mathbf{p}'/m_D$.

On the lattice we start by computing the ratio of correlation functions

$$R_{V_i/V_0}^{B\to D}(t,\mathbf{p}') \equiv \frac{C^{DV_iB}(t,\mathbf{p}',\mathbf{0})}{C^{DV_0B}(t,\mathbf{p}',\mathbf{0})}. \tag{29}$$

In the limit of well-separated currents, the time dependence flattens,

$$R_{V_i/V_0}^{B\to D}(t,\mathbf{p}') \to \frac{\langle D(\mathbf{p}')|V_i|B(\mathbf{0})\rangle}{\langle D(\mathbf{p}')|V_0|B(\mathbf{0})\rangle} = \frac{v_i'}{2}\left[1 - \frac{h_-^{B\to D}(w)}{h_+^{B\to D}(w)}\right] \times \left[1 - \frac{1}{2}\left(1 - \frac{h_-^{B\to D}(w)}{h_+^{B\to D}(w)}\right)(w-1)\right]. \tag{30}$$

The last step holds for small $\mathbf{v}'^2$ and suppresses radiative corrections. Because the velocity inherits statistical uncertainties from the D's kinetic mass, it is further useful to define a double ratio

$$R_{V_i/V_0}^{(B\to D)/(D\to D)}(t,\mathbf{p}') \equiv R_{V_i/V_0}^{B\to D}(t,\mathbf{p}')/R_{V_i/V_0}^{D\to D}(t,\mathbf{p}'). \tag{31}$$

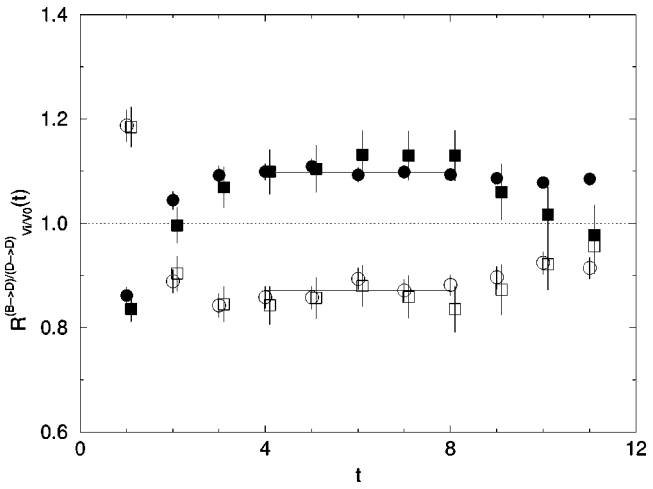

FIG. 6. $R_{V_i/V_0}^{(B\to D)/(D\to D)}(t)$ for the final state momentum (1,0,0) (circles) and (2,0,0) (squares). The heavy quark hopping parameters for the initial and final mesons are $(\kappa_b,\kappa_c) = (0.089, 0.119)$ (solid symbols) and (0.119,0.089) (open symbols). The light quark corresponds to the strange quark, $\kappa_l = 0.1405$. The solid lines represent a constant fit for the momentum (1,0,0) with $4 \leqslant t \leqslant 8$.

Then, for large time separations,

$$R_{V_i/V_0}^{(B\to D)/(D\to D)}(t,\mathbf{p}') \to \frac{\langle D(\mathbf{p}')|V_i|B(\mathbf{0})\rangle}{\langle D(\mathbf{p}')|V_0|B(\mathbf{0})\rangle}\frac{\langle D(\mathbf{p}')|V_0|D(\mathbf{0})\rangle}{\langle D(\mathbf{p}')|V_i|D(\mathbf{0})\rangle} = \left[1 - \frac{h_-^{B\to D}(w)}{h_+^{B\to D}(w)}\right]\left[1 + \frac{h_-^{B\to D}(w)}{2h_+^{B\to D}(w)}(w-1)\right]. \tag{32}$$

The final expression is simplified using the property $h_-^{D\to D}(w) = 0$. Provided that $|h_+^{B\to D}(w)|$ is obtained sufficiently precisely in the previous sections, the relation (32) can be used to extract $h_-^{B\to D}(w)$. The part proportional to $w-1$ gives only a small contribution, since the coefficient $h_-(w)/h_+(w)$ is itself a small quantity of order $(m_B - m_D)/(m_B + m_D)$.

Figure 6 shows the t dependence of the ratio $R_{V_i/V_0}^{(B\to D)/(D\to D)}(t,\mathbf{p}')$ for final state momenta $L\mathbf{p}'/2\pi = (1,0,0)$ (circles) and (2,0,0) (squares). Solid symbols represent the $b\to c$ transition, while open symbols correspond to the reverse $c\to b$ transition. The plateau is reached around $t=4$, so that we can fit in the interval $4 \leqslant t \leqslant 8$, as with $|h_+(1)|^2$. The fit results for the momentum (1,0,0) are given by the solid lines.

Up to the small contribution of order $w-1$, this ratio gives the combination $1 - h_-^{B\to D}(w)/h_+^{B\to D}(w)$, in which $h_+^{B\to D}(w)$ is almost equal to 1. Looking at the solid symbols in Fig. 6, $h_-^{B\to D}(w)$ is roughly -0.1 and is almost independent of the final state momentum. Since $h_-(w)$ changes its sign under the exchange of initial and final states, it is consistent that the open symbols, which correspond to the transition $D\to B$, appear below 1.

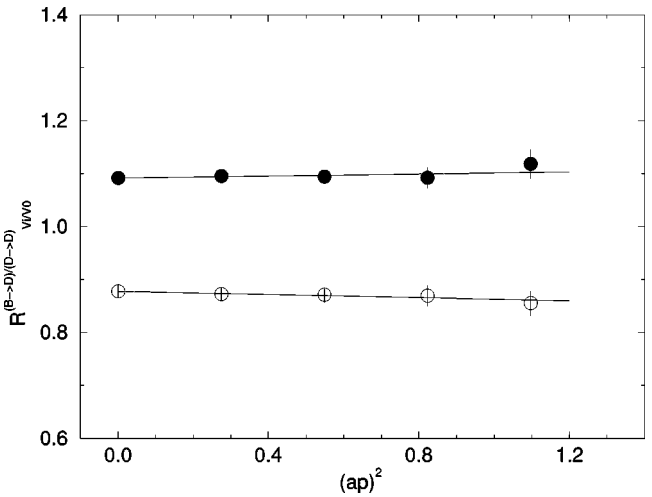

FIG. 7. Extrapolation of $R^{(B\to D)/(D\to D)}_{V_i/V_0}$ to the zero-recoil limit. The heavy quark hopping parameters for the initial and final mesons are $(\kappa_b,\kappa_c)=(0.089,0.119)$ (solid circles) and (0.119,0.089) (open circles). The light quark corresponds to the strange quark, $\kappa_l=0.1405$. Note that the lattice spacing a is held fixed here.

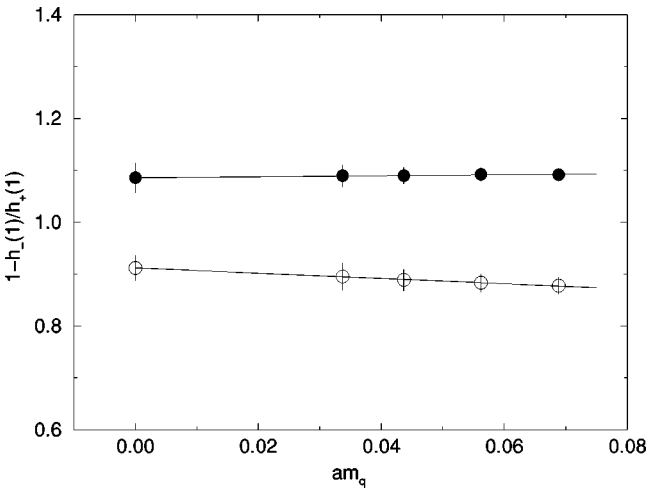

FIG. 8. Chiral extrapolation of $1-h_-(1)/h_+(1)$. The heavy quark hopping parameters for the initial and final mesons are $(\kappa_b,\kappa_c)=(0.089,0.119)$ (solid circles) and (0.119,0.089) (open circles).

To obtain the value of $h_-^{B\to D}(w)/h_+^{B\to D}(w)$ at the zero-recoil limit, we extrapolate the plateau values of $R^{(B\to D)/(D\to D)}_{V_i/V_0}$ for $\mathbf{p}'^2\to 0$. The small piece of order $w-1$ vanishes in this limit as well as the possible w dependence of form factors, so we obtain $h_-^{B\to D}(1)/h_+^{B\to D}(1)$ without further approximation. Figure 7 shows the extrapolation for the same mass values as in Fig. 6. There is no significant dependence on $(a\mathbf{p}')^2$. Thus, we simply apply a linear form to fit the data, shown in the figure. The numerical data in the zero-recoil limit are given in Table II.

The chiral extrapolation of $1-h_-(1)/h_+(1)$ is shown in Fig. 8, for the combinations $(\kappa_b,\kappa_c)=(0.089,0.119)$ and (0.119,0.089). As in the case of $|h_+(1)|^2$, the dependence on am_q is insignificant, but the (statistical) uncertainty increases, by a factor of 2.

IX. HEAVY QUARK MASS DEPENDENCE OF $h_-(1)$

As with $h_+(1)$, the heavy quark mass dependence of $h_-(1)$ can be described, in the heavy quark limit of QCD, with a $1/m_Q$ expansion. The form of the heavy quark expansion of $h_-^{B\to D}(1)$ is restricted by its anti-symmetry under the exchange of the initial and final states,

$$h_-(1)=-\left(\frac{1}{m_c}-\frac{1}{m_b}\right)_3\left[c_-^{(1)}-c_-^{(2)}\left(\frac{1}{m_c}+\frac{1}{m_b}\right)_2\right]+O(1/m_Q^3). \tag{33}$$

The meaning of the subscripts on the combinations of inverse masses is given below. The ratio $h_-(1)/h_+(1)$ obeys the same expansion up to the given order, since the correction to the $h_+(1)$ starts at order $1/m_Q^2$.

To take radiative corrections into account, we should note that the (lattice) ratio $R^{(B\to D)/(D\to D)}_{V_i/V_0}$ corresponds to $[1-h_-/h_+]/\rho_{V_i}$. The right-hand side of Eq. (33), on the other hand, is justified in HQET when radiative corrections are ignored. Thus, we should multiply the data of Table II by $\rho_{V_i}/(1-\beta_V)$. We shall not do this for two reasons. First, the one-loop contribution to ρ_{V_i} is not yet available, although a calculation is in progress [23]. Second, there is an indication from an analysis of renormalons that the series for β_V con-

TABLE II. Numerical data in the zero-recoil limit for $R^{(B\to D)/(D\to D)}_{V_i/V_0}$, which corresponds to $1-h_-(1)/h_+(1)$, at $\kappa_l=0.1405$. Rows (columns) are labeled by the value of κ_h in the initial (final) state. Combinations without data have not been calculated in this work. The diagonal elements are one by construction.

κ_h	0.062	0.089	0.100	0.110	0.119	0.125
0.062	1	1.067(12)	1.093(14)			1.181(21)
0.089	0.892(20)	1	1.033(04)	1.063(08)	1.095(11)	1.121(15)
0.100	0.836(27)	0.963(05)	1			1.092(11)
0.110		0.923(10)		1	1.034(04)	
0.119		0.878(15)		0.964(04)	1	
0.125	0.636(47)	0.837(20)	0.889(13)			1

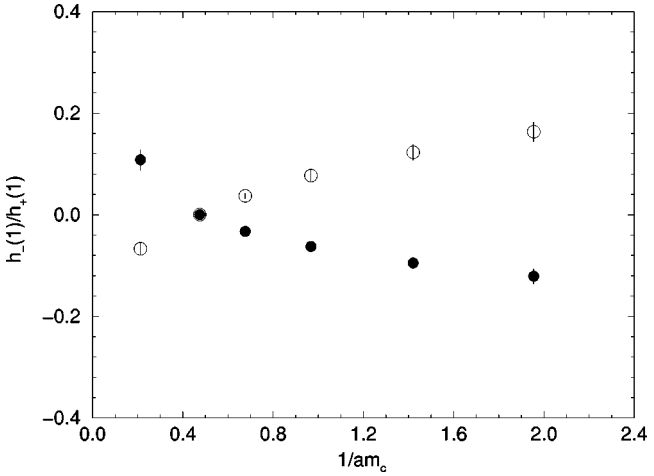

FIG. 9. $1/am_c$ dependence of $h_-(1)/h_+(1)$. The initial heavy quark mass is fixed at $\kappa_b=0.089$ (solid circles), which corresponds to $1/am_b=0.475$. The open circles are obtained by exchanging the initial and final states. The light quark corresponds to the strange quark, $\kappa_l=0.1405$.

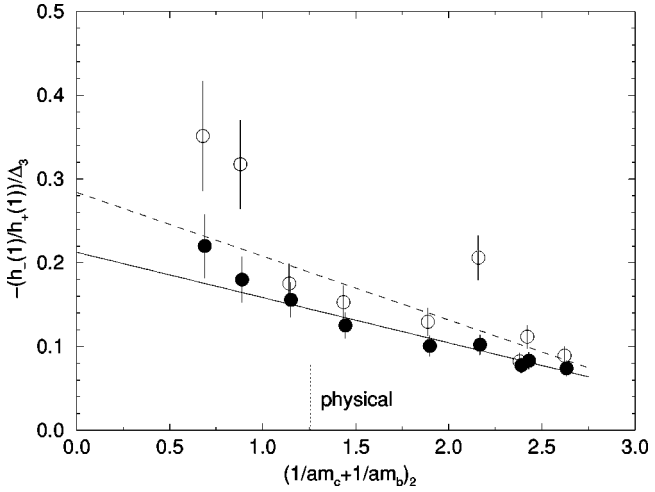

FIG. 10. $-[h_-(1)/h_+(1)]/\Delta_3$ vs $1/am_c+1/am_b$. Solid (open) symbols represent the ''heavier-to-lighter'' (''lighter-to-heavier'') decay results. The solid and dashed lines are fitted results to the solid and open data points, respectively. The dotted vertical line indicates the physical value of $1/am_c+1/am_b$. The light quark corresponds to the strange quark, $\kappa_l=0.1405$.

verges poorly [32]. With these points in mind, we omit radiative corrections and employ (33) as an Ansatz for interpolation.

The subscripts on the parentheses in Eq. (33) mean that the enclosed masses should be taken to be m_3 or m_2, introduced in Sec. III. The reasoning is as follows. The contribution to $h_-(1)$ of first order in $1/m_Q$ comes solely from the current [6], namely the $1/m_3$ terms in Eq. (17). The second-order contribution comes mainly from the first-order contribution iterated with the $1/m_Q$ corrections to the Hamiltonian [30,26], namely the $1/m_2$ and $1/m_B$ terms in Eq. (15). We can take $m_B=m_2$ because, with the clover action, the difference affects the interpolation negligibly. Tracing the $1/m_Q$ expansion in this way, and making use of the anti-symmetry under the exchange of initial and final states, leads to the heavy-quark expansion for the lattice data of the form given in Eq. (33).

In Fig. 9 we plot the $1/am_c$ dependence of $h_-(1)/h_+(1)$. The solid circles are obtained by fixing the initial-state quark mass to be $1/am_b=0.475$ and varying the final-state mass. The open circles are obtained by fixing the final-state mass and varying the initial-state mass. We can clearly observe the mass dependence, which makes it possible to extract the value of the form factor for physical masses.

To extract the coefficients $c_-^{(1)}$ and $c_-^{(2)}$ we plot, in Fig. 10,

$$\frac{R_{V_i/V_0}^{(B\to D)/(D\to D)}-1}{\Delta_3}=-\frac{h_-(1)/h_+(1)}{\Delta_3}=c_-^{(1)}-c_-^{(2)}\left(\frac{1}{am_c}+\frac{1}{am_b}\right)_2, \quad (34)$$

where now $\Delta_3=1/am_{3c}-1/am_{3b}$. Here the solid symbols represent the results from the ''heavier-to-lighter'' transitions and the open symbols from ''lighter-to-heavier'' transitions. The two sets of data are consistent with each other, except three points appearing well above the other points. These data involve the heaviest quark mass in our calculation, where the statistical noise is very large, and reliable fits become difficult. The data are well described by the linear form (34), and our results for its coefficients extracted with the ''heavier-to-lighter'' data are $c_-^{(1)}=0.212(31)$ and $c_-^{(2)}=0.054(11)$. In physical units, these coefficients are $c_-^{(1)}=0.246(37)$ GeV and $c_-^{(2)}=[0.27(3)\ \text{GeV}]^2$.

The data presented in the figures and in Table II are obtained with the rotation parameter $d_1=0$. One expects h_- to be sensitive to d_1, because d_1 is the coefficient of an operator of order v, and h_- parametrizes a matrix element of order v. From the discussion in Sec. IV, however, one sees that d_1 influences matrix elements through the mass m_3. Thus, our method of fitting compensates for the omitted rotation, provided we reconstitute the physical value of $h_-(1)$ using the physical values of the quark masses throughout. A bonus of this method is that the radiative correction factor ρ_{V_i} will be easier to compute when $d_1=0$.

We have checked the influence of the rotation by repeating the calculations with $d_1=\tilde{d}_1$; cf. Eq. (26). The primary effect of varying d_1 is through $1/m_3$ and, from Eqs. (20) and (33), is proportional to the difference $d_1^c-d_1^b$. A secondary effect is to modify the radiative corrections of the lattice currents.

With hopping parameters $(\kappa_b,\kappa_c)=(0.089,0.119)$, the difference $\tilde{d}_1^c-\tilde{d}_1^b$ nearly vanishes. Nevertheless, we find

$$R_{V_i/V_0}^{(B\to D)/(D\to D)}(\tilde{d}_1)-R_{V_i/V_0}^{(B\to D)/(D\to D)}(0)=0.0089\pm0.0012, \quad (35)$$

where we use the bootstrap method to obtain a statistical uncertainty that takes correlations into account. This differ-

ence must stem almost entirely from a change in the radiative corrections, because the change in the heavy quark expansion is, fortuitously, negligible. Thus, it provides an estimate of the uncertainty from omitting the radiative corrections.

Another check on the magnitude of the radiative corrections comes from comparing the heavier-to-lighter transition with the lighter-to-heavier. Because the physical form factor h_- is anti-symmetric under interchange of the initial and final states, the incomplete anti-symmetry of $R_{V_i/V_0}^{(B\to D)/(D\to D)}-1$, seen in Table II, can come only from radiative corrections. Near the physical region, these discrepancies are 10–20 % of $h_-(1)$. With these considerations to guide an estimate, we take the uncertainty in $h_-(1)$ owing to unknown radiative corrections to range from $+0.010$ to -0.030.

X. COMPARISON WITH THE QCD SUM RULES

In the past, the form factors $h_+(1)$ and $h_-(1)$ have been studied with QCD sum rules or the non-relativistic quark model. Here we make a comparison of our results for $c_+^{(2)}$ and $c_-^{(1)}$ with estimates obtained with those techniques.

From the zero-recoil sum rule, Shifman *et al.* obtain [5]

$$F_{B\to D}^2+\sum_X F_X^2=1-\frac{\mu_\pi^2-\mu_G^2}{4}\left(\frac{1}{m_c}-\frac{1}{m_b}\right)^2, \qquad (36)$$

where $F_{B\to D}$ corresponds to $h_+(1)$ and the F_X represent contributions of higher excited states. The hadronic parameters μ_π^2 and μ_G^2 are estimated with other sum rules, and recent results are $\mu_\pi^2=0.5(1)$ GeV^2 and $\mu_G^2=0.36$ GeV^2 [3]. The relation (36) gives an upper bound for $h_+(1)$,

$$h_+(1)<1-\frac{\mu_\pi^2-\mu_G^2}{2}\left(\frac{1}{m_c}-\frac{1}{m_b}\right)^2, \qquad (37)$$

provided that the contributions F_X^2 of higher excited states are strictly positive. This can be translated as a lower bound for the coefficient $c_+^{(2)}$:

$$c_+^{(2)}>\frac{1}{2}(\mu_\pi^2-\mu_G^2)=(0.26^{+0.09}_{-0.12}\ \text{GeV})^2. \qquad (38)$$

Our result $c_+^{(2)}=[0.20(4)\ \text{GeV}]^2$ is lower than the central value but still consistent within errors.

In [4,30] the authors used the non-relativistic quark model to estimate the coefficient $c_+^{(2)}$. Their results scatter in a range (0.2–0.4 GeV^2), strongly depending on the assumed shape of the quark-antiquark wave function and the value of the valence light quark mass.

The form factor $h_-(1)$ has been studied with QCD sum rules [34,35]. Applying their analysis to the heavy quark expansion (33) one finds

$$c_-^{(1)}=\frac{\bar{\Lambda}}{2}[1+\delta_1-2(1+\delta_2)\eta(1)], \qquad (39)$$

TABLE III. Tree level estimate of the form factors at zero recoil, with statistical errors only. The light quark corresponds to the strange quark, $\kappa_l=0.1405$. The entries for the form factors do not reflect radiative corrections.

am_b	am_c	$h_+(1)$	$h_-(1)$	$\mathcal{F}_{B\to D}(1)$
4.4	1.1	0.992(3)	−0.103(13)	1.041(8)
3.9	1.0	0.991(3)	−0.107(14)	1.042(8)
3.4	0.9	0.990(4)	−0.112(14)	1.043(8)

where $\bar{\Lambda}=m_B-m_b$, the δ_i are radiative corrections, and $\eta(1)$ represents a ratio of HQET form factors, at zero recoil. Neglecting radiative corrections, Neubert [34] finds $\eta(1)=1/3$ from a QCD sum rule. Taking $\bar{\Lambda}=0.5\pm0.1$ GeV and $\delta_1=\delta_2=0$, this implies $c_-^{(1)}=0.08(2)$ GeV. With radiative corrections in the sum rule, Ligeti *et al.* find $\eta(1)=0.6\pm0.2$ [35]. Taking now $\delta_1=0.11$ and $\delta_2=0.09$ [36], this implies $c_-^{(1)}=-0.05(10)$ GeV. Our result is significantly larger than both, but it is difficult to make a direct comparison. Our lattice calculation contains some of the radiative corrections automatically, and the remainder has not yet been calculated. When the lattice one-loop calculation is available, it should be possible to make a direct comparison. As we mentioned above, it is conceivable that these effects could change $c_-^{(1)}$ significantly, without a great effect on the value we extract for $h_-(1)$.

XI. RESULT FOR $\mathcal{F}_{B\to D}(1)$

In the previous sections we have investigated the heavy quark mass dependence of $h_+(1)$ and $h_-(1)$ and obtained the coefficients in the $1/m_Q$ expansions (24) and (33). To extract the value of $\mathcal{F}_{B\to D}(1)$ we input the physical values of m_c and m_b, which we adjust to give the physical meson masses. At $\beta=5.7$ these parameters are $am_c=1.0(1)$ and $am_b=3.9(5)$. The central value is fixed with the D and B meson masses with the lattice spacing $a^{-1}(1S\text{-}1P)$, and the error range reflects the uncertainty in the lattice spacing.

The values of physical $h_+(1)$ and $h_-(1)$ (without the matching factors) are given in Table III for three possible combinations of am_b and am_c. Since the systematic errors in am_b and in am_c are correlated, we consider the central and two limiting combinations only. The statistical errors on $h_+(1)$ and $h_-(1)$ are estimated with the jackknife method, so that the resulting precision is better than that obtained by adding in quadrature the errors on coefficients $c_\pm^{(n)}$. In the physical amplitude $\mathcal{F}_{B\to D}(1)$, which is the linear combination of $h_+(1)$ and $h_-(1)$ given in Eq. (5), the uncertainty from adjusting the quark masses largely cancels, and the value of $\mathcal{F}_{B\to D}(1)$ is very stable.

To obtain the physical result, we must now fold in the radiative correction ρ_{V_0}, relating the lattice current to the continuum. Two of us recently have calculated this factor to one loop [23], and at $am_b=3.9$ and $am_c=1.0$ they find $\rho_{V_0}=1+0.096\alpha_s$. The Lepage-Mackenzie scale q^* for the coupling $\alpha_s(q^*)$ [15] has also been calculated, and at the

same quark masses the result is $q^*=4.4/a$. At $\beta=5.7$, $\alpha_V(4.4/a)=0.168$ and the correction to $h_+(1)$ is $+0.016(3)$, taking the error of omitting higher orders to be 20% of the one-loop correction.

A similar one-loop calculation for ρ_{V_i}, which modifies $h_-(1)$, is not yet available. We allow, therefore, a systematic uncertainty for this effect.

Our results for the form factors are

$$h_+(1)=+1.007\pm0.006\pm0.002\pm0.003, \tag{40}$$

$$h_-(1)=-0.107\pm0.028\pm0.004^{+0.010}_{-0.030}, \tag{41}$$

where the error estimates are as follows. The first error comes from statistics, after the chiral extrapolation; the second from adjusting the heavy quark masses; and the third error from unknown radiative corrections, two loops and higher for h_+ and one loop and higher for h_-. The chiral extrapolations, which are shown in Figs. 3 and 8, double the statistical errors of Table III, without changing the central values.

Our main result is the value of the form factor entering the decay rate, at zero recoil. Inserting the physical values of the B and D meson masses and the results (40) and (41) into Eq. (5),

$$\mathcal{F}_{B\to D}(1)=1.058\pm0.016\pm0.003^{+0.014}_{-0.005}, \tag{42}$$

where errors are from statistics, heavy quark masses, and omitted radiative corrections. The last of these could be reduced substantially by calculating the radiative correction factor ρ_{V_i} to one loop.

Two sources of uncertainty have yet to be investigated carefully. They are the dependence on the lattice spacing and the effects of the quenched approximation. From our experience with f_B [18,19], we might suppose that these effects are a few percent and $\sim 15\%$, respectively. The ratios have been constructed so that all sources of error, including these, vanish for equal heavy quark masses. It is, therefore, our expectation that these percentages apply not to $\mathcal{F}(1)$ but to $\mathcal{F}(1)-1$. That means that these two sources of error should be under good control, just as we have found with the other sources of uncertainty.

XII. CONCLUSIONS

In this paper we have shown that precise lattice calculations of the zero-recoil form factors $h_+(1)$ and $h_-(1)$ are possible. The principal technical advance is to consider ratios of matrix elements, in which a large cancellation of statistical and systematic errors takes place. The numerical data are interpreted in a way mindful of heavy quark symmetry [17]. We find, therefore, that the dependence of the form factors on the heavy quark mass is well described by $1/m_Q$ expansions, and we obtain the coefficients in the expansions.

Our control over the heavy quark mass dependence allows us to determine the individual form factors $h_+(1)$ and $h_-(1)$, as well as the physical combination $\mathcal{F}_{B\to D}(1)$. The main results (40)–(42) account for most uncertainties, but not the dependence on the lattice spacing or the effect of the quenched approximation. Since our method is designed to yield the deviation of $\mathcal{F}_{B\to D}(1)$ from 1, we do not expect these qualitatively to spoil the quoted precision. With the proof of principle provided by this work, it should be possible, in the short term, to obtain $\mathcal{F}_{B\to D}(1)$ with control over all sources of uncertainty and an error bar that is small enough to be relevant to the determination of $|V_{cb}|$.

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