Plasma wave instabilities induced by neutrinos

Luís Bento

Centro de Física Nuclear da Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal

Quantum field theory is applied to study the interaction of an electron plasma with an intense neutrino flux. A connection is established between the field theory results and classical kinetic theory. The dispersion relation and damping rate of the plasma longitudinal waves are derived in the presence of neutrinos. It is shown that supernova neutrinos are never collimated enough to cause non-linear effects associated with a neutrino resonance. They only induce neutrino Landau damping, linearly proportional to the neutrino flux and G_F^2.

I. INTRODUCTION

Bingham *et al.* [1] have studied the interaction of a neutrino beam with an electron plasma and concluded that the neutrino fluxes produced in supernovae are intense enough to cause plasma instabilities with large growth rates. If true this would provide a physical mechanism of energy transfer from the neutrinos to the plasma that might explain the supernova explosions. The interaction between neutrinos and plasma was described with a ponderomotive force acting on the electrons and a neutrino wave function obeying a naive Klein-Gordon equation modified with a matter induced external potential. This non-standard treatment was not well established from the standard model of electroweak interactions and originated some controversy [2,3]. More recently [4], classic kinetic theory was applied to study the neutrino-plasma system where both neutrino and electron particles suffer each own ponderomotive force. Again the lepton spin and chiral structure of the weak interactions remain unnoticed. The dispersion relation derived for the plasma waves differed from the one previously obtained in [1]. Yet, the authors reiterated the claim that the neutrinos produce non-linear effects for certain resonant modes of plasma waves causing instabilities with large growth rates proportional, not to $G_F^2 n_\nu$, but to a smaller power of this quantity.

The aim of the present work presented here is to obtain a formulation of the problem based on field theory and standard model of electroweak interactions, an effort initiated in [5]. We confine to isotropic plasmas and longitudinal photon excitations (also called plasmons or Langmuir waves). The Čerenkov emission of longitudinal photons by *massless* standard model neutrinos in an isotropic plasma has been studied along the years [6–10]. This is a single neutrino decay but not a collective neutrino process. Hardy and Melrose [3] gave one step more by extending the work to spontaneous and *stimulated emission and absorption* of plasmons to study the so-called kinetic instabilities (see also [11]). They obtained an expression for the decay (growth) rate of the plasma waves induced by a neutrino flux. However, because the derivation was based on single neutrino processes the result is necessarily proportional to G_F^2 excluding *a priori* any possible non-linear effects. In the present paper we derive the neutrino contribution to the photon self-energy and obtain a modified dispersion relation for the longitudinal

waves. This allows one to study not only kinetic instabilities but also the possible existence of hydrodynamic instabilities and non-linear phenomena.

Next section we calculate the neutrino contribution to the electromagnetic polarization tensor and the dispersion relation of longitudinal photons. In Sec. III we establish a relationship with kinetic theory and in Sec. IV the dispersion relation and nature of neutrino induced instabilities are analyzed in detail. In the last section we summarize the main results.

II. ELECTROMAGNETIC POLARIZATION AND LONGITUDINAL WAVES

In a medium the Maxwell equations are modified by polarization effects. The electromagnetic waves obey the following equation:

$$(-k^2 g^{\mu\nu} + k^\mu k^\nu + \pi^{\mu\nu})\varepsilon_\nu = 0, \tag{1}$$

where ε is the wave polarization vector, k the linear momentum and $\pi^{\mu\nu}$ is the polarization tensor. The purely electromagnetic contribution, $\pi_{\text{EM}}^{\mu\nu}$, is well known for an electron plasma in first order of approximation and can be identified with the Feynman diagram of Fig. 1. At finite temperature the one-particle propagators possess additional terms related to the one-particle distribution functions of the background matter. In the real-time formalism [12,13,14], adopted here, the electron propagator in a homogeneous and unpolarized plasma is given by

$$(\not{p}+m)\left[\frac{i}{p^2-m^2} - 2\pi\delta(p^2-m^2)(\theta(p^0)f_e(p) + \theta(-p^0)f_{\bar{e}}(-p))\right], \tag{2}$$
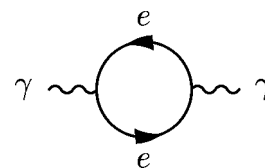

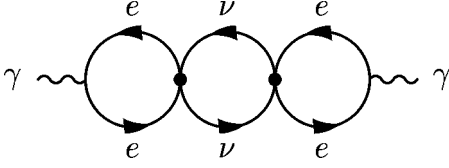FIG. 1. Electron plasma contribution to the photon self-energy, $i\pi_{\text{EM}}^{\mu\nu}$.

FIG. 2. Neutrino-plasma contribution to the photon self-energy, $i\pi_W^{\mu\nu}$.

where f_e, $f_{\bar{e}}$ are the electron and positron distribution functions respectively.

Gauge invariance implies that ε is defined up to a vector proportional to k and $k_\mu \pi^{\mu\nu}=0=\pi^{\mu\nu}k_\nu$. On the other hand, for an isotropic, homogeneous and unpolarized plasma the tensor $\pi_{EM}^{\mu\nu}$ is symmetric and can be written in terms of the metric tensor, momentum k and vector u defining the time direction in the plasma rest frame [13]. As a result,

$$\pi_{EM}^{i0}=\frac{\omega k^i}{\vec{k}^2}\,\pi_{EM}^{00} \qquad (3)$$

in the plasma frame ($\omega=k^0$) and the wave Eqs. (1) admit a purely electrostatic solution, $\varepsilon^\mu=(1,\vec{0})=u^\mu$ in the Coulomb gauge. The dispersion relation of these longitudinal waves is

$$\vec{k}^2+\pi_{EM}^{00}=0. \qquad (4)$$

In the low energy limit of the standard model of electroweak interactions the electron and neutrino interactions (both charged and neutral currents) are described by the effective Lagrangian

$$\mathcal{L}_{int}=e\,A_\mu\,\bar{e}\,\gamma^\mu e-\sqrt{2}\,G_F(\bar{\nu}_L\gamma_\mu\nu_L)\,\bar{e}\,\gamma^\mu(c_V'-c_A'\,\gamma_5)e, \qquad (5)$$

where e denotes the positron charge ($\alpha=e^2/4\pi$), G_F the Fermi constant and

$$c_V'=c_A'+2\sin^2\theta_W, \qquad (6)$$

with c_A' equal to $+1/2$ for ν_e and $-1/2$ for ν_μ,ν_τ. The neutrino flux contributes to the electromagnetic polarization through the diagram of Fig. 2. The diagrams with fermion self-energy corrections either in vacuum or in matter will be neglected as well as the weak interactions between electrons or nucleons of the medium. The expression of the ν_L propagators is similar to the electron propagator of Eq. (2):

$$G_\nu(p)=\frac{1-\gamma_5}{2}\,\not{p}\left[\frac{i}{p^2}-2\pi\,\delta(p^2)(\theta(p^0)f_\nu(p)\right.$$

$$\left.+\theta(-p^0)f_{\bar{\nu}}(-p))\right]. \qquad (7)$$

The only differences are that the ν masses are taken to be zero, there is only one spin degree of freedom, ν_L or $\bar{\nu}_R$, and the neutrino and anti-neutrino distribution functions, f_ν and $f_{\bar{\nu}}$, are not thermal, as they move along a privileged direction.
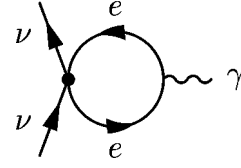

FIG. 3. Neutrino electromagnetic coupling induced by an electron plasma.

The neutrino loop gives a tensor,

$$-i\,\pi_{\alpha\beta}^N(k)=-\int\frac{d^4p}{(2\pi)^4}\mathrm{Tr}\{\gamma_\alpha G_\nu(p+k)\gamma_\beta G_\nu(p)\}, \qquad (8)$$

whose vacuum contribution is to be ignored. Only the contributions from the electron and neutrino loops that are linear in the respective particle densities will be retained. The electron loops are directly related to the neutrino electromagnetic coupling. Writing the $\nu\nu\gamma$ vertex in Fig. 3 as

$$-i\,\gamma_\nu\frac{1-\gamma_5}{2}\Gamma^{\nu\mu}(k), \qquad (9)$$

where k is the incoming photon momentum, the diagram of Fig. 2 is given by

$$i\,\pi_W^{\mu\nu}=i\,\Gamma^{\alpha\mu}(-k)\pi_{\alpha\beta}^N(k)\Gamma^{\beta\nu}(k). \qquad (10)$$

The $\nu\nu\gamma$ vertex separates in a pseudo-tensor proportional to c_A' and a tensor that is proportional to $\pi_{EM}^{\mu\nu}$, as follows [15,16]:

$$\Gamma^{\mu\nu}(k)=\frac{1}{e}\sqrt{2}G_F(c_V'\pi_{EM}^{\mu\nu}-c_A'\pi_5\,\varepsilon^{\mu\nu\alpha\beta}k_\alpha u_\beta). \qquad (11)$$

For future reference let us write the expressions of the susceptibility tensors,

$$\pi_{EM}^{\mu\nu}=-2e^2\int\frac{d^3p}{(2\pi)^3}\frac{f_e+f_{\bar{e}}}{E_e}$$

$$\times\frac{(k^\mu p^\nu+p^\mu k^\nu-k\cdot p\,g^{\mu\nu})k\cdot p-k^2 p^\mu p^\nu}{(k\cdot p)^2-(k^2/2)^2}, \qquad (12)$$

$$\pi_N^{\mu\nu}=-\int\frac{d^3p}{(2\pi)^3}\frac{f_\nu+f_{\bar{\nu}}}{E_\nu}$$

$$\times\frac{(k^\mu p^\nu+p^\mu k^\nu-k\cdot p\,g^{\mu\nu})k\cdot p-k^2 p^\mu p^\nu}{(k\cdot p)^2-(k^2/2)^2}$$

$$-\frac{i}{2}\varepsilon^{\mu\nu\alpha\beta}k_\alpha p_\beta\int\frac{d^3p}{(2\pi)^3}\frac{f_\nu-f_{\bar{\nu}}}{E_\nu}\frac{k^2}{(k\cdot p)^2-(k^2/2)^2}. \qquad (13)$$

If the neutrino distributions were isotropic the polarization tensor π_W would satisfy a relation like Eq. (3) and the longitudinal photons could still be described with a scalar

potential i.e., $\varepsilon^{\mu}=(1,\vec{0})$ in the plasma rest frame. That is not strictly the case in the presence of a neutrino flux but since the plasma is isotropic, as considered here, the vector components ε^i are expected to be suppressed with respect to ε^0 by a factor of $G_F^2 k^4$. We now prove that the components ε^i are in fact suppressed and their contribution to the dispersion relation only comes at G_F^4 order. First, note that the assumed plasma isotropy implies not only the relation (3) but also the structure [13]

$$\pi_{EM}^{ij}=-\pi_T P_{ij}+\pi_{EM}^{00}\frac{\omega^2}{\vec{k}^2}\frac{k^i k^j}{\vec{k}^2}, \qquad (14)$$

where P is the projector over the plane orthogonal to $\vec{k}$,

$$P_{ij}=\delta_{ij}-\frac{k^i k^j}{\vec{k}^2}. \qquad (15)$$

Because of gauge invariance only three of the wave Eqs. (1) are linearly independent. Working in the Coulomb gauge, $\vec{k}\cdot\vec{\varepsilon}=0$, the application of the projector P on the left of the last three Eqs. (1) gives

$$(k^2-\pi_T)\varepsilon^i=P_{ij}\pi_W^{j\nu}\varepsilon_\nu. \qquad (16)$$

The factor $k^2-\pi_T$ vanishes for transverse photons but not for longitudinal waves: in a non-relativistic gas, π_T is equal to ω_p^2, the square plasma frequency, and $k^2=\omega^2-\vec{k}^2$. The above equation can be used to calculate ε^i in an iterative way, as follows:

$$\varepsilon^i=P_{ij}\frac{\pi_W^{j0}}{k^2-\pi_T}\varepsilon^0+\cdots. \qquad (17)$$

It is clear that ε^i are suppressed by G_F^2 with respect to ε^0. When substituted in the first of the wave Eqs. (1),

$$(\vec{k}^2+\pi_{EM}^{00}+\pi_W^{00})\varepsilon^0=\pi_W^{0i}\varepsilon^i, \qquad (18)$$

it becomes evident that the vector components ε^i only reflect in the dispersion relation at G_F^4 order and can thus be safely neglected. The dispersion relation of longitudinal waves is so

$$\vec{k}^2+\pi_{EM}^{00}+\pi_W^{00}=0 \qquad (19)$$

at G_F^2 order. It remains to evaluate π_{EM}^{00} and π_W^{00}.

The electroweak interactions conserve the electric charge and lepton numbers L_e, L_μ, L_τ. This implies that at energies considered here, far below the muon mass, the electron, ν_e, ν_μ and ν_τ numbers are separately conserved which materializes in the conservation laws

$$k_\mu \pi_N^{\mu\nu}=k_\mu \Gamma^{\mu\nu}=0=\pi_N^{\mu\nu}k_\nu=\Gamma^{\mu\nu}k_\nu. \qquad (20)$$

Using repeatedly these relations together with Eq. (3) one obtains from Eqs. (10),(11)

$$\pi_W^{00}=\frac{2G_F^2 c_V'^2}{e^2}(\pi_{EM}^{00})^2\left(1-\frac{\omega^2}{\vec{k}^2}\right)^2\pi_N^{00}. \qquad (21)$$

The outcome is that the c_A' coupling does not contribute to the dispersion relation of longitudinal waves in an isotropic electron plasma. On the other hand, $c_V'^2$ is much smaller for $\nu_\mu, \nu_\tau (c_V'\approx-0.04)$ than for $\nu_e (c_V'\approx0.96)$ so only ν_e and $\bar{\nu}_e$ will be taken in consideration.

The expressions of the response functions are

$$\pi_{EM}^{00}=-2e^2\int\frac{d^3 p_e}{(2\pi)^3}(f_e+f_{\bar{e}})E_e\frac{\vec{k}^2-(\vec{k}\cdot\vec{v}_e)^2}{(k\cdot p_e)^2-(k^2/2)^2}, \qquad (22)$$

$$\pi_N^{00}=-\int\frac{d^3 p_\nu}{(2\pi)^3}(f_\nu+f_{\bar{\nu}})E_\nu\frac{\vec{k}^2-(\vec{k}\cdot\vec{v}_\nu)^2}{(k\cdot p_\nu)^2-(k^2/2)^2}, \qquad (23)$$

where E and $\vec{v}$ denote in each case the energy and velocity of the particle. The factor of 2 in front of the electromagnetic susceptibility stands for the number of electron spin states. The normalization of the distribution functions is fixed by the propagators (2) and (7) as follows: the number of particles $(e^-, e^+, \nu$ or $\bar{\nu})$ per unity of volume *and* spin degree of freedom is

$$n=\int\frac{d^3 p}{(2\pi)^3}f. \qquad (24)$$

Next section we seek a connection between field theory and classical kinetic theory in the limit of small frequencies and wavelengths. In Sec. IV we study the dispersion relation and possible neutrino induced instabilities.

III. KINETIC THEORY

The Feynman diagrams in Figs. 1 and 2 are a consequence of a certain field dynamics that couples boson fields with fermion densities. The electroweak interactions in particular only involve vector fields and currents. The ones relevant here are the electromagnetic field and current, A_μ and J_{EM}^μ, plus the neutrino and weak electron currents defined as

$$J_N^\mu=\bar{\nu}_L\gamma^\mu\nu_L, \qquad (25)$$

$$J_{We}^\mu=\sqrt{2}G_F\bar{e}\,\gamma^\mu(c_V'-c_A'\gamma_5)e. \qquad (26)$$

Behind those Feynman diagrams there is a set of equations relating the current fluctuations, $J^\mu(k)$ in momentum space, to each other:

$$J_{EM}^\mu=-\pi_{EM}^{\mu\nu}(k)A_\nu+J_{N\alpha}\Gamma^{\alpha\mu}(-k), \qquad (27)$$

$$J_{N\alpha}=-\pi_{\alpha\beta}^N(k)J_{We}^\beta, \qquad (28)$$

$$J_{We}^\beta=\Gamma^{\beta\nu}(k)A_\nu+\pi_{We}^{\beta\nu}(k)J_{N\nu}. \qquad (29)$$

Here, π_{We} is an electron loop, suppressed by G_F^2, that couples the weak electron and neutrino currents. One obtains from this a relation between the electromagnetic current and A_μ. Using matrix notation and the definition $\Gamma'^{\mu\alpha}(k) = \Gamma^{\alpha\mu}(-k)$ one gets

$$J_{EM} = -(\pi_{EM} + \Gamma' \pi_N (1 + \pi_{We}\pi_N)^{-1}\Gamma)A \quad (30)$$

$$\simeq -(\pi_{EM} + \Gamma' \pi_N \Gamma)A, \quad (31)$$

where the corrections of G_F^4 order associated with π_{We} are neglected in the last equation. When this electromagnetic current is put in the Maxwell equations,

$$(-k^2 g^{\mu\nu} + k^\mu k^\nu)A_\nu = J_{EM}^\mu, \quad (32)$$

they give rise to the wave equations (1) with a polarization tensor given by the Feynman diagrams of Figs. 1 and 2.

We have seen in the previous section that for an isotropic plasma the c_A' coupling and electron axial-current do not play a significant role. If that current is dropped out of Eqs. (27)–(29) they reduce to

$$J_{EM}^\mu = \frac{1}{e}\pi_{EM}^{\mu\nu}(-eA_\nu + \sqrt{2}G_F c_V' J_{N\nu}), \quad (33)$$

$$J_{N\alpha} = \frac{1}{e}\pi_{\alpha\beta}^N(\sqrt{2}G_F c_V' J_{EM}^\beta) \quad (34)$$

which now have a classical counterpart in the sense that the fermion polarization does not appear explicitly. This is the kind of relations also obtained in classical kinetic theory, a framework that has been used by some authors [4,17]. That fact motivated us to formulate our own classic theory in order to better understand the differences between the results based on quantum field theory and their works.

In kinetic theory a system is described by distribution functions on the phase space of the particles, in particular, the single particle distribution functions $f(t,\vec{x},\vec{p})$. In low dense plasmas the collisions are less important than the collective interactions and the time evolution of the distribution functions is given by the Vlasov equations [18]:

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{x}} + \vec{F}\cdot\frac{\partial f}{\partial \vec{p}} = 0. \quad (35)$$

The functions velocity, $\vec{v}$, and force, $\vec{F}$, have to be specified for each of the particles species.

If one ignores the electron polarization and c_A' coupling, the effective Lagrangian of Eq. (5) reduces to

$$\mathcal{L}_{int} = e A_\mu J_e^\mu - \sqrt{2}G_F c_V' J_{N\mu}J_e^\mu. \quad (36)$$

It clearly admits a classical limit where the neutrino vector current J_N^μ is equal to the difference between the ν and $\bar{\nu}$ current densities, $J_N^\mu = j_\nu^\mu - j_{\bar\nu}^\mu$, and $J_e^\mu = j_e^\mu - j_{\bar e}^\mu$ is the analogous current for electrons and positrons. The next step is to write down the interaction Lagrangian for a classical electron or neutrino particle respectively [5],

$$L_e = (e A^\mu - \sqrt{2}G_F c_V' J_N^\mu)\dot{x}_\mu, \quad (37)$$

$$L_\nu = -(\sqrt{2}G_F c_V' J_e^\mu)\dot{x}_\mu. \quad (38)$$

They are symmetric to the positron and anti-neutrino Lagrangians. Notice that the vector current J_N^μ couples to an electron particle in exactly the same way as the electromagnetic potential A_μ and in turn the neutrinos interact with a vector potential as well, proportional to J_e^μ. Therefore, the electroweak forces are a straightforward generalization of the electromagnetic Lorentz force [5]. The total force acting on an electron is

$$\vec{F}_e = -e(\vec{E} + \vec{v}_e \wedge \vec{B}) + \sqrt{2}G_F c_V'(\vec{E}_N + \vec{v}_e \wedge \vec{B}_N) \quad (39)$$

with weak-electric and weak-magnetic fields given by

$$\vec{E}_N = -\vec{\nabla}J_N^0 - \frac{\partial \vec{J}_N}{\partial t}, \qquad \vec{B}_N = \vec{\nabla}\wedge\vec{J}_N. \quad (40)$$

The positron force is $-\vec{F}_e$. In a similar fashion, the neutrinos suffer a weak force

$$\vec{F}_\nu = \sqrt{2}G_F c_V'(\vec{E}_e + \vec{v}_\nu \wedge \vec{B}_e), \quad (41)$$

$(-\vec{F}_\nu$ for anti-neutrinos) where

$$\vec{E}_e = -\vec{\nabla}J_e^0 - \frac{\partial \vec{J}_e}{\partial t}, \qquad \vec{B}_e = \vec{\nabla}\wedge\vec{J}_e. \quad (42)$$

Above, the velocity and linear momentum are related to each other by $\vec{v} = \vec{p}/\sqrt{\vec{p}^2 + m^2}$ with a zero mass in the neutrino case.

These weak forces differ from the ones employed before [1,4,17]. In particular, the ponderomotive forces considered in [1,4] only contain the terms proportional to the gradients, $\vec{\nabla}J_N^0$ and $\vec{\nabla}J_e^0$, of the neutrino and electron densities but not the terms that go with the vector currents $\vec{J}_N$ and $\vec{J}_e$. In the case of the longitudinal waves, the weak-magnetic forces are suppressed by an additional power of G_F^2 but the time derivatives of $\vec{J}_N$ and $\vec{J}_e$ still contribute at the same level as the density gradients. This fact will manifest in the dispersion relation itself.

The proper modes of a system are usually investigated by expanding the distribution function around a static and uniform function f^0: i.e.,

$$f(x,\vec{p}) = f^0(\vec{p}) + \delta f(x,\vec{p}). \quad (43)$$

The Vlasov equations (35) are then approximated to the linearized form

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\frac{\partial}{\partial \vec{x}}\right)\delta f + \vec{F}\cdot\frac{\partial f^0}{\partial \vec{p}} = 0, \quad (44)$$

which become [18]

$$-i(\omega-\vec{k}\cdot\vec{v})\delta f(k,\vec{p})+\vec{F}(k,\vec{p})\cdot\frac{\partial f^0}{\partial \vec{p}}=0 \qquad (45)$$

after Fourier analysis, where $k^\mu=(\omega,\vec{k})$ denote the frequency and wave vector components. It is convenient to write this in a relativistic covariant notation. In terms of

$$F_4^\mu=p^0\frac{dp^\mu}{dt}, \qquad (46)$$

that transforms as a 4-vector, the equation above reads as

$$-ik\cdot p\,\delta f(k,p)+F_4^\mu(k,p)\frac{\partial f^0}{\partial p^\mu}=0. \qquad (47)$$

It does not matter whether f^0 depends on p^0 or it is evaluated on-shell ($p^2=m^2$) because $F_4^\mu\,\partial/\partial p^\mu$ is a total derivative when applied on functions that do not depend on the space coordinates.

Under an external perturbation the current densities

$$j^\mu(x)=\int \frac{d^3p}{(2\pi)^3E}f(x,p)\,p^\mu \qquad (48)$$

(E is the kinetic energy) suffer fluctuations equal to

$$j^\mu(k)=-i\int \frac{d^3p}{(2\pi)^3E}\frac{\partial f^0}{\partial p^\nu}\frac{F_4^\nu(k,p)\,p^\mu}{k\cdot p}. \qquad (49)$$

To compare with the field theory results it is convenient to integrate by parts obtaining

$$j^\mu(k)=i\int \frac{d^3p}{(2\pi)^3E}f^0\frac{\partial}{\partial p^\nu}\left(\frac{F_4^\nu\,p^\mu}{k\cdot p}\right). \qquad (50)$$

It must be kept in mind that here $\partial F_4^\nu/\partial p^\nu$ identifies with $E\,\partial\vec{F}/\partial\vec{p}$ after setting the on-shell condition $p^0=E$. In the cases of interest to us the particles interact with a vector potential, V_μ, and the 4-force is

$$F_4^\mu=(\partial^\mu V^\nu-\partial^\nu V^\mu)p_\nu. \qquad (51)$$

Equation (49), (50) yield a linear relation, $j^\mu=-\pi^{\mu\nu}V_\nu$, with susceptibility given by

$$\pi^{\mu\nu}[f^0]=\int \frac{d^3p}{(2\pi)^3E}\frac{\partial f^0}{\partial p^\alpha}\frac{k^\alpha p^\nu-k\cdot p\,g^{\alpha\nu}}{k\cdot p}p^\mu \qquad (52)$$

or, after integrating by parts,

$$\pi^{\mu\nu}[f^0]=-\int \frac{d^3p}{(2\pi)^3}\frac{f^0}{E}$$

$$\times\frac{(k^\mu p^\nu+p^\mu k^\nu-k\cdot p\,g^{\mu\nu})k\cdot p-k^2p^\mu p^\nu}{(k\cdot p)^2}. \qquad (53)$$

For instance, to obtain the electromagnetic polarization tensor it suffices to make $J_{EM}^\mu=q\,j^\mu$ and $V_\nu=qA_\nu$ for each charged particle (charge q) in the relation $J_{EM}^\mu=-\pi_{EM}^{\mu\nu}A_\nu$. The result is a sum over all particles and spin states of the terms $q^2\pi^{\mu\nu}$. For the electron plasma in particular (2 spin states)

$$\pi_{EM}^{\mu\nu}=2e^2\pi^{\mu\nu}[f_e+f_{\bar{e}}]. \qquad (54)$$

It can be extended to the neutrino-plasma system simply by taking the vector potentials indicated by Eqs. (37), (38) namely, $-eA^\mu+\sqrt{2}G_Fc_V'J_N^\mu$ for electrons and $\sqrt{2}G_Fc_V'J_e^\mu$ for neutrinos. In this way, one obtains the same relations as Eqs. (33), (34) but with classic theory susceptibilities given by

$$\pi_N^{\mu\nu}=\pi^{\mu\nu}[f_\nu+f_{\bar{\nu}}] \qquad (55)$$

and the tensor $\pi_{EM}^{\mu\nu}$ above. The differences between this and the field theory results (12), (13) only appear, vacuum corrections apart, in the particle propagators and parity-violating terms. However, in the limit of frequencies and wavenumbers much lower than the electron and neutrino energies, field theory delivers the same results as classical kinetic theory. The dispersion relation of the longitudinal waves will be analyzed next section.

IV. WAVES AND PLASMA INSTABILITIES

Bingham *et al.* [1,4] conceived a mechanism of energy transfer from a neutrino beam to an electron plasma in which certain plasma wave modes acquire large growth rates as a result of a neutrino resonant effect. In order that those resonant waves be not electron Landau damped the electron plasma has to be in a non-relativistic regime. Let $\omega_{pl}(\vec{k})$ designate the frequency of the longitudinal waves as a function of $\vec{k}$ in a plasma without neutrinos. In the case of a Maxwellian distribution ω_{pl} is given by [18]

$$\omega_{pl}^2(\vec{k})=\omega_p^2+3\frac{\omega_p^2}{k_D^2}\vec{k}^2 \qquad (56)$$

for wave numbers much smaller than k_D, the Debye wave number. ω_p is the plasma frequency and $\omega_p^2=4\pi\alpha n_e/m_e$, $k_D^2=4\pi\alpha n_e/T_e$ for an electron density and temperature equal to n_e and T_e, respectively. More important to what follows is that for $k\ll k_D$ (either degenerate or non-degenerate gas) the frequency ω_{pl} does not vary much with k and $\pi_{EM}^{00}/\vec{k}^2$ is approximately equal to $-\omega_{pl}^2/\omega^2$, as long as ω/k is much larger than $v_{eT}=\sqrt{T_e/m_e}$, the electron

thermal velocity. Using this relation in Eqs. (19), (21), (23) one obtains the dispersion relation in the presence of neutrinos as

$$\omega^2 - \omega_{pl}^2(\vec{k}) = -\frac{G_F^2 c_V'^2}{2\pi\alpha}\frac{\omega^2}{\vec{k}^2}(\vec{k}^2 - \omega^2)^2 \pi_N^{00}, \qquad (57)$$

up to terms of G_F^4 order (recall that $e^2 = 4\pi\alpha$), with

$$\pi_N^{00} = -\int \frac{d^3 p_\nu}{(2\pi)^3}(f_\nu + f_{\bar\nu})E_\nu \frac{\vec{k}^2 - (\vec{k}\cdot\vec{v}_\nu)^2}{(k\cdot p_\nu)^2 - (k^2/2)^2}. \qquad (58)$$

The expressions above put the real impact of the neutrino flux in perspective. Keeping only the main factors,

$$\omega^2 - \omega_{pl}^2 \propto \omega_{pl}^2 \frac{G_F n_\nu}{E_\nu}G_F \vec{k}^2 \qquad (59)$$

clearly indicates that the neutrino contribution is severely suppressed by G_F^2. The only potential exception is a strong neutrino resonance effect. The poles in the neutrino propagators represent kinematic conditions for a massless neutrino with momentum p to emit or absorb a plasmon with momentum k: $(p\pm k)^2 = 0 = p^2$. A necessary condition for such a Čerenkov process is that k be a space-like vector, i.e., $\omega < |\vec{k}|$. That is quite possible for modes that are not Landau damped, $|\vec{k}| < k_D$, because the Debye wavenumber k_D of a non-relativistic plasma is much larger than the plasma frequency ω_p.

The frequency and wave numbers of interest are much smaller than the electron and neutrino single particle energies. Therefore, it makes sense to neglect k^2 in front of $k\cdot p$ in the neutrino propagators so that

$$\pi_N^{00} = -\int \frac{d^3 p_\nu}{(2\pi)^3}\frac{f_\nu + f_{\bar\nu}}{E_\nu}\frac{\vec{k}^2 - (\vec{k}\cdot\vec{v}_\nu)^2}{(\omega - \vec{k}\cdot\vec{v}_\nu)^2}. \qquad (60)$$

This is nothing but the classic theory result contained in the Eqs. (53), (55), which gives also, taking Eq. (52) in account,

$$\pi_N^{00} = \int \frac{d^3 p_\nu}{(2\pi)^3}\frac{1}{\omega - \vec{k}\cdot\vec{v}_\nu}\left(\frac{\partial f_\nu}{\partial\vec{p}} + \frac{\partial f_{\bar\nu}}{\partial\vec{p}}\right)\cdot\vec{k}. \qquad (61)$$

Together with Eq. (57) it constitutes the dispersion relation predicted with classical kinetic theory. Our result differs from others [4,17] simply because the forces assumed there to account for the weak interactions are different from the Lorentz kind of force we derived in Sec. III. Bingham *et al.* in particular [1,4], only considered the terms proportional to the electron and neutrino density gradients and so obtained a factor of $(1 - \omega^2/\vec{k}^2)^2$ less in the dispersion relation [4]. This factor comes from the time derivatives of the currents $\vec{J}_\nu$ and $\vec{J}_e$ in the forces (39) and (41): $\vec{J} = \omega\vec{k}J^0/\vec{k}^2$ for *longitudinal* waves and the same type of relation holds for the polarization tensor, Eq. (3), as a result of plasma isotropy and current

conservation. Altogether, it makes a factor of $1 - \omega^2/\vec{k}^2$, one for neutrinos and one for electrons, as can be learned from Eqs. (33), (34).

The kinetic instability is analogous to the electron Landau damping [18] and takes place if the neutrino spectrum crosses from one side to the other of the resonance ($\vec{k}\cdot\vec{v}_\nu = \omega$). Then, the integral in Eq. (61) separates into a principal part and an imaginary quantity that is evaluated with the so-called Landau prescription [14] in this case, with $\omega - \vec{k}\cdot\vec{v}_\nu + i0^+$ in the denominator. The neutrino contribution to the damping rate comes then as

$$\gamma_W = -\frac{G_F^2 c_V'^2}{4\alpha}\frac{\omega}{\vec{k}^2}(\vec{k}^2 - \omega^2)^2 \int \frac{d^3 p_\nu}{(2\pi)^3}\delta(\omega - \vec{k}\cdot\vec{v}_\nu)$$

$$\times\left(\frac{\partial f_\nu}{\partial\vec{p}} + \frac{\partial f_{\bar\nu}}{\partial\vec{p}}\right)\cdot\vec{k}. \qquad (62)$$

Hardy and Melrose [3] obtained this result starting from the decay rate of single neutrinos into longitudinal photons. However, by the very nature of such calculation the full neutrino contribution to the dispersion relation was not derived, which might be important to investigate possible reactive instabilities. Before going to that we just note that contrary to the interpretation in [3], the factor of $(1 - \omega^2/\vec{k}^2)^2$ is not due to the chiral nature of weak interactions but rather to particle number conservation and plasma isotropy, as discussed above.

γ_W goes as G_F^2 and is exceedingly small when compared with the plasma collisional damping for instance. If however all neutrinos were on the top of the resonance they could generate an hydrodynamic instability. The claim was [1,4] that this occurs in the conditions of supernova neutrino emission causing much larger growth rates, proportional to $G_F^{2/3}$ rather than G_F^2, or to G_F if electron-ion collisions are taken into consideration. The question we raise is, in the end one has to check whether or not the entire neutrino flux lies in the resonance, i.e., whether $|\omega - \vec{k}\cdot\vec{v}_\nu|$ is confined to the calculated resonance width $|\gamma|$. If that is not so one falls in the Landau damping case and result (62).

The hydrodynamic limit is obtained by assuming that $\omega - \vec{k}\cdot\vec{v}_\nu$ is approximately constant over the neutrino spectrum in Eq. (60) and then solving the dispersion relation for a complex ω. A solution with positive imaginary part represents a reactive instability. From Eqs. (57), (60) one writes

$$\omega^2 - \omega_{pl}^2(\vec{k}) \simeq \frac{G_F^2}{2\pi\alpha}\left\langle\frac{n_\nu}{E_\nu}\right\rangle\frac{\omega^2}{\vec{k}^2}(\vec{k}^2 - \omega^2)^2\frac{\vec{k}^2 - (\vec{k}\cdot\vec{v}_\nu)^2}{(\omega - \vec{k}\cdot\vec{v}_\nu)^2}, \qquad (63)$$

where n_ν means the joint ν_e and $\bar\nu_e$ flux and we have made $c_V' = 1$. The largest growth rates ($\text{Im}\{\omega\} = -\gamma > 0$) are obtained for resonant modes ($\vec{k}\cdot\vec{v}_\nu \approx \omega_{pl}$) with magnitudes around

$$\Gamma = \omega_{pl} \left\{ \frac{G_F n_\nu}{E_\nu} \frac{G_F n_e}{m_e} \right\}^{1/3} \frac{\vec{k}^2 - \omega_{pl}^2}{\omega_{pl}^{4/3} \vec{k}^{2/3}}. \qquad (64)$$

But this assumes that $\vec{k} \cdot \vec{v}_\nu$ covers an interval of width $\Delta \vec{k} \cdot \vec{v}_\nu$ not greater than Γ. The neutrino velocities only spread in direction. Far away from the neutrinosphere of radius R_ν they essentially move in the radial direction yet, the velocity cone at a radius r has a finite aperture, $\theta_\nu \approx 2R_\nu/r$. Hence,

$$\Delta \vec{k} \cdot \vec{v}_\nu \approx \theta_\nu k \sin\theta \approx \theta_\nu \omega_{pl} |\tan\theta| \qquad (65)$$

for the resonant modes, where θ is the angle $\vec{k}$ makes with the radial direction. Γ increases with $\tan^{4/3}\theta$ so the best chance of satisfying the requirement $\Gamma > \Delta \vec{k} \cdot \vec{v}_\nu$ is to have an angle θ as close as possible to $\pi/2$. It must however not get any closer than $\pi/2 \pm \theta_\nu$ otherwise $\Delta \vec{k} \cdot \vec{v}_\nu$ becomes larger than ω_{pl} and neutrino Landau damping cannot be avoided. That puts an upper limit on the ratio $\Gamma / \Delta \vec{k} \cdot \vec{v}_\nu$ which implies the necessary condition

$$\hat{\gamma}_1 = \left\{ \frac{G_F n_\nu}{E_\nu} \frac{G_F n_e}{m_e} \right\}^{1/3} \theta_\nu^{-4/3} > 1. \qquad (66)$$

Another upper limit and necessary condition comes from that k must be smaller than the Debye wave number. The necessary condition is

$$\hat{\gamma}_2 = \left\{ \frac{G_F n_\nu}{E_\nu} \frac{G_F n_e}{m_e} \right\}^{1/3} \frac{k_D}{\theta_\nu} > 1. \qquad (67)$$

Knowing how small the energies $G_F n_\nu$ and $G_F n_e$ are those conditions look quite disfavored by data. Take a neutrino luminosity [19] $L_\nu = 10^{53}$ ergs/s, $E_\nu = 10$ MeV, neutrinosphere radius $R_\nu = 30$ km and a generous electron density $n_e = 10^{30}$ cm^{-3}, barely compatible with the non-relativistic regime. Recalling that $\theta_\nu = 2R_\nu/r$ and $n_\nu = L_\nu / 4\pi r^2$ one gets

$$\hat{\gamma}_1 = 1.26 \left(\frac{r}{10^{14} \text{ km}} \right)^{2/3}. \qquad (68)$$

It means that for an electron density as high as 10^{30} cm^{-3} a strong ν resonance effect could only take place at a radius of 10^{14} km or larger. That is clearly absurd. The conclusion we draw is the neutrinos are never collimated enough to cause an hydrodynamic instability. The resonance width is far too small for that. It slices the neutrino velocity cone in two pieces, one above and one below the resonance $\vec{k} \cdot \vec{v}_\nu = \omega$. The neutrinos only induce Landau damping with a rate given by the expression (62). This was calculated by Hardy and Melrose [3] for an electron density of 10^{30} cm^{-3} and they found that the growth rates are too small for the time duration of neutrino emission in Supernovae. When one tries to push the electron density to increase the growth rates one approaches the relativistic regime but then electron Landau damping takes over. In addition, the phase space of plasma waves with phase velocity less than one ($\omega < |\vec{k}|$, necessary

for Čerenkov neutrino emission) becomes vanishing small thus suppressing the relative factor $(1 - \omega^2/\vec{k}^2)^2$ in the dispersion relation. On the other hand, the collisional damping in a non-relativistic plasma is many orders of magnitude larger than the neutrino Landau damping. To conclude, it looks that the neutrinos are not capable of transferring any significant energy to the medium through plasma wave instabilities.

V. CONCLUSIONS AND DISCUSSION

We applied the techniques of finite temperature field theory to study the longitudinal modes of the electromagnetic waves in a plasma crossed by an intense neutrino flux. It is shown that for an isotropic plasma the electron axial-vector couplings and polarization effects are suppressed by G_F^4 in the dispersion relation. Only the electron vector couplings (weak and electromagnetic) contribute at G_F^2 order. In addition, in the limit of frequencies and wavenumbers much smaller than the individual electron and neutrino energies, the susceptibility tensors derived from field theory are well approximated by the results obtained with classic kinetic theory provided that the electroweak interactions are described with the appropriate forces. As a result of the vectorial nature of the interactions, the weak forces are of the same type as the Lorentz electromagnetic force. They diverge however from the forces employed by other authors [1,2,4,17] in particular the ponderomotive force of Bingham et al. [1,4]. That explains the difference between the dispersion relation derived by us and the ones obtained in [4] and [17] using classic kinetic theory.

In the early papers of Bingham et al. [1] the neutrinos were treated with a wave function obeying a sort of Klein-Gordon equation with an external potential accounting for the interaction with the medium. The collective effects were attributed there to a puzzling phase coherence between the neutrino wave functions. There is nothing wrong with that: it simply means that under an external plasma wave of wave number k^μ the neutrino wave function fluctuations share a phase factor, common to all neutrinos ($e^{-ik \cdot x}$), on top of the arbitrary initial phases of each neutrino wave function. The problem was rather that in this Klein-Gordon sort of equation (which could in principle be derived by squaring the Dirac equation) all the terms that might depend on the external potential derivatives or electron 3-vector current were completely discarded. That is fine to study problems like neutrino oscillations but not for neutrino effects on plasma waves because their very nature concerns the variations of the medium densities on the time and length scales of the wave period and wavelength.

The other problem concerns the claimed [1,4] non-linear effects and plasma instabilities induced by resonant neutrinos ($\vec{k} \cdot \vec{v}_\nu = \omega$) on a non-relativistic plasma. We compared the calculated damping (growth) rate of the plasma waves with the phase space occupied by the neutrino flux and concluded that the angular dispersion of the neutrino velocity due to the finite size of the neutrinosphere exceeds by far the assumed resonance width, at least a factor of 10^8 even in the case of

the most optimistic growth rate, no matter how large is the distance to the neutrinosphere. This is of course due to the severe weakness of the neutrino interactions. As a consequence no hydrodynamic (reactive) instabilities can be induced by neutrinos. What is left is just neutrino Landau damping, a linear effect proportional to G_F^2. The corresponding growth rates are however too small for the time duration of neutrino emission in supernovae [3]. In addition they are overcome by either collisional damping in a non-relativistic plasma or by electron Landau damping in a relativistic one.

In conclusion, wave instabilities induced by neutrinos do not seem to be a viable mechanism of substantial energy transfer from neutrinos to a supernova plasma.

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