Lorentz- and *CPT*-violating Chern-Simons term in the functional integral formalism

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We show that in the functional integral formalism, with the weaker condition of gauge invariance alone, one cannot determine the (finite) coefficient of the induced, Lorentz- and *CPT*-violating Chern-Simons term, arising from the Lorentz- and *CPT*-violating fermion sector. [S0556-2821(99)03220-8]

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In recent work [1], the following question has been posed: is a Lorentz- and *CPT*-violating Chern-Simons term induced by the Lorentz- and *CPT*-violating term $\bar{\psi}\not{b}\gamma_5\psi$ (b_μ is a constant four-vector) in the conventional Lagrangian of QED. In the paper by Jackiw and Kostelecký [2], this problem has been discussed. Their results depend on whether one uses a nonperturbative formalism or a perturbative formalism. In a nonperturbative formalism, radiative corrections arising from the axial vector term in the fermion sector induce a definite and nonzero Chern-Simons term, while when a perturbative formalism is used, radiative corrections are finite but undetermined.

The purpose of this paper is to view this undeterminicity[1] of the finite radiative corrections in the functional integral formalism.

Let us consider the following functional integral

$$Z(A)=\int d\bar{\psi}d\psi\ \exp[iI(A)], \tag{1}$$

where

$$I(A)=\int d^4x[i\bar{\psi}\not{D}\psi-m\bar{\psi}\psi-b_\mu j_5^\mu],$$

with $j_5^\mu(x)=\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)-$[(possibly finite) local counterterms]. The gauge field A in the covariant derivative $\not{D}$ is external. By changing the field variables

$$\psi(x)\rightarrow\exp[i\alpha(x)\gamma_5]\psi(x),$$

$$\bar{\psi}(x)\rightarrow\bar{\psi}(x)\exp[i\alpha(x)\gamma_5],$$

we see that the integration measure of Eq. (1) changes by

$$d\bar{\psi}d\psi\rightarrow d\bar{\psi}d\psi\ \exp\left[-i\int\frac{\alpha(x)}{8\pi^2}{}^*F^{\mu\nu}(x)F_{\mu\nu}(x)\right],$$

where this form for the anomaly is obtained when $\bar{\psi}\not{D}\psi$ is defined in a gauge-invariant manner. The action changes into

$$I(A)\rightarrow\int d^4x[i\bar{\psi}\not{D}\psi-(\partial_\mu\alpha)\bar{\psi}\gamma^\mu\gamma_5\psi-m\bar{\psi}e^{2i\alpha(x)\gamma_5}\psi-b_\mu j_5^\mu], \tag{2}$$

where the second term $[(\partial_\mu\alpha)\bar{\psi}\gamma^\mu\gamma_5\psi]$ in the integrand of Eq. (2) comes from transformation of the gauge invariant kinetic term, hence the axial vector current $\bar{\psi}\gamma^\mu\gamma_5\psi$ *is* gauge invariant, and does not necessarily equal to j_5^μ, which need not be gauge invariant. However, we must insist that $\int d^4x j_5^\mu$ is gauge invariant, i.e., the density need not be gauge invariant but its space-time integral—the action—is gauge invariant. Since the Chern-Simons term $\frac{1}{2}\epsilon^{\mu\alpha\beta\gamma}F_{\alpha\beta}A_\gamma={}^*F^{\mu\nu}A_\nu$ behaves precisely in this same way under gauge transformation, we may use it to represent gauge noninvariant portion of j_5^μ. Thus we write

$$j_5^\mu(x)=\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)-c{}^*F^{\mu\nu}A_\nu,$$

where c is an unknown constant. By choosing $\alpha(x)=-x^\mu b_\mu$, the gauge invariant current $\bar{\psi}\gamma^\mu\gamma_5\psi$ cancels and we arrive at

$$Z(A)=\exp\left(-i\int\frac{d^4x}{4\pi^2}b_\mu{}^*F^{\mu\nu}A_\nu\right)\times\exp\left(i\ c\int d^4x b_\mu{}^*F^{\mu\nu}A_\nu\right)\int d\bar{\psi}d\psi\times\exp\left(i\int d^4x[i\bar{\psi}\not{D}\psi-m\bar{\psi}e^{2i\gamma_5x^\mu b_\mu}\psi]\right).$$

Now let us calculate the vacuum polarization for the theory governed by the action in the functional integral. The propagator $G(x,y)$ satisfies the following equation:

$$(i\not{\partial}_x-me^{2i\gamma_5x\cdot b})G(x,y)=i\delta(x-y)$$

and its formal solution can be expressed as

$$G(x,y)=\frac{i}{i\not{\partial}_x-me^{2i\gamma_5x\cdot b}}\delta(x-y),$$

which to first order in b becomes

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[1]See also Refs. [3,4].

$$\frac{i}{i\not\partial_x - m - 2im\gamma_5 x^\alpha b_\alpha}\delta(x-y)$$
$$\approx S(x-y) - i\int S(x-z)2im\gamma_5 z^\alpha b_\alpha S(z-y)d^4z$$
$$\equiv S(x-y) + \Delta G(x,y),$$

where $S(x-y)$ is the free propagator

$$\frac{i}{i\not\partial_x - m}\delta(x-y).$$

This decomposition of the propagator splits the vacuum poraization tensor into three parts:

$$\Pi^{\mu\nu} = \Pi_0^{\mu\nu} + \Pi_b^{\mu\nu} + \Pi_{bb}^{\mu\nu}.$$

The first term $\Pi_0^{\mu\nu}$ is the usual lowest-order vacuum polarization tensor of QED, and the last term $\Pi_{bb}^{\mu\nu}$ is at least quadratic in b. Our main concern here is to calculate the middle term $\Pi_b^{\mu\nu}$, which is linear in b. It is expressed as

$$\Pi_b^{\mu\nu}(x,y) = \mathrm{tr}[\gamma^\mu S(x-y)\gamma^\nu \Delta G(y,x)] + \mathrm{tr}[\gamma^\mu \Delta G(x,y)\gamma^\nu S(y-x)] \equiv \Pi_{b1}^{\mu\nu}(x,y) + \Pi_{b2}^{\mu\nu}(x,y).$$

The Fourier transform of $\Pi_{b1}^{\mu\nu}(x,y)$ is readily obtained as

$$\Pi_{b1}^{\mu\nu}(p,q) = \int d^4x d^4y e^{-ipx}e^{iqy}\Pi_{b1}^{\mu\nu}(x,y) = \int d^4x d^4y d^4z e^{-ipx}e^{iqy}\,\mathrm{tr}[\gamma^\mu S(x-y)\gamma^\nu(-i)S(y-z)2im\gamma_5 z^\alpha b_\alpha S(z-x)]$$
$$= -2imb_\alpha\,\mathrm{tr}\left[\int d^4x d^4y d^4z \frac{d^4l}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}e^{i(-p-l+k)x}e^{i(l-r+q)y}e^{i(r-k)z}\gamma^\mu\frac{1}{\not l - m}\gamma^\nu\frac{1}{\not r - m}\gamma_5 z^\alpha\frac{1}{\not k - m}\right].$$

After carrying out x, y, z, r, and k integrations, we are left with a simple momentum integration:

$$\Pi_{b1}^{\mu\nu}(p,q) = -2imb_\alpha\,\mathrm{tr}\left[\int \frac{d^4l}{(2\pi)^4}\partial_q^\alpha\delta(q-p)\gamma^\mu\frac{1}{\not l - m}\gamma^\nu\frac{1}{\not l + \not q - m}\gamma_5\frac{1}{\not l + \not p - m}\right]$$
$$= 2imb_\alpha\delta(q-p)\mathrm{tr}\left[\int\frac{d^4l}{(2\pi)^4}\gamma^\mu\frac{1}{\not l - m}\gamma^\nu\partial_q^\alpha\frac{1}{\not l + \not q - m}\gamma_5\frac{1}{\not l + \not p - m}\right].$$

Note that in spite of the explicit x dependence in the action, the vacuum polarization is translation invariant [proportional to $\delta(p-q)$ in momentum space].

Using the identity

$$\partial_q^\alpha\frac{1}{\not l + \not q - m} = -\frac{1}{\not l + \not q - m}\gamma^\alpha\frac{1}{\not l + \not q - m},$$

we have

$$\Pi_{b1}^{\mu\nu}(p,q) = -2imb_\alpha\delta(q-p)\mathrm{tr}\left[\int\frac{d^4l}{(2\pi)^4}\gamma^\mu\frac{1}{\not l - m}\gamma^\nu\frac{1}{\not l + \not q - m}\gamma^\alpha\frac{1}{\not l + \not q - m}\gamma_5\frac{1}{\not l + \not p - m}\right],$$

which can be written as

$$\Pi_{b1}^{\mu\nu}(p,q) = 2imb_\alpha\delta(q-p)\mathrm{tr}\left[\int\frac{d^4l}{(2\pi)^4}\gamma^\mu\frac{1}{\not l - m}\gamma^\nu\frac{1}{\not l + \not p - m}\gamma^\alpha\gamma_5\frac{1}{(l+p)^2 - m^2}\right] \equiv \delta(p-q)b_\alpha\tilde{\Pi}_{b1}^{\mu\nu\alpha}(p).$$

If we observe that $\Pi_{b2}^{\mu\nu}(x,y) = \Pi_{b1}^{\nu\mu}(y,x)$, then we readily obtain that the momentum conserving Fourier transform of $\Pi_{b2}^{\mu\nu}(x,y)$ is given by

$$\tilde{\Pi}_{b2}^{\mu\nu\alpha}(p) = \tilde{\Pi}_{b1}^{\nu\mu\alpha}(-p).$$

The calculation of the integral determining $\tilde{\Pi}_{b1}^{\mu\nu\alpha}(p)$ can be found in Ref. [2]:

$$\tilde{\Pi}_{b1}^{\mu\nu\alpha}(p) = \frac{-i\epsilon^{\mu\nu\alpha\beta}p_\beta}{4\pi^2}\frac{\theta}{\sin\theta},$$

where $\theta \equiv 2\sin^{-1}(\sqrt{p^2}/2m)$ and $p^2 < 4m^2$. Therefore we obtain the induced, parity violating, nonlocal action, which agrees with [2],

$$\frac{1}{4\pi^2}\int \frac{d^4p}{(2\pi)^4} b_\mu \, {}^*F^{\mu\nu}(p)\left[\frac{\theta}{\sin\theta} - 1 + 4\pi^2 c\right] A_\nu(-p).$$

Since $(\theta/\sin\theta)_{p^2=0}=1$, we obtain the Chern-Simons term with an undetermined strength c. This completes our argument.

We have shown in the functional integral formalism that the (finite) coefficient of the induced, Lorentz- and CPT-violating Chern-Simons term arising from the Lorentz- and CPT-violating fermion sector is undetermined. Our result is basically a consequence of the weaker condition for gauge invariance: since $\bar{\psi}\gamma_\mu\gamma_5\psi$ does not couple to any other field, physical gauge invariance is maintained provided $\bar{\psi}\gamma_\mu\gamma_5\psi$ is gauge invariant at zero four-momentum [2].

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