One-loop flavor changing electromagnetic transitions

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We discuss the effects of the external fermion masses in the flavor-changing radiative transitions of a heavy fermion (quark or lepton) to a lighter fermion at the one-loop level, and point out an often overlooked crucial difference in the sign of the charge factor between transitions of the down type $s \to d\gamma$ and the up type $c \to u\gamma$. We give formulas for the $F \to f\gamma$ effective vertex in various approximations and the exact formula for $t \to c\gamma$ and $\tau \to \mu\gamma$.

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I. INTRODUCTION

Flavor-changing radiative transitions $F \to f\gamma$ are processes in which a fermion (quark or lepton) undergoes a flavor change accompanied by the emission of a real or virtual photon. They are the result of an interplay between the weak and electromagnetic interactions at the loop level, often enhanced by QCD effects. As such, they have considerable theoretical and experimental interest, not only because they provide an excellent test of the standard model but also because they hold the promise of being sensitive to new physics. They are operating in many different situations, for example in radiative weak decays of hyperons, such as $\Sigma^+ \to p\gamma$ and $\Xi^0 \to \Lambda\gamma$, in rare B meson decays ($B \to X_s\gamma$), and in rare D meson decays ($D \to V\gamma$). They may also occur in rare processes involving leptons, such as $K \to \pi e^+e^-$, $\tau \to \mu\gamma$, and $\nu_1 \to \nu_2 e^+ e^-$.

At the quark or lepton level, all flavor-changing electromagnetic transitions may be divided into two categories: one involving the upper components of weak isospin doublets and the other their lower components, as shown in Table I. In one loop processes, the initial fermion F creates an intermediate state of a boson and a fermion f_l which, after emitting a photon, returns to a fermionic state f as depicted in Figs. 1 and 2 for an example of quark transition.

The amplitudes of these penguin processes were evaluated many years ago [1,2]. In particular, a simple analytic formula obtained by Inami and Lim [1] has often served as the starting point for several more advanced calculations of QCD corrections [3–5] or properties of detailed particle models [5,6]. The interested reader is referred to a recent review [7] for further discussion. Unfortunately several authors have overlooked the fact that this formula applies only to amplitudes of the type $sd\gamma$, which were the primary object of interest to Inami and Lim [1], and have used it unchanged to study processes of the type $cu\gamma$. As we shall see, this oversight will cause an error in sign and magnitude (by a factor of 5) in the amplitudes of the up-type transitions. Moreover, in the $b \to s\gamma$ transition the intermediate state is dominated by the contribution of the top quark and therefore it is well justified to neglect the effects of the external fermion masses as it was done before [1], but it is doubtful that such an approximation will hold in other cases, especially in $c \to u\gamma$ and $\tau \to \mu\gamma$, not to mention $t \to c\gamma$, in which the mass of the initial particle is comparable or larger than any internal (boson or quark) mass.

In this paper, we will correct the abovementioned shortcomings and proceed on to examine the effect of the external fermionic masses in flavor-changing radiative transitions. After writing down the rules and conventions of calculations in Sec. II, we will evaluate in Sec. III the effective vertex for $F \to f\gamma$ assuming first small but nonvanishing external fermionic masses. Although the methods we follow are applicable to both quarks and leptons, we refer specifically to quarks for convenience. We calculate the inclusive rates for $q_i \to q_j\gamma$ and examine the effects of the external masses on the decay rates in some detail; we also give an estimate of the QCD-corrected branching ratio of $c \to u\gamma$. In Sec. IV, we derive an exact formula, with $m_c = 0$ but nonvanishing m_t, for the effective $t \to c\gamma$ vertex and demonstrate the importance of the initial quark mass. We conclude with Sec. V.

II. FEYNMAN RULES IN THE 'T HOOFT–FEYNMAN GAUGE

The standard electroweak model formulated in any renormalizable gauge R_ξ contains, in addition to the usual physi-

TABLE I. Elementary flavor-changing radiative processes.

F	f	f_l
ν_1	ν_2	e^-, μ^-, τ^-
c	u	d, s, b
t	$c(u)$	d, s, b
τ^-	$\mu^-(e^-)$	ν_j, $j = 1,2,3$
s	d	u, c, t
b	$s(d)$	u, c, t

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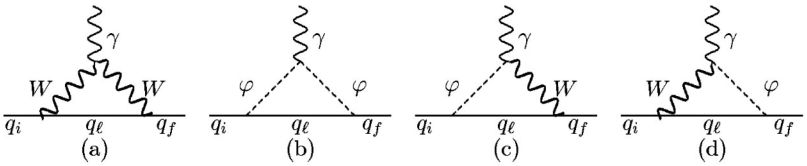

FIG. 1. Contributions to $q_i \to q_f \gamma$ vertex: emission from bosons.

cal particles (the photon A_μ, bosons $W^\pm_\mu$, and quarks q_i), the unphysical would-be Goldstone scalar bosons $\varphi^\pm, \varphi^0$ coming from the spontaneous symmetry breaking (as well as the Faddeev–Popov ghosts which, however, do not contribute to processes considered here because the external particles are fermions). In writing down the following Feynman rules we use the convention that defines *ingoing* momenta for particles entering the vertex and *outgoing* momenta for particles leaving the vertex. Quark flavors will be denoted by i or j, and the left and right projection operators by $L=(1-\gamma_5)/2$ and $R=(1+\gamma_5)/2$.

The rules for the elementary vertices are

$$q_i \to q_i A_\mu : -ieQ_i\gamma_\mu \quad \text{with } Q_i = \frac{2}{3}(u), -\frac{1}{3}(d),$$

$$W^\pm_\lambda(p) \to W^\pm_\nu(p')A_\mu : +ieQ_W[(p+p')_\mu g_{\lambda\nu} + (p'-2p)_\nu g_{\lambda\mu} + (p-2p')_\lambda g_{\mu\nu}]$$

$$\text{with } Q_W = \pm 1 (W^\pm),$$

$$\varphi^\pm(p) \to \varphi^\pm(p')A_\mu : -ieQ_S(p+p')_\mu$$

$$\text{with } Q_S = \pm 1(\varphi^\pm),$$

$$W^\pm_\nu \to \varphi^\pm A_\mu : +ieM_W g_{\mu\nu},$$

$$q_i \to q_j W^\pm_\mu : \frac{-ig}{\sqrt{2}}\gamma_\mu L U^{(I_i)}_{ji},$$

$$q_i \to q_j \varphi^\pm : \frac{-ig}{\sqrt{2}M_W} I_i(m_j L - m_i R)U^{(I_i)}_{ji}.$$

The symbol I_i which appears in the last two rules is related to the weak isospin of the initial quark. For $I_i = +1$ (u type quark) the Cabibbo-Kobayashi-Maskawa (CKM) matrix element is $U^{(+)}_{ji} = V^*_{ij}$, and for $I_i = -1$ (d type quark) $U^{(-)}_{ji} = V_{ji}$. Note that $I_i = Q_W$ (or Q_S) and thus, with the exception of the φAW and qqW vertices, all other vertices depend on the sign of the charge of the quark or boson involved. Finally, e is the positive unit of the electric charge, $e>0$.

The relevant boson propagators in the 't Hooft–Feynman ($\xi=1$) gauge are

$$\frac{i}{p^2 - M_W^2} \quad \text{(scalar)},$$

$$\frac{-ig_{\mu\nu}}{p^2 - M_W^2} \quad (W \text{ boson}).$$

III. CONTRIBUTIONS TO THE EFFECTIVE VERTEX

In the lowest order, i.e., at the one-loop level, the processes in which a quark undergoes a flavor change accompanied by the emission of a real or virtual photon are represented by the diagrams in Figs. 1(a)–1(d) and Figs. 2(a),2(b). In the first group, the photon is emitted by a scalar or a vector boson in the intermediate state, whereas in the second it arises from an internal fermion. As we are eventually interested in real, transversally polarized photon emission, the transition should take place in a magnetic mode characterized by the operator $i\sigma_{\mu\nu}q^\nu$, and we need not concern ourselves with diagrams in Fig. 3 since they are all proportional to γ_μ.

We perform our calculations in n dimensions and let $n = 4-2\omega$. For convenience we also introduce an arbitrary scale parameter with the dimension of mass μ so that the coupling constant g remains dimensionless even in arbitrary n dimensions. The kinematic variables to be used are defined in Fig. 4. In particular, the initial quark has momentum P and mass M; the final quark has momentum p and mass m. The internal quark, with mass m_l, may have flavors u,c, or t for an initial quark of the d type, and s,d, or b for an initial quark of the u type. It is understood that the various partial or total vertex operators obtained in the following for $q_i \to q_f\gamma$ are to be inserted between the initial state $u_i(P)$ and the final state $\overline{u_f}(p)\varepsilon^{\mu *}(q)$, where u_i and u_f are the fermion spinors, and $\varepsilon^\mu(q)$ is the photon polarization vector with momentum $q = P - p$. A summation over all allowed quark flavors l and an integration over the loop n-dimensional k momentum are both implicit.

From the rules given above, the expression corresponding to Fig. 1(a) is

$$iT^\mu_{1(\mathrm{a})} = \mu^{2\omega}\left(\frac{-ig}{\sqrt{2}}\right)^2 \lambda_l \gamma^\rho L \frac{i}{\not k - m_l}\gamma^\sigma L \times \frac{(-i)^2(ieQ_W X^\mu{}_{\rho\sigma})}{[(p-k)^2 - M_W^2][(P-k)^2 - M_W^2]}, \tag{1}$$

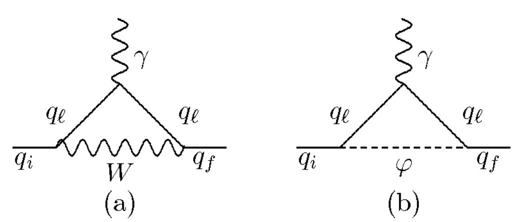

FIG. 2. Contributions to $q_i \to q_f\gamma$ vertex: emission from internal quark.

where the following shorthand notations have been used:

$$X_{\mu\rho\sigma}=(P+p-2k)_\mu g_{\rho\sigma}+(P+k-2p)_\sigma g_{\rho\mu} +(p+k-2P)_\rho g_{\mu\sigma} \tag{2}$$

and

$$\begin{aligned} i=u \text{ type:} \quad & Q_W=+1, \lambda_l=V_{fl}V_{il}^*, \\ i=d \text{ type:} \quad & Q_W=-1, \lambda_l=V_{lf}^*V_{li}. \end{aligned} \tag{3}$$

It is convenient to put Eq. (1) in the form

$$iT^\mu_{1(a)}=\frac{-g^2e}{2}\mu^{2\omega}Q_W\lambda_l\frac{N^\mu_{1(a)}}{D_1} \tag{4}$$

with the following expressions for the numerator and denominator:

$$\begin{aligned} N^\mu_{1(a)}&=\gamma^\rho L(\not k+m_l)\gamma^\sigma L X^\mu{}_{\rho\sigma}, \\ D_1&=(k^2-m_l^2)[(p-k)^2-M_W^2][(P-k)^2-M_W^2]. \end{aligned} \tag{5}$$

Corresponding to Fig. 1(b) is the transition operator

$$\begin{aligned} iT^\mu_{1(b)}&=\mu^{2\omega}\lambda_l\left[\frac{igQ_S}{\sqrt{2}M_W}(mL-m_lR)\right]\frac{i}{\not k-m_l}\left[\frac{-igQ_S}{\sqrt{2}M_W}\right. \\ &\quad\times\left.(m_lL-MR)\right]\frac{i^2(-ieQ_S)(P+p-2k)^\mu}{[(p-k)^2-M_W^2][(P-k)^2-M_W^2]} \\ &=\frac{-g^2e}{2}\mu^{2\omega}Q_S\lambda_l\frac{N^\mu_{1(b)}}{D_1}. \end{aligned} \tag{6}$$

If the initial quark flavor q_i is of the u type, $Q_S=+1$; otherwise $Q_S=-1$. The expression for the denominator D_1 remains the same as defined above, whereas in the numerator one has

$$N^\mu_{1(b)}=\frac{1}{M_W^2}(mL-m_lR)(\not k+m_l)(m_lL-MR)(P+p-2k)^\mu. \tag{7}$$

For Fig. 1(c), the Feynman rules yield

$$\begin{aligned} iT^\mu_{1(c)}&=\mu^{2\omega}\lambda_l\left(\frac{-ig}{\sqrt{2}}\gamma_\mu L\right)\frac{i}{\not k-m_l}\left[\frac{-igQ_S}{\sqrt{2}M_W}(m_lL\right. \\ &\quad\left.-MR)\right]\frac{(ieM_W)(-i)(i)}{[(p-k)^2-M_W^2][(P-k)^2-M_W^2]} \\ &=\frac{-g^2e}{2}\mu^{2\omega}Q_S\lambda_l\frac{N^\mu_{1(c)}}{D_1}, \end{aligned} \tag{8}$$

with the numerator given by

$$N^\mu_{1(c)}=\gamma^\mu L(\not k+m_l)(MR-m_lL). \tag{9}$$

The diagram in Fig. 1(d) is similar to that in Fig. 1(c), with the scalar and vector bosons exchanging roles in the intermediate state. The corresponding vertex operator is

$$\begin{aligned} iT^\mu_{1(d)}&=\mu^{2\omega}\lambda_l\left[\frac{+igQ_S}{\sqrt{2}M_W}(mL-m_lR)\right]\frac{i}{\not k-m_l}\left(\frac{-ig}{\sqrt{2}}\gamma_\mu L\right) \\ &\quad\times\frac{(ieM_W)(i)(-i)}{[(p-k)^2-M_W^2][(P-k)^2-M_W^2]} \\ &=\frac{-g^2e}{2}\mu^{2\omega}Q_S\lambda_l\frac{N^\mu_{1(d)}}{D_1}, \end{aligned} \tag{10}$$

with the following expression in the numerator:

$$N^\mu_{1(d)}=(mL-m_lR)(\not k+m_l)\gamma^\mu L. \tag{11}$$

The photon may be emitted also from the intermediate quark as shown in Figs. 2(a),2(b). The diagram in Fig. 2(a) represents the transition operator

$$\begin{aligned} iT^\mu_{2(a)}&=\mu^{2\omega}\lambda_l\left(\frac{-ig}{\sqrt{2}}\gamma^\rho L\right)\frac{i}{\not p-\not k-m_l}(-ieQ_l\gamma^\mu) \\ &\quad\times\frac{i}{\not P-\not k-m_l}\left(\frac{-ig}{\sqrt{2}}\gamma_\rho L\right)\frac{-i}{k^2-M_W^2} \\ &=\frac{-g^2e}{2}\mu^{2\omega}Q_l\lambda_l\frac{N^\mu_{2(a)}}{D_2}, \end{aligned} \tag{12}$$

where $N^\mu_{2(a)}$ and D_2 stand for

$$\begin{aligned} N^\mu_{2(a)}&=\gamma^\rho(\not p-\not k+m_l)\gamma^\mu(\not P-\not k+m_l)\gamma_\rho L, \\ D_2&=(k^2-M_W^2)[(p-k)^2-m_l^2][(P-k)^2-m_l^2]. \end{aligned} \tag{13}$$

Similarly for Fig. 2(b), we have

$$\begin{aligned} iT^\mu_{2(b)}&=\mu^{2\omega}\lambda_l\left[\frac{+igQ_S}{\sqrt{2}M_W}(mL-m_lR)\right]\frac{i}{\not p-\not k-m_l} \\ &\quad\times(-ieQ_l\gamma^\mu)\frac{i}{\not P-\not k-m_l} \\ &\quad\times\left[\frac{-igQ_S}{\sqrt{2}M_W}(m_lL-MR)\right]\frac{i}{k^2-M_W^2} \\ &=\frac{-g^2e}{2}\mu^{2\omega}Q_l\lambda_l\frac{N^\mu_{2(b)}}{D_2}, \end{aligned} \tag{14}$$

together with the expression for the numerator:

$$\begin{aligned} N^\mu_{2(b)}&=\frac{1}{M_W^2}(mL-m_lR)(\not p-\not k+m_l) \\ &\quad\times\gamma^\mu(\not P-\not k+m_l)(m_lL-MR). \end{aligned} \tag{15}$$

FIG. 3. Emission from initial or final quark.

Thus, all partial contributions to the effective radiative vertex may be written in the form

$$iT^{\mu}_{p\alpha}=\frac{-g^2e}{2}\mu^{2\omega}Q_p\lambda_l\frac{N^{\mu}_{p\alpha}}{D_p}\quad(p=1,2;\alpha=a,b,\ldots)\tag{16}$$

with the charge factor

$$Q_1=Q_B(Q_W\ \text{or}\ Q_S),\quad Q_2=Q_l.\tag{17}$$

After Feynman parameterization of the denominators, the integration over n-dimensional momentum space is performed in the standard way. Consider for example $T^{\mu}_{1(\mathrm{a})}$. The terms independent of the integrated momentum k are evaluated as follows:

$$\begin{aligned}\mu^{2\omega}\int\frac{d^nk}{(2\pi)^n}\frac{1}{D_1}&=\mu^{2\omega}\int\frac{d^nk}{(2\pi)^n}\int_0^1dx\int_0^{1-x}dy\\&\quad\times\frac{2}{[a(1-x-y)+bx+cy]^3}\\&=\frac{-i}{(4\pi)^2}\int_0^1dx\int_0^{1-x}dy\frac{1}{M_W^2\Lambda_1}\\&\quad\times\left(\frac{4\pi\mu^2}{-M_W^2\Lambda_1}\right)^{\omega}\Gamma(1+\omega)\end{aligned}$$

for $a=(p-k)^2-M_W^2$, $b=k^2-m_l^2$, $c=(P-k)^2-M_W^2$, and

$$M_W^2\Lambda_1=M_W^2(1-x)+m_l^2x-P^2xy-p^2xz-q^2yz,$$

$$z=1-x-y.$$

Terms with explicit k-dependent integrands are handled in a similar way. With the summation over the internal quarks now explicit, the k integration therefore carries Eq. (1) into

$$\begin{aligned}iT^{\mu}_{1(\mathrm{a})}&=\frac{-g^2e}{2}Q_W\mu^{2\omega}\sum_l\lambda_l\int\frac{d^nk}{(2\pi)^n}\frac{N^{\mu}_{1(\mathrm{a})}}{D_1}\\&=\frac{ig^2e}{32\pi^2M_W^2}Q_W\sum_l\lambda_l\int_0^1dx\int_0^{1-x}dy\frac{1}{\Lambda_1}\\&\quad\times R\{[-4xP^{\mu}+2(1-2z)q^{\mu}][(1-x)\not{P}-z\not{q}]\\&\quad+(1-x)(\not{q}\gamma^{\mu}\not{P}-\not{P}\gamma^{\mu}\not{q})+(-x\not{P}-z\not{q})[(1-x)\not{P}\\&\quad-z\not{q}]\gamma^{\mu}+\gamma^{\mu}[(1-x)\not{P}-z\not{q}][-x\not{P}+(1-z)\not{q}]\\&\quad+4(1-n)\gamma^{\mu}\mathcal{F}_1\}.\end{aligned}\tag{18}$$

Here $\mathcal{F}_1$ comes from the singular part of the quadratic k-dependent integrand. This result will considerably simplify when the external particles go on shell, which is all we need. When the initial and final quarks are on the mass shell, we may use the Dirac equations $\not{P}u_i(P)=Mu_i(P)$ and $\overline{u}_f(p)\not{p}=m\overline{u}_f(p)$ to express the operator $T_{1(\mathrm{a})\mu}$ in terms of the four-vectors P_μ, q_μ, and γ_μ or, alternatively, $i\sigma_{\mu\nu}q^{\nu}$, q_μ, and γ_μ, the two bases being related by

$$\begin{aligned}2P_\mu R&=R(i\sigma_{\mu\nu}q^{\nu}+q_\mu+M\gamma_\mu)+Lm\gamma_\mu,\\2P_\mu L&=L(i\sigma_{\mu\nu}q^{\nu}+q_\mu+M\gamma_\mu)+Rm\gamma_\mu.\end{aligned}\tag{19}$$

These relations are understood as being sandwiched between $\overline{u}(p)$ and $u(P)$. The induced complete $q_iq_f\gamma$ vertex will then assume the general form

$$T_\mu\sim R(q^2\gamma_\mu-q_\mu\not{q})F+i\sigma_{\mu\nu}q^{\nu}(RMG+LmH),\tag{20}$$

where F, G, and H are appropriate form factors. For an on-shell photon, $q^2=0$, we may set $q_\mu\varepsilon^{\mu}(q)=0$, and the coefficient of F should vanish. The singular terms, such as $\mathcal{F}_1$, from all diagrams likewise completely cancel out [8]. Hence, with real photon radiative processes in mind, we may drop all such terms in the individual contributions. When the above simplifications are applied, $T^{\mu}_{1(\mathrm{a})}$ reduces on the mass shell to

$$\begin{aligned}T^{\mu}_{1(\mathrm{a})}&=\frac{eG_{\mathrm{F}}}{4\sqrt{2}\pi^2}Q_{\mathrm{W}}\sum_l\lambda_l\int_0^1dx\int_0^{1-x}dy\frac{1}{\Lambda_1(r_l)}\\&\quad\times\{RM[1-x+y(1-2x)]\\&\quad+Lm[1-x+z(1-2x)]\}i\sigma^{\mu\nu}q_\nu,\end{aligned}\tag{21}$$

where $G_{\mathrm{F}}/\sqrt{2}=g^2/(8M_W^2)$ and

$$\Lambda_1(r_l)=1-x+r_lx-r_ixy-r_fxz\tag{22}$$

with $r_l=m_l^2/M_W^2$, $r_i=M^2/M_W^2$, and $r_f=m^2/M_W^2$. The contributions from all the other diagrams may be reduced to this standard form.

Let us sum up the partial contributions in each group ($p=1,2$) of diagrams to get the transition operators

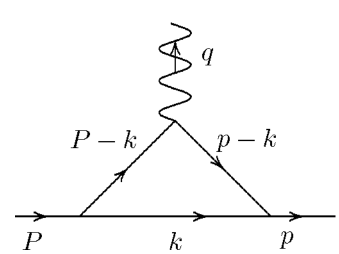

FIG. 4. Kinematic variables in one-loop diagrams.

$$T_p^\mu = \frac{eG_F}{4\sqrt{2}\pi^2} Q_p \sum_l \lambda_l [RMG_p(r_l) + LmH_p(r_l)] i\sigma^{\mu\nu} q_\nu, \tag{23}$$

where we have used Q_p defined in Eq. (17), and introduced the functions

$$G_p(r) = \int_0^1 dx \int_0^{1-x} dy \frac{1}{\Lambda_p} g_p(r,x,y),$$

$$H_p(r) = \int_0^1 dx \int_0^{1-x} dy \frac{1}{\Lambda_p} h_p(r,x,y)$$

$$(p=1,2), \tag{24}$$

where $\lambda_1(r)$ is given in (22) and

$$\lambda_2(r) = x + r(1-x) - r_i xy - r_f xz,$$

$$g_1(r,x,y) = 1 - x + z + y(1-2x) + rx(1-y) - r_f xz,$$

$$h_1(r,x,y) = 1 - x + y + z(1-2x) + rx(1-z) - r_i xy,$$

$$g_2(r,x,y) = -2x(1-y) + r(-1+x+xy) + r_f xz,$$

$$h_2(r,x,y) = -2x(1-z) + r(-1+x+xz) + r_i xy. \tag{25}$$

These expressions display a symmetry between g_p and h_p, for a given p , under simultaneous interchange of y, r_i and z, r_f. They reduce to

$$g_1(r,x,y) = 2 + (r-2)x - (r+2)xy - r_f xz,$$

$$h_1(r,x,y) = 2 - 4x + (r+2)x^2 + (r+2-r_i)xy,$$

$$g_2(r,x,y) = -r + (r-2)x + (r+2)xy + r_f xz,$$

$$h_2(r,x,y) = -r + 2rx - (r+2)x^2 - (r+2-r_i)xy. \tag{26}$$

An obstacle to a simple analytic expression for the effective vertex is the x integration in Eq. (24). However, if both initial and final quark masses are small compared with the gauge boson masss, i.e., if $r_i, r_f \ll 1$, one may consider making a linear approximation in r_i and r_f:

$$\frac{1}{\Lambda_p(r)} \approx \frac{1}{\Lambda_p^{(0)}(r)} + \frac{1}{(\Lambda_p^{(0)}(r))^2} x(r_i y + r_f z) \quad (p=1,2)$$

$$\text{for } \Lambda_1^{(0)}(r) = 1 - x + r_l x, \quad \Lambda_2^{(0)}(r) = x + r(1-x). \tag{27}$$

In this approximations of Λ_p, the functions G_p and H_p may be written as

$$G_p(r) = F_p(r) + r_f K_p(r) + r_i L_p(r),$$

$$H_p(r) = F_p(r) + r_i K_p(r) + r_f L_p(r) \quad (p=1,2), \tag{28}$$

with a symmetry reflecting that between g_p and h_p. Here F_p, K_p, and L_p are functions of r only. Writing $g_p = g_p^{(0)} + r_f g_p^{(1)}$ and $h_p = h_p^{(0)} + r_i h_p^{(1)}$, we see that F_p comes from the xy integral $\int g_p^{(0)}/\Lambda_p^{(0)}$ (or its equal $\int h_p^{(0)}/\Lambda_p^{(0)}$), while K_p and L_p arise from various terms linear in r_i or r_f in g_p, h_p, or Λ_p. Together the functions F_p, K_p, and L_p determine G_p and H_p for $p=1,2$ to first order in r_i, r_f. They are given by $(d=r-1)$

$$F_1(r) = \frac{r}{4d^3}(1-5r-2r^2) + \frac{3r^3}{2d^4}\ln r,$$

$$K_1(r) = \frac{r}{72d^5}(13-11r-195r^2+17r^3-4r^4) + \frac{r^2}{6d^6}(-3+13r+5r^2)\ln r,$$

$$L_1(r) = \frac{r}{36d^5}(7-65r-141r^2+23r^3-4r^4) + \frac{r^3}{3d^6}(13+2r)\ln r,$$

$$F_2(r) = \frac{r}{4d^3}(-2-5r+r^2) + \frac{3r^2}{2d^4}\ln r,$$

$$K_2(r) = \frac{r}{72d^5}(41-237r+39r^2-29r^3+6r^4) + \frac{r}{6d^6}(-2+8r+9r^2)\ln r,$$

$$L_2(r) = \frac{r}{18d^5}(-5-105r+33r^2-16r^3+3r^4) + \frac{r}{6d^6}(-1+22r+9r^2)\ln r. \tag{29}$$

We have omitted in the above expressions terms that are numerical constants. Such terms independent of r_l will give vanishing contributions as long as F_p, K_p, and L_p are used under the sums carried out over all allowed l, as in Eq. (23), for $\Sigma_l \lambda_l = 0$ by the unitarity of the CKM matrix.

It is understood that for $s \to d$ and $b \to s$, we have $Q_1 = -1$, $Q_2 = 2/3$ and $l = 1,2,3$ correspond, respectively, to u,c,t; whereas for $c \to u$ and $t \to c$, we take $Q_1 = +1$, $Q_2 = -1/3$, and $l = d,s,b$.

Inami and Lim [1] have entirely neglected the external masses in their calculations, so that $G_p = H_p = F_p$. For the stated purpose of their work, they have given the explicit expression for $-F_1 + (2/3)F_2$ valid for the down-type transitions, such as $s \to d\gamma$ or $b \to s\gamma$; namely,

$$-F_1+\frac{2}{3}F_2=\frac{r}{12(r-1)^4}[(1-r)(7-5r-8r^2)+6r(2-3r)\ln r]\xrightarrow{r\to 0}\frac{7}{12}r. \tag{30}$$

But for the up-type transitions such as $c\to u\gamma$, one should use instead the expression

$$F_1-\frac{1}{3}F_2=\frac{-r}{12(r-1)^4}[(1-r)(5-10r-7r^2)+6r(1-3r)\ln r]\xrightarrow{r\to 0}\frac{-5}{12}r, \tag{31}$$

and not the following combination [5,6] or its limiting value [4]

$$-F_1-\frac{1}{3}F_2=\frac{r}{12(r-1)^4}[(1-r)(1-20r-5r^2)-6r(1+3r)\ln r]\xrightarrow{r\to 0}\frac{1}{12}r. \tag{32}$$

As for the leptonic transitions listed in Table I, the corresponding amplitude for the heavy neutrino decay [8,9] $\nu_1\to\nu_2\gamma(Q_1=1,Q_2=-1)$ is

$$F_1-F_2=\frac{3r}{4d^3}(1-r^2+2r\ln r), \tag{33}$$

whereas for $\tau^-\to\mu^-\gamma(Q_1=-1,Q_2=0)$ it is simply $-F_1$.

To examine the relative importance of the external masses, we calculate the decay rates of $q_i\to q_f\gamma$, using the linear approximations in r_i, r_f, as defined in Eqs. (23)–(29). For this purpose, it is convenient to define the amplitudes

$$S_p(r)=\frac{1}{2}[G_p(r)+\sqrt{r_f/r_i}H_p(r)],$$

$$P_p(r)=\frac{1}{2}[G_p(r)-\sqrt{r_f/r_i}H_p(r)], \tag{34}$$

so that Eq. (23), when summed over both groups of diagrams, may be rewritten as

$$T_\mu=\frac{eG_{\mathrm{F}}M_W\sqrt{r_i}}{4\sqrt{2}\pi^2}(A+B\gamma_5)i\sigma_{\mu\nu}q^\nu, \tag{35}$$

where

$$A=\sum_{l=1}^{3}\lambda_l\sum_{p=1}^{2}Q_pS_p(r_l)=\sum_{l=1}^{2}\lambda_l\sum_{p=1}^{2}Q_p[S_p(r_l)-S_p(r_1)], \tag{36}$$

TABLE II. Widths for $q_i\to q_f\gamma$ transitions.

Mode	Γ_0 (GeV)	Γ (GeV)
$s\to d\gamma$	7.48×10^{-29}	7.53×10^{-29}
$b\to s\gamma$	5.06×10^{-17}	5.10×10^{-17}
$c\to u\gamma$	1.50×10^{-28}	1.50×10^{-28}
$t\to c\gamma$	4.51×10^{-14}	2.73×10^{-12}

$$B=\sum_{l=1}^{3}\lambda_l\sum_{p=1}^{2}Q_pP_p(r_l)=\sum_{l=1}^{2}\lambda_l\sum_{p=1}^{2}Q_p[P_p(r_l)-P_p(r_1)]. \tag{37}$$

In the last step we have made use of the unitarity of the Cabibbo-Kobayashi-Maskawa (CKM) matrix to replace λ_1 with $-(\lambda_2+\lambda_3)$. In terms of these amplitudes the decay rate for $q_i\to q_f\gamma$ is given by

$$\Gamma(q_i\to q_f\gamma)=\frac{G_{\mathrm{F}}^2\alpha}{64\pi^4}M_W^5(\sqrt{r_i})^5\left(1-\frac{r_f}{r_i}\right)^3(|A|^2+|B|^2). \tag{38}$$

The results of our calculations based on Eq. (38) are shown in Table II together with Γ_0, the corresponding values of the width when the masses of the external quarks in the amplitudes are neglected. We have used $m_u=5$ MeV, $m_c=1.5$ GeV, $m_t=174$ GeV, $m_d=11$ MeV, $m_s=150$ MeV, and $m_b=4.9$ GeV for the quark masses, $M_W=80$ GeV for the W boson mass, and the central values of the CKM matrix elements as given in the Review of Particle Physics [10].

We see that, with one exception, generally $\Gamma\approx\Gamma_0$, i.e., apart from the kinematic factor $(\sqrt{r_i})^5$, the external masses can be safely neglected in calculating the width for $q_i\to q_f\gamma$. The reason is that the mass correction functions K_p and L_p enter the amplitudes S_p and P_p, respectively, accompanied by the mass factors $\sqrt{r_ir_f}$ and $[(\sqrt{r_i})^3+(\sqrt{r_f})^3]/(\sqrt{r_i}+\sqrt{r_f})$, which are much smaller than 1, the coefficient of F_p [cf. Eqs. (28) and (34)]. The exceptional case is the $t\to c\gamma$ transition. Since $r_t\approx4.73$, neglecting the mass of the initial quark is not justified, and even keeping just terms linear in r_t as in Eqs. (27),(28) is not enough and misleading: the approximation has broken down. However, the fact that $\Gamma\approx100\Gamma_0$ for this transition clearly indicates the importance of the effect of the external top mass. In Table III we list the contributions of the intermediate quarks (d,s, and b) to the amplitudes $S=S_1-(1/3)S_2$ and $P=P_1-(1/3)P_2$ of the $t\to c\gamma$ transition for both $m_c=0$ and $m_c=1.5$ GeV.

TABLE III. Contributions to $t\to c+\gamma$ in the linear approximation in m_c^2 and m_t^2.

Quark	$S=P(m_c=0)$	$S(m_c\neq0)$	$P(m_c\neq0)$
d	6.12×10^{-8}	6.20×10^{-8}	6.05×10^{-8}
s	8.12×10^{-6}	9.02×10^{-6}	8.81×10^{-6}
b	5.95×10^{-3}	6.00×10^{-3}	5.90×10^{-3}
Γ(GeV)	2.7269×10^{-12}		2.7273×10^{-12}

These data show that the mass of the final quark is completely negligible in this case as well, and the difference between Γ and Γ_0 for this transition can be entirely attributed to the top mass.

Before leaving this section we will make estimates of the branching fraction for the $c\to u\gamma$ transition (i.e., its inclusive rate scaled to that of the semileptonic decay $c\to ql^+\nu$). The correct expression $\overline{F}=F_1-(1/3)F_2$ which we use differs in magnitude and sign from the wrong combination $\overline{F'}=-F_1-(1/3)F_2$ for the relevant values of the argument r_l. The QCD-uncorrected branching we have obtained, $B^{(0)}=3.90\times10^{-16}$, is about 28 times larger than an estimate [5] based on $\overline{F'}$. To calculate the QCD corrections, one begins with the Wilson coefficients evaluated at the W mass scale; in particular the coefficient

$$c_7(M_W)=-\frac{1}{2}\left[\frac{\lambda_s}{\lambda_b}\overline{F}(r_s)+\overline{F}(r_b)\right] \tag{39}$$

yields 2.065×10^{-3}, to be compared with -0.414×10^{-3} when it is calculated with $\overline{F'}$. The Wilson coefficients are then evolved down from M_W to the renormalization scale $\mu=m_c$ to give the effective coefficient $c_7^{\rm eff}(m_c)$. The resulting QCD-corrected branching fraction obtained is reduced by 1% from the estimate of Burdman *et al.* [5]. The reason for the smallness of this effect is that $c_7(M_W)$ makes a very small contribution to $c_7^{\rm eff}(m_c)$ which is completely dominated by $c_2(M_W)$. Nevertheless, the point being made is that the correct input functions F_p or, even better, G_p and H_p are crucial for a proper evaluation of the Wilson coefficients.

IV. MASS EFFECT IN THE TOP QUARK DECAY

Since $m_c\ll m_t$, and G_p and H_p must be comparable in magnitude, the amplitude in Eq. (23) reduces to RMG_p and we need only to perform an accurate evaluation of the integrals

$$G_p(r)=\int_0^1 dx\int_0^{1-x}dy\,\frac{1}{\Lambda_p}g_p(r,x,y), \tag{40}$$

where $\Lambda_1=1-x+rx-r_ixy$ and $\Lambda_2=x+r(1-x)-r_ixy$, with $r_i=r_t$, $r=r_l$, and $l=d,s,b$. After integration over y, which yields logarithm functions, the x integration leads to

$$G_1(r)=\frac{(2+r)}{2r_i}+\frac{1}{r_i^2}[(r_i-1)(2-r)+r^2]J(r)+\frac{1}{r_i^2}[2(1-r_i)+r]\left[\mathcal{L}_2(1-r)-\mathcal{L}_2\left(\frac{1}{\alpha}\right)-\mathcal{L}_2\left(\frac{1}{\beta}\right)\right], \tag{41}$$

TABLE IV. Values of G_1 and G_2 for $d,s,$ and b in $t\to c+\gamma$.

	G_1	G_2	S
d	$0.84375-i0.80176$	$-0.19858+i0.82598$	$0.45497-i0.53855$
s	$0.84375-i0.80177$	$-0.19860+i0.82600$	$0.45498-i0.53855$
b	$0.84541-i0.80377$	$-0.20823+i0.83341$	$0.45742-i0.54079$
		$\Gamma(\text{GeV})=8.45\times10^{-13}$	

$$G_2(r)=-\frac{(2+r)}{2r_i}+\frac{1}{r_i^2}[(r_i-1)(2-r)+r^2]J(r)+\frac{1}{r_i^2}[r(r_i-r-2)]\left[\mathcal{L}_2\left(1-\frac{1}{r}\right)-\mathcal{L}_2\left(\frac{1}{1-\alpha}\right)-\mathcal{L}_2\left(\frac{1}{1-\beta}\right)\right]. \tag{42}$$

The parameters α,β are defined by

$$\alpha=\frac{1}{2r_i}(1+r_i-r-\sqrt{\Delta}),\quad \beta=\frac{1}{2r_i}(1+r_i-r+\sqrt{\Delta}), \tag{43}$$

$$\Delta=(1+r_i^2+r^2)-2(r_i+r+r_ir).$$

For the transition under discussion, they satisfy the conditions $0<\alpha<\beta<1$. *J(r)* stands for

$$J(r)=-1+\ln r_i+\frac{r}{1-r}\ln r+\alpha\ln\alpha+(1-\alpha)\ln(1-\alpha)+\beta\ln\beta+(1-\beta)\ln(1-\beta)+i\pi\frac{\sqrt{\Delta}}{r_i}, \tag{44}$$

and $\mathcal{L}_2$ is the dilogarithm (Spence integral) [11] represented by

$$\mathcal{L}_2(x)=-\int_0^x\frac{\ln(1-t)}{t}dt, \tag{45}$$

which admits the expansion series $\mathcal{L}_2(x)=\Sigma_{k=1}^{\infty}x^k/k^2$ for $|x|\leq1$.

With a top mass larger than the mass of the W boson and the internal quark, Λ_1 and Λ_2 may change signs over the range of the x,y integrations, and hence the arguments of the various logarithm functions that appear in the course of the integrations may become negative, and imaginary parts will arise. Physically they signal the presence of the intermediate states on the mass shell. Thus, for each internal quark $l=d,s,b$, the imaginary parts of $G_p(r_l)$ correspond to the real emission process $t\to W^++q_l+\gamma$.

Values of $G_1(r_l)$, $G_2(r_l)$, and $S(r_l)=\frac{1}{2}[G_1(r_l)-\frac{1}{3}G_2(r_l)]$ for relevant r_l in $t\to c\gamma$ are recorded in Table IV. They depend weakly on the mass of the internal quark, being dominated by the large mass of the initial top quark.

Thus, the Glashow-Iliopoulos-Maiani mechanism works effectively to suppress the amplitude

$$A=B=\frac{1}{2}\sum_{l=d,s,b}\lambda_l\left[G_1(r_l)-\frac{1}{3}G_2(r_l)\right]. \tag{46}$$

In contrast, in the zeroth or first order approximation (in r_i,r_f) of the transition amplitudes, the functions $S(r_l)$ and $P(r_l)$ depend strongly on the internal quark mass, as shown in Table III. The contribution of the intermediate b quark is then overwhelmingly dominant and the GIM mechanism hardly operative. However, this conclusion turns out to be erroneous, as is clear from the data presented in Table IV. Finally, the exact G_1 and G_2 scale as $r_t^{-1}\sim 1$ [see Eqs. (41),(42)], whereas their approximate values scale at best as $r_t r_b \sim r_b \sim 10^{-2}$ [see Eqs. (28),(29)], which explains the striking differences in magnitudes that we see in Tables III and IV.

With the amplitude given in Eq. (46), we find the width $\Gamma(t\to c\gamma)\approx 8.45\times 10^{-13}$ GeV, which falls between the values presented in Table II based on the approximation $r_i=r_f=0$ and the linear approximation (38). In brief, the effect of the initial fermionic mass in $t\to c\gamma$ and, by extension, $\tau\to\mu\gamma$ is both subtle and substantial.

V. CONCLUSIONS

We have supplemented the Inami-Lim formula for the $s\to d\gamma$ effective vertex with an equally simple formula for the $c\to u\gamma$ vertex [Eq. (31)] under the same assumptions that the external fermionic masses are negligible. We then proceeded on to reexamine the external mass effects and obtain the corresponding formulas [Eqs. (23), (28),(29)] valid to linear order in M^2 and m^2, the results of which are shown in Table II. Since this approximation breaks down (see Table III) in the cases such as $t\to c\gamma$, in which the initial fermion mass is much larger than any internal mass, we derive the exact one-loop formula for such transitions [Eqs. (41)–(45)] and evaluate the corresponding rate of the $t\to c\gamma$ transition in Table IV. Given the importance of flavor-changing electromagnetic transitions in testing the standard model and probing new physics, it is essential to have reliable results for these processes at the lowest, one-loop level since they are the key components in building up more sophisticated calculations.

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