# Nonlinear dynamics in three-dimensional QED and nontrivial infrared structure

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In this work we consider a coupled system of Schwinger-Dyson equations for self-energy and vertex functions in $QED_3$. Using the concept of a semiamputated vertex function, we manage to decouple the vertex equation and transform it in the infrared into a nonlinear differential equation of the Emden-Fowler type. Its solution suggests the following picture: in the absence of infrared cutoffs there is only a trivial infrared fixed-point structure in the theory. However, the presence of masses, for either fermions or photons, changes the situation drastically, leading to a mass-dependent nontrivial infrared fixed point. In this picture a dynamical mass for the fermions is found to be generated consistently. The nonlinearity of the equations gives rise to highly nontrivial constraints among the mass and effective (''running'') gauge coupling, which impose lower and upper bounds on the latter for dynamical mass generation to occur. Possible implications of this to the theory of high-temperature superconductivity are briefly discussed. [S0556-2821(99)02524-2]

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## I. INTRODUCTION

Three-dimensional quantum electrodynamics ($QED_3$), with an even number of fermion flavors, apart from serving as a toy model for studying chiral-symmetry breaking and confinement, also constitutes a physically interesting theory *per se*, in view of its possible applications in modeling novel (high-temperature) superconductors [1–4].

Chiral symmetry breaking or, equivalently, dynamical mass generation for fermions, in even-flavor $QED_3$ has still many unresolved issues. One of those is the existence of a (dimensionless) critical coupling, above which dynamical mass generation for the fermions occurs [5]. In the context of large-$N$ treatment, which at present constitutes the only well-studied approach, the role of the dimensionless coupling[1] is played by the inverse of the fermion flavor number $N$. The issue of the existence of a critical coupling in $QED_3$ is a delicate one [6]. Many of the original approximations [5] leading to its existence have been questioned, in particular, the fact that wave-function renormalization effects have not been properly accounted for. Recently, however, the incorporation of such effects, still within the large-$N$ context, appears to corroborate [7–9] the qualitative picture advocated in [5]. In addition, the latter is also supported by lattice simulations [10].

Nonetheless, the situation is far from being conclusive. The fact that the critical $N$, below which dynamical mass generation occurs, is found to be of order 3 (in a four-component notation for the fermions) provides motivation for searches beyond the large-$N$ treatment. Moreover, up to now the gauge coupling in the infrared has been treated as an arbitrary parameter, whose size has not been restricted by an additional dynamical constraint. At present we are lacking a self-consistent treatment of the dynamical Schwinger-Dyson (SD) equations involving the vertex function on an equal footing with the self-energy and gap functions. In all the approaches so far, at least within the large-$N$ treatment that we are aware of, one invokes a specific ansatz for the vertex, by the sole requirement of satisfying some truncated form of the Ward-Takahashi identity stemming from gauge invariance [8–11]. The lack of dynamical information for the coupling poses problems; for instance, its size in the infrared is treated as an arbitrary parameter, being assumed to merely exceed a critical value, if one wishes to trigger chiral symmetry breaking.

Another point, related to the above, which is already familiar from studies in the case of four-dimensional non-Abelian gauge theories, is whether chiral-symmetry breaking is associated with confinement of charges [12]. This issue acquires physical importance in view of the condensed-matter applications. In particular, it may shed more light in the dynamics of spin-charge separation, by analogy with the physics of strong interactions [13].

In this work we shall not deal with issues of confinement, which exists in $QED_3$ despite its Abelian nature. Instead, we shall attempt a novel approach to chiral symmetry breaking, independently of a large-$N$ treatment, by studying the *coupled* fermion and photon self-energies and vertex SD equations in the context of a method first introduced for the case of four-dimensional QCD [14]. The novel ingredient is that we concentrate on the semi-amputated vertex, defined in Sec. II, which is the correct gauge-invariant quantity to determine a physically meaningful ''running'' coupling (''effective charge''). The fact that $QED_3$ is superrenormalizable

[1]In three dimensions the coupling $e^2$ has dimensions of mass. One can still define, however, dimensionless couplings by dividing with a dynamically generated scale, which in the large-$N$ treatment arises by demanding that [5] $e^2N=8\alpha$, with the scale $\alpha$ kept fixed as $N\to\infty$. The dimensionless coupling is then defined as the ratio $e^2/8\alpha=1/N$.

 

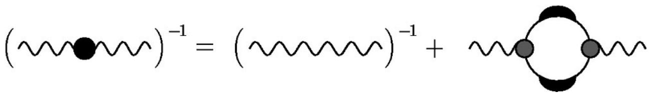

FIG. 1. Schematic form of the SD equation for the gauge field propagator in resummed perturbation theory. The blobs on the right-hand-side indicate the full (non-perturbative) vertex corrections.

FIG. 2. Schematic form of the SD equation for the full fermion propagator. Blobs indicate full non-perturbative quantities. Notice the full vertex appears on both ends of the internal photon line.

in the ultraviolet does not preclude the possibility of defining such a quantity, having nontrivial structure in the infrared. As we show in Secs. II and III, the existence of a nontrivial infrared fixed point is a property *only* of the infrared-regularized theory, as conjectured in [11]. In other words, in the absence of fermion (or photon) masses, the infrared singularities of the vertex equation will force the effective charge to vanish at zero momentum transfer, thereby excluding the possibility of an interesting infrared behavior. From a condensed-matter application point of view this would correspond to what is usually called the Landau-Fermi liquid theory (trivial infrared fixed point) [15,11]. In the presence of masses, and in particular fermion masses, we show in Sec. III that there is a non-trivial infrared fixed point structure, stemming from the fact that the effective charge obtained as a self-consistent solution of the *non-linear* vertex SD equation, is driven to a finite positive value, which can be large enough to trigger dynamical generation of a fermion mass. This implies that the phenomenon of chiral symmetry breaking is intimately associated with deviations from the trivial infrared fixed-point structure.

We should stress that, as a result of the non-linearity of the vertex equation, there are delicate constraints between the fermion mass and the effective charge, which are responsible for the appearance of regions of the latter for which dynamical generations occurs. At present, these restrictions appear as a consequence of mathematical self-consistency of the truncated equations. It is not clear to us, whether the upper bounds on the effective charge, imposed by the present cubic approximations for the vertex corrections, will survive the inclusion of higher orders. In contrast, we believe that the lower bounds will survive such a treatment, thereby indicating the existence of a critical coupling above which dynamical mass generation will occur. This is physically appealing, given that one would not expect a weakly coupled theory to be capable of breaking dynamically chiral symmetry.

The layout of this article is as follows: in Sec. II we set up the SD equations that we wish to study, and discuss in detail the approximations employed. We demonstrate that, under certain assumptions to be justified retrospectively in Sec. IV, the equation for the semi-amputated vertex decouples from the rest, and hence can be solved separately. Moreover, we establish the absence of a non-trivial infrared fixed point rigorously (within the cubic approximation for the vertex), by casting the SD equation for the effective charge in a form of a non-linear differential equation, known as Emden-Fowler equation [16,17]. In Sec. III we study the equation for the vertex in the presence of a fermion mass, acting as an infrared regulator. We derive the appropriate non-linear differential equation describing the infrared behavior of the running coupling, and solve it to demonstrate the existence of a non-trivial infrared fixed point. The so-derived running charge is a monotonically decreasing function of the momentum, tending asymptotically to a constant positive value in the ultraviolet. In Sec. IV, we solve the equations for the photon and fermion self-energies, which in our approximations decouple from each other and depend only on the semi-amputated vertex function. We then verify the self-consistency of the approach. In Sec. V we examine the self-consistency of the dynamical generation of the fermion mass, by solving the appropriate SD equation upon substituting the solution for the semi-amputated vertex found in previous sections. The self-consistency of the approach restricts the allowed regions of the effective charge, implying the existence of a lower bound (critical coupling) but also of an upper one. In Sec. VI, we examine an alternative type of infrared cutoff, namely that of a (bare) covariant photon mass term. This case also exhibits a non-trivial infrared fixed-point structure but, in contrast to the monotonic decrease of the effective charge in the case of fermion masses, here the coupling initially increases in the infrared, then displays a local maximum, and eventually decreases, tending asymptotically to a constant value in the ultraviolet. Some possible applications of this behavior, inspired by condensed-matter physics, are briefly discussed. Finally, in Sec. VII we present our conclusions and outlook.

## II. THE SD EQUATION FOR THE SEMIAMPUTATED VERTEX

In this section we will first set up the SD equations for the photon and electron self-energies, and the photon-electron vertex; then we will define the semi-amputated vertex and derive its corresponding SD equation. As we will explain, the latter governs the behavior of the effective coupling in the infra-red. The derivation of the SD equations for the photon propagator $\Delta_{\mu\nu}$, the electron propagator $S_F$, and the photon-electron vertex $\Gamma_\mu$ proceeds following standard methods [18,19] (see Figs. 1,2,3).

The full photon propagator $\Delta_{\mu\nu}$, its inverse $\Delta^{-1}_{\mu\nu}$, and the full vacuum polarization $\Pi_{\mu\nu}$ in Euclidean space are related by

$$\begin{aligned}
\Delta^{-1}_{\mu\nu}(q) &= \Delta^{-1}_{0\mu\nu}(q) + \Pi_{\mu\nu}(q), \\
\Delta^{-1}_{0\mu\nu}(q) &= q^2\delta_{\mu\nu} - \left(1-\frac{1}{\xi}\right)q_\mu q_\nu, \\
\Pi_{\mu\nu}(q) &= (q^2\delta_{\mu\nu} - q_\mu q_\nu)\Pi(q), \\
\Delta_{\mu\nu}(q) &= (\delta_{\mu\nu} - q_\mu q_\nu/q^2)[q^2 + q^2\ \ \Pi(q)]^{-1} \\
&\quad + \xi q_\mu q_\nu/q^4, \qquad (2.1)
\end{aligned}$$

where $\xi$ is the gauge fixing parameter (in covariant gauges). The corresponding SD equation reads

$$\Delta_{\mu\nu}^{-1}(q)=\Delta_{0\mu\nu}^{-1}(q)+e^2\int\frac{d^3k}{(2\pi)^3}\mathrm{Tr}[\Gamma_\mu S_F\Gamma_\nu S_F]+\cdots. \tag{2.2}$$

The SD equation for the electron propagator $S_F$ is given by

$$S_F^{-1}(p)=-i\not{p}-e^2\int\frac{d^3k}{(2\pi)^3}\Gamma_\mu S_F\Gamma_\nu\Delta^{\mu\nu}+\cdots. \tag{2.3}$$

Finally, the SD equation for the photon-electron vertex $\Gamma_\mu$ has the form

$$\Gamma_\mu(p_1,p_2,p_3)=\gamma_\mu-e^2\int\frac{d^3k}{(2\pi)^3}\Gamma_\alpha S_F(k-p_1)\times\Gamma_\mu S_F(k)\Gamma_\beta\Delta^{\alpha\beta}(k+p_2)+\cdots \tag{2.4}$$

with $p_1+p_2+p_3=0$. The ellipses on the right-hand side of Eqs. (2.2)–(2.4), denote the infinite set of terms containing the two-particle irreducible four-point function [18,19]. Although we are not working in the context of a large $N$ analysis, we note that the above truncation is compatible with working to leading order in resummed $1/N$ expansion.

We next define the scalar quantities $A$, $B$, and $\mathcal{G}$ as follows:

$$S_F(k)=\frac{1}{A(k)\not{k}}, \tag{2.5}$$

$$\Delta_{\mu\nu}(k)\equiv\frac{\delta_{\mu\nu}}{B(k)k^2}, \tag{2.6}$$

and

$$\Gamma_\mu(p_1,p_2,p_3)=\mathcal{G}(p_1,p_2,p_3)\gamma_\mu. \tag{2.7}$$

The quantity $B$ is related to $\Pi$ defined in Eq. (2.1) by $B(q)=1+\Pi(q)$. The definition in Eq. (2.6) implies that the longitudinal pieces of the photon propagator will be discarded in what follows. Of course, there are no rigorous field-theoretic arguments justifying their omission or inclusion. The correct treatment of such terms necessitates a formalism which would allow for the self-consistent truncation of the SD series in a manifestly gauge-invariant way; unfortunately, no such formalism exists to date. The standard lore when writing down SD equations is to use in Eq. (2.3) the form of $\Delta_{\mu\nu}$ given in Eq. (2.1), setting $\xi=0$ (Landau gauge). While in four-dimensional quantum electrodynamics ($QED_4$) this choice renders the vertex corrections unimportant in the ultraviolet, it appears to be less compelling in the context of the superrenormalizable $QED_3$. In addition, it is known that, while the conventionally defined fermion self-energy and photon-fermion vertex depend explicitly on the gauge-fixing parameter $\xi$, it is possible to construct —at least at one-loop — a $\xi$-independent fermion self-energy and vertex, by resorting to the diagrammatic rearrangement of the $S$-matrix known as the pinch technique [20]. It turns out that the $\xi$-independent fermion self-energy and vertex so constructed *coincide* with their conventional counterparts, if we choose for the latter the special value $\xi=1$ (Feynman-'t Hooft gauge) [21]. Furthermore, as has been formally shown in [22], *all* longitudinal pieces appearing in Eq. (2.1) vanish from *physical* observables, such as $S$-matrix elements, to all orders in perturbation theory. Thus, one is led to a generalized form of the Feynman-'t Hooft gauge, known as the ''stagnant gauge,'' where only the $\delta_{\mu\nu}$ part of the vacuum polarization contributes, to *all* orders in perturbation theory. This gauge will be adopted throughout the present article.

Following [14] and [19] we define the semi-amputated vertex $\hat{G}$ as

$$\hat{G}(p_1,p_2,p_3)\equiv Z(p_1,p_2,p_3)\mathcal{G}(p_1,p_2,p_3) \tag{2.8}$$

with

$$Z(p_1,p_2,p_3)=B^{-1/2}(p_1)A^{-1/2}(p_2)A^{-1/2}(p_3). \tag{2.9}$$

This definition proves very useful in reducing the complexity of the set of equations (2.3) and (2.4), under certain approximations to be discussed in detail in the next sections. In addition, the quantity

$$g_R(p_1,p_2,p_3)\equiv e\hat{G}(p_1,p_2,p_3) \tag{2.10}$$

provides a natural generalization of the concept of the running or effective charge in the context of superrenormalizable gauge theories [23], such as $QED_3$. This running of the coupling should be understood as a Wilsonian rather than Gell-Mann-Low type, in the sense that it is not associated with ultraviolet infinities; instead, it expresses a non-trivial infrared structure of the theory [11].

Note that in $QED_4$ the effective charge $e_{\rm eff}^2$ is defined in terms of the photon vacuum polarization as

$$e_{\rm eff}^2(q^2)=e^2[1+\Pi(q^2)]^{-1}, \tag{2.11}$$

and is a gauge-, scale-, and scheme-independent quantity [24] to all orders in perturbation theory. $e_{\rm eff}^2$ depends explicitly on $q^2$ *and* the masses of the fermions inside the vacuum polarization loop. In the limit where the fermion masses can be neglected, $e_{\rm eff}^2(q^2)$ coincides with the running coupling obtained by the $\beta$ function of $QED_4$, i.e. the solution of the usual renormalization-group differential equation. An advan-

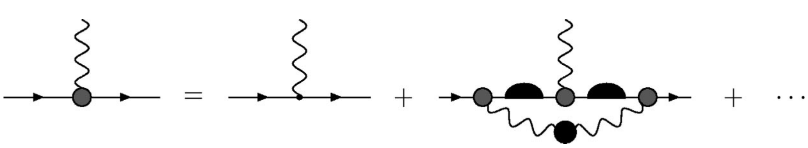


FIG. 3. The SD equation for the vertex $\Gamma_\mu$.

tage of the definition given in Eq. (2.10) is that it captures the running coupling even in the case of scalar theories [19], where, due to the absence of Ward-Takahashi identities, the role of the gauge boson self-energies is not as prominent as in gauge theories. As we shall see in Sec. V, the interpretation of $g_R$ defined in Eq. (2.10) as a running coupling is also justified by the form of the SD for the fermion mass gap.

The equation for the semi-amputated vertex $\hat{G}$ may be obtained from Eq. (2.4) by multiplying both sides by the factor $Z(p_1,p_2,p_3)$, i.e.,

$$\hat{G}(p_1,p_2,p_3)\gamma_\mu = Z(p_1,p_2,p_3)\gamma_\mu - e^2\int \frac{d^3k}{(2\pi)^3}\hat{G}^3 \times \gamma^\alpha \frac{1}{\not{k}-\not{p}_1}\gamma_\mu \frac{1}{\not{k}}\gamma_\alpha \frac{1}{(k+p_2)^2}, \tag{2.12}$$

where

$$\hat{G}^3 \equiv \hat{G}(p_3,k+p_2,p_1-k)\hat{G}(p_1,-k,k-p_1) \times \hat{G}(p_2,k,-k-p_2). \tag{2.13}$$

In what follows we shall restrict ourselves to the case where the photon momentum is vanishingly small, and thus one is left with a single momentum scale $p$. One can then define a renormalization-group $\beta$ function from this ''running'' coupling $G(p)$ by setting

$$\beta \equiv p\frac{d}{dp}\hat{G}(p). \tag{2.14}$$

In order to further simplify the SD equation for $G(p)$ we make the additional approximation that $\hat{G}^3 = \hat{G}^3(k)$, i.e., a cubic power of a single $\hat{G}(k)$ depending only on the integration variable $k$. This approximation will be justified by the self-consistency of the solutions.

Carrying out the gamma-matrix algebra using the formulas $\gamma_\mu\gamma_\mu = -d$, and $\gamma_\mu\gamma_\rho\gamma_\mu = (d-2)\gamma_\rho$ valid for $4\times 4$ gamma matrices in $d(=3)$-dimensional Euclidean space, one obtains in a straightforward manner:

$$\hat{G}(p) = Z(p) + \frac{1}{3}e^2\int \frac{d^3k}{(2\pi)^3}\hat{G}^3(k)\frac{1}{k^2}\frac{1}{(k-p)^2}. \tag{2.15}$$

Several remarks are now in order. First, one observes that $Z(p)\to 1$ for $p\to\infty$, where perturbation theory is valid. This is inferred from the fact that in such a case, as can be readily verified, the functions $A(p), B(p)\to 1+\mathcal{O}(e^2/p)$. In addition, in the ultraviolet region, $p\to\infty$, gauge invariance requires $\mathcal{G}(p)\sim A(p)$. Second, from Eq. (2.15) one observes that, if $\hat{G}$ stays positive, which is expected for any physical theory, then, as a result of the positivity of the integrand, $\hat{G}(p)\geqslant Z(p)$ for any $p$. Thus, one has the following basic properties of $\hat{G}(p)$, which stem directly from the integral equation (2.15):

$$\hat{G}(p)\geqslant Z(p), \quad \text{for all } p, \quad \hat{G}(p)\sim B^{-1/2}(p)\to 1, \quad p\to\infty. \tag{2.16}$$

Notice that $\hat{G}(p)$ in the ultraviolet is thereby given by $[1+\Pi(p)]^{-1/2}$ as in $QED_4$. This is the perturbative result, which, as we shall see later, is modified non-trivially in the infrared, in a way consistent with gauge invariance. Moreover, as we shall see later, the self-consistency of the approximations employed will require $\hat{G}(0)\geqslant\sqrt{3/2}$.

Our next assumption is that in the infrared regime, $k/\alpha \ll 1$, which is of interest to us here, the effects of the inhomogeneous term $Z(p)$ can be ignored. This assumption will be justified later on, when we consider a non-trivial self-consistency check of the solutions. We now remark that by ignoring the effects of $Z(p)$ one can decouple the equation for the (gauge-invariant) *amputated* vertex from the equations for $A(p), B(p)$. As we shall discuss in subsequent sections, these latter equations also decouple from each other, depending only on the vertex function $\hat{G}(p)$.

Because of this, we commence our analysis from the SD equation for the vertex $\hat{G}(p)$, which we solve upon ignoring the effects of the inhomogeneous $Z(p)$ term. Thus we arrive at the homogeneous equation

$$\hat{G}(p) = \frac{1}{3}e^2\int \frac{d^3k}{(2\pi)^3}\hat{G}^3(k)\frac{1}{k^2}\frac{1}{(k-p)^2}. \tag{2.17}$$

This integral equation involves only one unknown function, namely $\hat{G}$, which must be self-consistently determined. Note that this equation is invariant under the rescaling $\hat{G}\to\hat{G}/e$. This indicates a straightforward extension of the analysis to a large-$N$ treatment, given that $N$ can be absorbed in a redefinition of $e^2$.

It is easy to see that, written in the form (2.17), the equation does not admit *physically acceptable* solutions, i.e., solutions with $\hat{G}\geqslant 0$ and *finite*.[2] Indeed, setting $p=0$ one obtains after the (trivial) angular integration:[3]

$$\hat{G}(0) = \frac{e^2}{12\pi^2}\int_0^\infty \frac{dk}{k^2}\hat{G}^3(k). \tag{2.18}$$

Finiteness of $\hat{G}(0)$ requires that the integrand of the right-hand side of Eq. (2.18) converges at $y\to 0$ and $\infty$, where $y\equiv k/\alpha$. The ultraviolet limit does not present a problem, because the kernel vanishes like $y^{-2}$, which is consistent with the superrenormalizability of the theory as well as the fact that the amputated vertex tends to 1. In the infrared limit

[2]Solutions that blow up in any point of the integration region are discarded.

[3]These arguments remain unaffected even in the presence of an inhomogeneous term $Z(p)$, such that $Z(0)$ is non-negative and finite. The non-negative nature of $Z(p)$ Eq. (2.9) stems from that of $A(p)$, which is guaranteed from general renormalization-group arguments [25].

$y \to 0$, however, the kernel blows up. For the integral to remain finite at that point, as required by the finiteness assumption for $\hat{G}(0)$, $G^3(y)$ must approach zero as $y^\alpha, \alpha > 1/3$, thereby implying that $\hat{G}(0)=0$. However, for that to happen the integrand in Eq. (2.18) must change sign, which would in turn imply that $\hat{G}(y)$ itself must change sign somewhere in $y$. According to our assumption above this is not a physically acceptable situation.

To show rigorously that there are no *physically* acceptable solutions leading to a non-trivial infrared structure we next convert the integral equation into a non-linear differential equation of the type known in the mathematical literature as the *Emden-Fowler* equation [16,17]. To this end, we perform the angular integration in Eq. (2.17), to arrive at the equation:

$$\hat{G}(p) = \frac{2}{3\pi^2}\frac{\alpha}{p}\int_0^\infty \frac{dk}{k}\hat{G}(k)^3 \ln\left|\frac{k+p}{k-p}\right|, \tag{2.19}$$

where we have set $e^2 \equiv 8\alpha$ to make contact with the usual large-$N$ definition [5]. For us, however, the number of fermion flavors is not assumed to be necessarily large. In fact, for brevity we set $N=1$ (in a four-component notation for the fermions) throughout this work. Next we introduce the dimensionless variables $x \equiv p/\alpha$ and $y \equiv k/\alpha$. Since we are interested in the infrared behavior of the model we consider the limit $x \ll 1$, for which one obtains by expanding the logarithms in the integrand:

$$\begin{aligned}\hat{G}(x) &= \frac{2}{3\pi^2 x}\int_0^\infty \frac{dy}{y}\hat{G}^3(y)\ln\left|\frac{y+x}{y-x}\right| \\ &\simeq \frac{4}{3\pi^2 x^2}\int_0^x dy\,\hat{G}(y)^3 + \frac{4}{3\pi^2}\int_x^\infty \frac{dy}{y^2}\hat{G}^3(y).\end{aligned} \tag{2.20}$$

Differentiating appropriately with respect to $x$, we arrive at the following differential equation for small $x$:

$$x^3\frac{d^2\hat{G}}{dx^2} + 3x^2\frac{d\hat{G}}{dx} + \frac{8}{3\pi^2}\hat{G}^3(x) = 0, \quad x \ll 1. \tag{2.21}$$

It is convenient to rescale $\hat{G}$ by setting

$$G \equiv \sqrt{\frac{8}{3\pi^2}}\hat{G}. \tag{2.22}$$

Then, Eq. (2.21) becomes

$$x^3\frac{d^2 G}{dx^2} + 3x^2\frac{dG}{dx} + G^3(x) = 0, \quad x \ll 1. \tag{2.23}$$

Upon the change of variables $\xi = 1/2x^2, G = 2^{3/4}\eta(\xi)$, the equation becomes of the Emden-Fowler type [16,17]:

$$\frac{d^2}{d\xi^2}\eta(\xi) + \xi^{-3/2}\eta^3(\xi) = 0, \quad \xi \to +\infty. \tag{2.24}$$

As discussed in the mathematical literature [16], the *only non-trivial real solutions* of Eq. (2.24), as $\xi \to +\infty$, are *oscillatory* about zero of simple sinusoidal form, which, however, oscillate infinitely rapidly as $x \to 0$. The amplitude of the above solutions behaves like $x^{-1/2}$, for $x \ll 1$. We interpret this behavior as indicating an *instability* of the massless-fermion ground state.

It is interesting to notice that the only power-law solution of Eq. (2.23) for $x \ll 1$, is *purely imaginary*, i.e.,

$$G(x) = i\frac{\sqrt{5}}{2}x^{1/2}, \quad x \ll 1. \tag{2.25}$$

This solution would imply a ''trivial infrared fixed point structure'' given that its associated $\beta$ function vanishes at $x=0$. However, the fact that Eq. (2.25) is *purely imaginary* would again suggest instability.

The above analysis constitutes a rigorous proof that, within the context of *proper* (i.e., finite and with finite-derivatives) solutions, and modulo the approximations discussed, no non-trivial infrared-fixed point is possible in $QED_3$ in the absence of an infrared cutoff. This was conjectured in Ref. [11], but here we have given an analytic proof. This motivates one to look for the existence of a possible non-trivial infrared fixed-point structure in the presence of fermion and/or photon masses. In the next section we shall discuss the case when the fermions develop a mass $m$. As we shall show, the existence of a non-trivial infrared fixed point is guaranteed due to the form of the resulting equations.

## III. EQUATION FOR THE VERTEX IN THE CASE OF NON-ZERO FERMION MASS

As a first kind of infrared cutoff in the integral equation (2.17) we shall consider the case of a fermion mass gap $m(p) = \Sigma(p)/A(p)$, where $\Sigma(p)$ is the fermion self-energy. In that case the fermion propagator $S_F$ becomes

$$S_F(k) = \frac{i}{A(k)[\not{k} + m_f(k)]}. \tag{3.1}$$

For our purposes below we assume that $m_f(p) \simeq m_f(0) \equiv m_f \neq 0$. In that case the integral equation (2.17) becomes

$$\begin{aligned}\hat{G}(p) &= \frac{1}{3}e^2\int \frac{d^3k}{(2\pi)^3}\hat{G}^3(k)\frac{1}{(k^2+m_f^2)(k-p)^2} \\ &\quad + \frac{2m_f^2}{3}\int \frac{d^3k}{(2\pi)^3}\hat{G}^3(k)\frac{1}{(k^2+m_f^2)^2(k-p)^2}.\end{aligned} \tag{3.2}$$

Notice that the effects of the fermion mass are not given simply by just adding a mass squared term in the fermion denominators, but they result in additional structures in the integral equation.

Performing the angular integrations one arrives at

$$\hat{G}(x)=\frac{2}{3\pi^2 x}\int dy f(y)\ln\left|\frac{y+x}{y-x}\right|\hat{G}^3(y), \tag{3.3}$$

where $x\equiv p/\alpha, m\equiv m_f/\alpha$ are dimensionless, and

$$f(y)\equiv y\frac{y^2+3m^2}{(y^2+m^2)^2}\geqslant 0. \tag{3.4}$$

Differentiation with respect to $x$ yields

$$x\frac{d}{dx}\hat{G}(x)=-\frac{2}{3\pi^2 x}\int_0^\infty dy f(y)\left(\ln\left|\frac{y+x}{y-x}\right|+\frac{2xy}{x^2-y^2}\right)\hat{G}^3(y). \tag{3.5}$$

One observes that formally as $x\to 0$ the right-hand-side vanishes, provided that $\hat{G}$ is finite. This indicates the existence of a fixed point. As we shall show below this is confirmed analytically by converting the integral equation into a nonlinear differential equation.

An additional feature which one would have hoped to study already at the level of the integral equation (3.5) is the monotonicity of $\hat{G}$. Unfortunately the kernel in Eq. (3.5) is not manifestly positive to allow for such an analytic proof at the level of the integral equation for generic values of $x$, and one has to resort to numerical treatments, which fall beyond the scope of this article. However, one can already infer from Eq. (3.5) that, for high momenta $x\gg 1$, a monotonically decreasing $\hat{G}$ is consistent with the expectation that in this regime $\hat{G}$ is essentially given by its perturbative expression which asymptotes to 1. The analysis is omitted because it is straightforward.

For low momenta, on the other hand, the behavior of $\hat{G}(x)$ will also be shown to be monotonically decreasing, starting from a nontrivial fixed point. This will be achieved by converting the integral equation into a differential one. Unfortunately, at present, we cannot analytically derive the monotonicity for intermediate momenta.

To derive the differential equation from Eq. (3.3) we follow a similar analysis to the one leading to Eq. (2.20). First, one expands the logarithms for small $x\ll 1$, thereby writing the equation as

$$\hat{G}(x)\simeq\frac{4}{3\pi^2x^2}\int_0^x dy\hat{G}(y)^3\frac{y^2}{y^2+m^2}+\frac{4}{3\pi^2}\int_x^\infty\frac{dy}{y^2+m^2}\hat{G}^3(y)$$
$$+\frac{8m^2}{3\pi^2x^2}\int_0^x dy\hat{G}(y)^3\frac{y^2}{(y^2+m^2)^2}$$
$$+\frac{8m^2}{3\pi^2}\int_x^\infty\frac{dy}{(y^2+m^2)^2}\hat{G}^3(y). \tag{3.6}$$

Differentiating with respect to $x$ one arrives at

$$x(x^2+m^2)^2\frac{d^2}{dx^2}\hat{G}(x)+3(x^2+m^2)^2\frac{d}{dx}\hat{G}(x)$$
$$+\frac{8}{3\pi^2}(x^2+3m^2)\hat{G}^3(x)=0. \tag{3.7}$$

The above equation can be solved numerically, to which we return later on. However, in the infrared region $x\ll m$ the equation accepts an analytic treatment, as we discuss below. In this region Eq. (3.7) is approximated by

$$x\frac{d^2}{dx^2}\hat{G}(x)+3\frac{d}{dx}\hat{G}(x)+\frac{8}{\pi^2m^2}\hat{G}^3(x)=0. \tag{3.8}$$

It is immediate to see that a special power-law solution is given by [for positive $G(x)$] by

$$\hat{G}(x)=m\pi\frac{\sqrt{3}}{4\sqrt{2}}x^{-1/2}. \tag{3.9}$$

Notice the infrared divergence of this type of solution *even* in the presence of a (bare) fermion mass. The associated renormalization-group $\beta$ function (2.14) for this case reads

$$\beta(x)=-\frac{1}{2}\hat{G}\sim x^{-1/2}\to+\infty,\ \text{as}\ x\to 0, \tag{3.10}$$

indicating the absence of an infrared fixed point. The associated operator appears to be *relevant* (negative scaling dimension), which implies the possibility that the theory be driven to a non-trivial fixed point.

However, in the infrared regime $x\ll 1$, one can find a different type of solution [16]:

$$\hat{G}=m\pi\frac{\sqrt{3}}{2\sqrt{2}}\frac{c}{1+c^2x},\quad x\to 0, \tag{3.11}$$

where $c$ is a constant of integration to be fixed by the boundary condition at $x=0$ implied by the integral equation, to be discussed later on. For physical solutions $c$ is assumed positive.

This type of solution has a renormalization-group $\beta$-function (2.14) of the form

$$\beta=-\hat{G}(x)+\frac{2\sqrt{2}}{\sqrt{3}\pi mc}\hat{G}^2(x)\sim-\frac{x}{(1+c^2x)^2}\to 0,\quad x\to 0 \tag{3.12}$$

from which we observe the existence of a non-trivial (nonperturbative) infrared fixed point at $\hat{G}^*=\pi m\sqrt{3}c/2\sqrt{2}>0$. Such a fixed point is the result of the dynamical generation of a parity-invariant, chiral-symmetry breaking fermion mass [5], indicating the connection of the phenomenon of chiral symmetry breaking in $QED_3$ to a non-trivial infrared fixed point structure, in agreement with the expectations of Ref. [11].

The non-trivial fixed-point solution (3.11) *is not* compatible with the integral equation (3.2) for *any* value of the fermion mass $m$. Indeed one can derive a *boundary condition* for $\hat{G}(0)$ from Eq. (3.2), which reads

$$\hat{G}(0)=\frac{4}{3\pi^2}\int_0^\infty dy\,\frac{1}{y^2+m^2}\hat{G}^3(y)+\frac{8m^2}{3\pi^2}\int_0^\infty dy\,\frac{1}{(y^2+m^2)^2}\hat{G}^3(y). \qquad (3.13)$$

In contrast to the massless case (2.18), $\hat{G}(0)$ is now a finite constant, $\pi mc\sqrt{3/8}$, as seen from Eq. (3.11), and this allows for a compatibility of the solution (3.11) with Eq. (3.13), *provided* that $m\hat{G}(0)$ satisfies certain conditions to be specified below.

To this end, we split the $y$-integration in Eq. (3.13) into two intervals: (i) $y\in[0,m)$, where the form (3.11) is valid to a good approximation, and (ii) $y\in[m,\infty)$, where we approximate $\hat{G}(x)$ by its perturbative asymptotic value $\hat{G}\simeq 1$. By this latter approximation we overestimate the actual value of the integration, given that $\hat{G}$ is actually slightly smaller than unity for finite $p$, approaching it only asymptotically [cf. Eq. (2.16)]. However, this is sufficient for our qualitative purposes of demonstrating the existence of constraints on the fermion mass implied by the boundary condition (3.13).

With these in mind, the boundary condition (3.13) reduces to

$$1=\frac{mc^2}{2}\int_0^1 dy\,\frac{1}{y^2+1}\,\frac{1}{(1+mc^2y)^3}\left(1+\frac{2}{y^2+1}\right)+\frac{4}{3\pi^2m^2c}\left(1-\frac{1}{\pi}\right)\sqrt{\frac{2}{3}}. \qquad (3.14)$$

To obtain the condition imposed on $m$ by the boundary condition (3.13) it suffices to observe that the first term on the right-hand-side is a function of $mc^2$ alone, and that, after the (elementary) $y$ integration, the resulting function of $mc^2$ asymptotes rapidly to the value 3/4 (see Fig. 4).

This implies the following inequality:

$$m\hat{G}(0)<\frac{8}{3\pi}\left(1-\frac{1}{\pi}\right). \qquad (3.15)$$

As already mentioned this bound is overestimated, given that in the actual situation the function $\hat{G}(y)$ is not exactly 1 immediately after the region $y\geqslant m$.

We next remark that $m$ should actually be determined self-consistently from a solution of the pertinent gap equation. This will be done in Sec. V. However, at the moment, and for completeness, we shall assume that $m$ is determined by its approximate form derived within the context of a large-$N$ treatment [5,1]. Compatibility of the dynamical solution with the constraint (3.15) will then lead to further restrictions on the range of the allowed masses $m$. As we shall see in Sec. V, there is good agreement between the allowed fermion-mass ranges, obtained within the context of a large-$N$ treatment, and those obtained from a self-consistent solution of the mass gap equation within our approach.

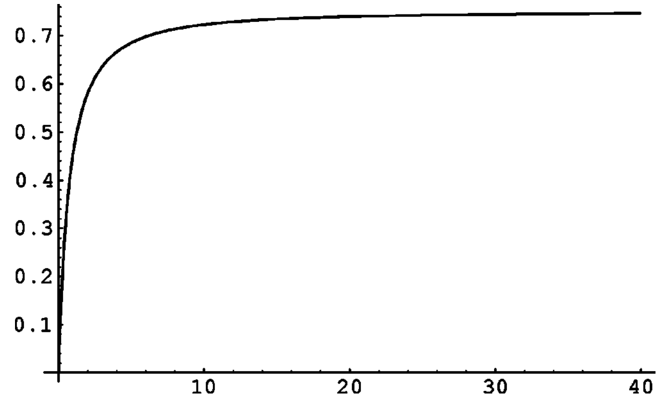


FIG. 4. Plot of the function $f(z)=z/2\int_0^1 dy\,1/y^2+1[1/(1+zy)^3](1+2/y^2+1)$, where $z=mc^2$. The function asymptotes rapidly to 3/4.

In the context of a large $N$ treatment, and to leading order in $1/N$ resummation, the following solution for the dynamically generated $m$ is found [5,1]:

$$m\sim\mathcal{O}(1)\exp\left(-\frac{2\pi}{\sqrt{\frac{g^2}{g_c^2}-1}}\right), \qquad (3.16)$$

where $g_c^2=\pi^2/32$ is the critical coupling, above which dynamical mass generation occurs [5]. Compatibility of the solution (3.16) with the constraint (3.15) implies the existence of an *upper bound* on fermion masses, $m<m_{max}$, where $m_{max}$ is defined through the intersection of the curves (3.15) and (3.16) in the $(m,g)$ plane (see Fig. 5). This yields $m_{max}\simeq 0.3$.

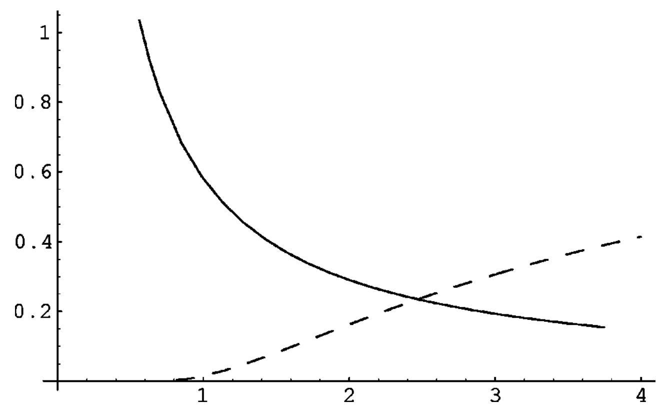


FIG. 5. Fermion mass versus the infrared-value of the coupling $\hat{G}(0)$. The solid curve represents the condition derived from the integral equation for the vertex, whereas the dashed line represents the solution obtained from the standard gap equation in the large-$N$ treatment.

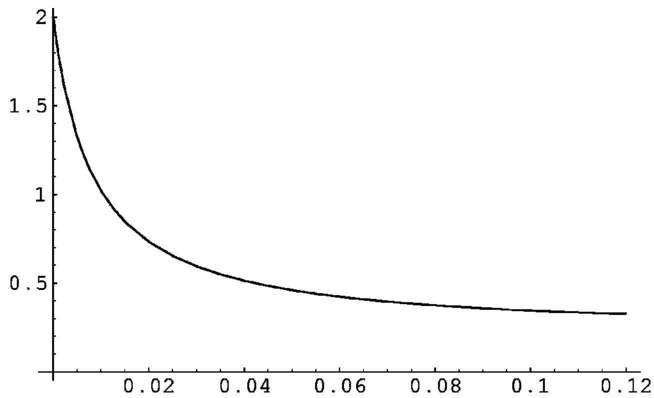


FIG. 6. Numerical solution of Eq. (3.7) versus $p/\alpha$, for a typical set of values $m \sim 0.1$ and $\hat{G}(0)=2$; the solution decreases monotonically and asymptotes quickly to a positive constant value.

On the other hand, for large momenta, we know that $\hat{G} \rightarrow 1$. Physically one expects *a monotonic decrease* of $\hat{G}(x)$ over the *entire* range of $x \in [0,\infty)$. This would occur in our case if and only if $\hat{G}(0)>1$, which, in the context of the large-$N$ result of [5,1], implies a *minimum* bound for the fermion masses $m > m_{min} \simeq 0.03$. Actually, as we shall argue in the next section, $\hat{G}(0)$ should be comfortably larger than $\sqrt{3/2}$ for self-consistency of our approximations.

Hence, we see that the monotonicity of the running coupling can be achieved in the context of a large-$N$ treatment, if the mass $m$ lies in the following regime:

$$0.03 \lesssim m \lesssim 0.3 \tag{3.17}$$

or equivalently if the coupling at the infrared point $\hat{G}(0)$ is restricted in the regime (see Fig. 4):

$$1 < \hat{G}(0) < 2.5. \tag{3.18}$$

At this point it is useful to turn to a numerical study of Eq. (3.7), supplemented with the boundary conditions imposed by the solutions of the form (3.11), specifically:

$$\hat{G}(0)=m\frac{\pi}{2}\sqrt{\frac{3}{2}}c; \quad \hat{G}'(0)=-\frac{8\hat{G}^3(0)}{3\pi^2 m^2}, \tag{3.19}$$

where the constant $c$ should be chosen in such a way that the bound (3.15) be satisfied. The numerical solution of Eq. (3.7) is given in Fig. 6, for a typical case, where $m=0.1$, and $\hat{G}(0)=2$. As we observe, the figure clearly demonstrates *the monotonic decrease* of $\hat{G}$ in the small $x$ interval, where Eq. (3.7) is valid. We also note that the solution asymptotes quickly to a constant positive value, smaller than 1 in contrast to Eq. (2.16), which would require $\hat{G}(x) \sim 1$, for $x \gg 1$. This, however, presents no contradiction, because Eq. (3.7) was based on ignoring the inhomogeneous term $Z(x)$. As one goes to a region of larger $x$, this assumption is no longer valid. In that case the inhomogeneous term $Z(x)$ becomes important, and Eq. (3.7) receives additive (positive) contributions, resulting in $\hat{G}(p) \sim 1$, for large $p$, as required by Eq. (2.16).

## IV. EQUATIONS FOR PHOTON AND FERMION SELF-ENERGIES

As a consistency check of our assumption about ignoring the effects of the inhomogeneous term $Z(p)$ as $p \rightarrow 0$, next to $\hat{G}$, we now turn our attention to the equations that determine the fermion and photon self-energies $A(p)$ and $B(p)$, respectively. As shown below, upon neglecting $Z(p)$ in the infrared, these equations decouple from each other and both functions can be determined solely from knowledge of $\hat{G}(x)$. It should also be stressed that in the massless case the system does not admit a self-consistent solution. In contrast, the presence a fermion mass term changes the situation drastically by yielding self-consistent solutions for $A(p),B(p)$ which are such that $Z(p) \rightarrow 3\sqrt{6}/5$, but at a rate slower than the one with which $\hat{G}$ approaches $\hat{G}(0)$ as $x \rightarrow 0$. Thus the approximation of ignoring the effects of $Z(x)$ in the region $x \ll 1$ is qualitatively correct. This is a highly non-trivial check of our approach, and justifies fully the approximations used above.

To this end, we begin from the integral equation for $A(x)$:

$$A(p)\not{p}=\not{p}-e^2\int\frac{d^3k}{(2\pi)^3}\mathcal{G}^2(k) \times \gamma_\mu \frac{i}{A(k)\not{k}}\gamma_\nu \frac{\delta_{\mu\nu}}{B(k-p)(k-p)^2}, \tag{4.1}$$

which in terms of the semi-amputated vertex $\hat{G}$ becomes

$$A(p)\ \ \not{p}=\not{p}-e^2 A(p)\int\frac{d^3k}{(2\pi)^3}\hat{G}^2(k)\frac{\not{k}}{k^2(k-p)^2}. \tag{4.2}$$

To arrive at the above equation we have carried out the $\gamma$-matrix algebra and we have used the approximation that $\hat{G}(k,p)=\hat{G}(k)$. This approximation is equivalent to the one made for $\hat{G}$ in Eq. (2.12). It will be justified below, by the self-consistency of the solution.

Next we introduce a (constant) fermion mass $m_f$ in the appropriate parts of the integrand in the right-hand side of Eq. (4.2). This effectively amounts to using Eq. (3.1) for the fermion propagator in the loop. In terms of the dimensionless $x,m$ parameters introduced previously, Eq. (4.2) becomes

$$x^4=A^{-1}(x)x^4-\frac{2}{\pi^2}\int_0^x dy\frac{y^4\hat{G}^2(y)}{y^2+m^2}-\frac{2x^4}{\pi^2}\int_x^1 dy\frac{\hat{G}^2(y)}{y^2+m^2}. \tag{4.3}$$

Differentiating twice with respect to $x$ one obtains the following differential equation:

$$\frac{d}{dx}\left[x^{-3}\frac{d}{dx}(x^4A^{-1}(x))\right]+\frac{8}{\pi^2}\frac{\hat{G}^2(x)}{x^2+m^2}=0. \tag{4.4}$$

Restricting our attention to the infrared region $x\to 0$, one may ignore $x^2$ in front of $m^2$, and use the form (3.11) for $\hat{G}(x)$. After elementary integrations one then finds for $A(x)^{-1}$:

$$\begin{aligned}A^{-1}(x)&=\frac{3}{2}\left[-\frac{2}{c^7}x^{-3}+\frac{1}{c^5}x^{-2}-\frac{2}{3c^3}x^{-1}\right.\\&\left.\quad+\frac{2}{c^9}x^{-4}\ln(1+c^2x)\right]+\frac{1}{4}c'\\&\simeq\frac{1}{4}\left(c'-\frac{3}{c}\right)+\frac{3}{5}cx-\frac{1}{2}c^3x^2+\mathcal{O}(x^5),\quad x\to 0,\end{aligned} \tag{4.5}$$

where $c>0$ is the integration constant of the solution (3.11), which obeys the restriction (3.15). We choose the constant of integration $c'=3/c$, so that

$$A(x)^{-1}\simeq\frac{3}{5}cx-\frac{1}{2}c^3x^2+\cdots \tag{4.6}$$

as $x\to 0$. As we shall show later, this choice is compatible with the boundary behavior $Z(x)\to$const, as $x\to 0$, advocated for the inhomogeneous term.

Next we turn our attention to the equation for the photon function $B(p)$. Following similar steps to the ones leading to Eq. (4.2), the pertinent integral equation reads

$$\begin{aligned}B(p)(p^2\delta_{\mu\nu}-p_\mu p_\nu)&=p^2\delta_{\mu\nu}-p_\mu p_\nu\\&\quad+B(p)e^2\int\frac{d^3k}{(2\pi)^3}\hat{G}^2(k)\\&\quad\times\mathrm{Tr}\left[\gamma_\mu\frac{\not{k}}{k^2}\gamma_\nu\frac{\not{k}-\not{p}}{(k-p)^2}\right].\end{aligned} \tag{4.7}$$

To transform the above equation into a scalar equation for $B(p)$ it is necessary to take the trace on both sides. In doing so one assumes that the integral term on the right-hand side is also transverse. This is expected from gauge invariance. In our truncated scheme it can be demonstrated that this is indeed the case, provided one makes our working hypothesis that $\hat{G}(k,p)=\hat{G}(k)$. Specifically if one contracts both sides of Eq. (4.7) by $p^\mu$, the right-hand-side vanishes (Ward identity for the photon) only upon making the above assumption for the semi-amputated vertex, and shifting appropriately the integration variable.

Introducing a fermion mass, and following standard manipulations similar to the ones above, one arrives at

$$\frac{d}{dx}\left[x^{-1}\frac{d}{dx}\left(x^{-1}\frac{d}{dx}(x^4B^{-1})\right)\right]+\frac{32}{\pi^2}\frac{1}{x^2+m^2}\hat{G}^2(x)=0,\quad x\to 0. \tag{4.8}$$

Making the same approximations in the infrared $x\ll 1$ as in the case of $A(x)$, and substituting Emden's solution (3.11) for $\hat{G}$ on the right-hand-side of Eq. (4.8), we obtain after some elementary integrations, upon setting their constants but one ($c_1$) to zero

$$\begin{aligned}x^4B^{-1}(x)&=\frac{6}{c^2}x^2-\frac{6}{c^6}x+\frac{3}{c^4}x^2+\frac{6}{c^8}\ln(1+c^2x)\\&\quad-\frac{6x^2}{c^4}\ln(1+c^2x)+\frac{1}{8}c_1x^4,\end{aligned} \tag{4.9}$$

which for $x\ll 1$ yields

$$B^{-1}(x)\simeq\frac{6}{c^2}x^{-2}\left[1-\frac{2}{3}x+\left(\frac{c_1}{8}+\frac{3}{2}\right)\frac{c^2}{6}x^2+\cdots\right]. \tag{4.10}$$

This form of $B(x)$ gives rise to the following form for the photon propagator in the infrared $k\to 0$,

$$\Delta_{\mu\nu}\sim\frac{\delta_{\mu\nu}}{k^4},\quad k\to 0. \tag{4.11}$$

Notice, as a consistency check, that the photon continues to be massless in the chiral-symmetry broken phase. The corresponding static effective potential is given by the appropriate Fourier transform of the 00 (temporal) component of the photon propagator for $k_0=0$. In the case (4.11) this yields formally an effective potential scaling like $R^2$ for large distances $R$, suggestive of confining behavior.[4] The fact that this behavior is found in the chiral-symmetry broken case is consistent with general (four-dimensional) arguments [12,14] that confinement is a sufficient but not necessary condition for chiral-symmetry breaking.

An important feature of the expressions (4.6) and (4.8) is that, although they are derived only in case where a fermion mass $m\neq 0$, however they do not explicitly depend on the magnitude of the mass. From Eqs. (4.6) and (4.10) one obtains for $Z(x)$ in the infrared region $x\to 0$:

$$\begin{aligned}Z(x)&=B^{-1/2}(x)A^{-1}(x)\\&\simeq\frac{3\sqrt{6}}{5}-\frac{2\sqrt{6}}{5}x\left(1+\frac{5}{4}c^2\right)+\mathcal{O}(x^3)\ \ldots,\quad x\to 0,\end{aligned} \tag{4.12}$$

where we have chosen the constant of integration $c_1$ such that there are no $\mathcal{O}(x^2)$ terms.

[4]However, this is a formal result, given that the corresponding momentum integrals are infrared divergent and hence need proper regularization, which falls beyond our scope here. For standard treatments see discussions in [12].

In order to check under which conditions the omission of the $x$-dependent parts of $Z$ next to those of $\hat{G}(x)$ is justified, we must compare the corresponding linear terms in $x$ of both quantities. Using that

$$\hat{G}(x) \sim \sqrt{\frac{3}{2}}\left( m\frac{\pi}{2}c - m\frac{\pi}{2}c^3 x + \cdots \right), \quad x \to 0 \tag{4.13}$$

this requires

$$\sqrt{8/3}\hat{G}^3(0) - 2\hat{G}^2(0) - (3\pi^2/5)m^2 \gg 0. \tag{4.14}$$

This inequality furnishes a condition on $\hat{G}(0)$ under the assumption $m \ll 1$, which, as we shall see in Sec. V, turns out to be correct. The condition is

$$\hat{G}(0) \gg \sqrt{\frac{3}{2}} \simeq 1.22. \tag{4.15}$$

In the above formulas the symbol $\gg$ means actually at least an order of magnitude. As we shall see below, such a regime for $\hat{G}(0)$ arises self-consistently. Note that the condition (4.15) is in agreement with Eq. (2.16) given that $Z(0) = 3\sqrt{6}/5 \simeq 1.47$ [cf. Eq. (4.12)] is only slightly larger than $\sqrt{3/2}$.

In view of the existence of a non-zero constant value of $Z(x)$ in the infrared region, our analysis on the restrictions (3.15), (3.18) has to be repeated, given that these restrictions stem from the integral equation. From Eq. (4.14), and the value $Z(0) = 3\sqrt{6}/5$ it becomes clear that the boundary condition (3.14) is now modified to

$$1 = \frac{12}{5\pi mc} + \frac{mc^2}{2}\int_0^1 dy \frac{1}{y^2+1}\frac{1}{(1+mc^2y)^3}\left(1 + \frac{2}{y^2+1}\right) + \frac{4}{3\pi^2 m^2 c}\left(1 - \frac{1}{\pi}\right)\sqrt{\frac{2}{3}}. \tag{4.16}$$

The first term continues to asymptote to 3/4, but now one obtains the following restrictions on the coupling:

$$\hat{G}(0) < \frac{12\sqrt{6}}{5} + \frac{8}{3}\left(1 - \frac{1}{\pi}\right)\frac{1}{\pi m}. \tag{4.17}$$

From the above relation it becomes clear that, if $\hat{G}(0) < 12\sqrt{6}/5$ there is no restriction on $m$. However, for this regime of the couplings the condition (4.15), necessary for safely ignoring the effects of the inhomogeneous term $Z(p)$ in the infrared next to $\hat{G}(p)$, is only marginally satisfied. On the other hand, if one allows $\hat{G}(0)$ to exceed this value, which is physically acceptable, then the following upper bound on $m$ as a function of $\hat{G}(0)$ is obtained:

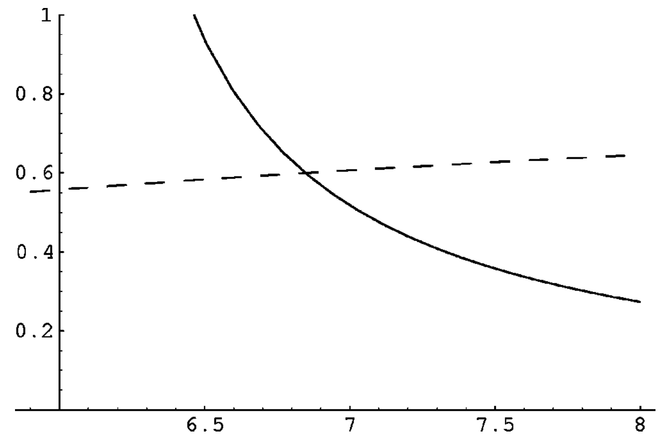

FIG. 7. Fermion mass versus the infrared-value of the coupling $\hat{G}(0)$ for the case where the effects of the inhomogeneous term (almost constant) $Z(x)$ have been taken into account in the infrared. The solid curve represents the condition derived from the integral equation for the vertex, whereas the dashed line represents the solution obtained from the standard gap equation in the large-$N$ treatment.

$$m < \frac{\frac{8}{3\pi}\left(1 - \frac{1}{\pi}\right)}{\hat{G}(0) - \frac{12\sqrt{6}}{5}}, \quad \hat{G}(0) > \frac{12\sqrt{6}}{5} \simeq 5.88, \tag{4.18}$$

which replaces Eq. (3.15).

One may repeat the comparison with the large $N$ limit results [5], as in the previous section, to determine the new region of allowed values of the mass. The analysis is given in Fig. 7.

From Fig. 7 we then obtain the following allowed range of the coupling $\hat{G}(0)$:

$$5.88 < \hat{G}(0) < 7 \tag{4.19}$$

to be compared with Eq. (3.18) associated with the monotonicity requirement for $\hat{G}(p)$. This analysis shows that the new allowed region for the fermion masses, in case (4.18) is

$$0.55 < m < 0.6, \tag{4.20}$$

where the lowest bound is the value of $m$ corresponding to $\hat{G} = 5.88$ from the large-$N$-analysis formula (3.16). The region (4.20) is to be compared with Eq. (3.17). As we shall see in Sec. V, the dynamically generated mass in our scenario, which does not resort to a large-$N$ treatment, lowers significantly the upper bound in Eq. (4.20).

It is important to emphasize that, as a consequence of Eqs. (4.5),(4.10), the non-amputated vertex $\mathcal{G}(p)$, defined in Eq. (2.7), approaches a non-zero *finite* constant value in the infrared $p = 0$. Notice that this constant behavior is compatible with the Ward identity in the infrared limit $p = 0$, thereby implying that the information obtained about a non-trivial infrared fixed-point structure is gauge invariant, as it should be.

This completes our analysis of including the effects of an almost constant $Z(x)\simeq Z(0)$ in the infrared. The above considerations justify retrospectively the omission of $Z$ in deriving the differential equation (3.7). Indeed, a constant shift of $\hat{G}$ inside the derivative operators in that equation, which would account for a constant $Z$, does not affect its form. These results, therefore, constitute a highly non-trivial check of our approach.

## V. DYNAMICAL DERIVATION OF THE FERMION MASS GAP

In the previous section we have assumed the presence of a finite fermion mass, which we have treated effectively as an arbitrary parameter of the model. In this section we turn to the full problem, and study the dynamical generation of this mass, by deriving it self-consistently from the corresponding SD mass-gap equation.

The equation for the gap $\Sigma(p)$ is derived from the graphs of Fig. 2, which yields

$$A(p)\not{p}+\Sigma(p)=\not{p}+A(p)e^2\int\frac{d^3k}{(2\pi)^3}\times\hat{G}(k)^2\gamma_\mu\frac{1}{\not{k}+M(k)}\gamma^\mu\frac{1}{(k-p)^2}, \tag{5.1}$$

where $M(k)\equiv\Sigma(k)/A(k)$ is the mass function, and we have pulled out factors of $A(p)$ appropriately so as to be able to define an amputated vertex function $\hat{G}(k)$. The consistency of the approach will be demonstrated below by the existence of solutions.

Taking the trace in the above equation, and performing the angular integration, one arrives easily at

$$M(p)=3e^2\int\frac{d^3k}{(2\pi)^3}\hat{G}^2(k)\frac{M(k)}{k^2+M^2(k)}\frac{1}{(k-p)^2}=\frac{6\alpha}{\pi^2 p}\int dk\frac{k}{k^2+M^2(k)}\hat{G}(k)^2M(k)\ln\left|\frac{k+p}{k-p}\right|. \tag{5.2}$$

From the middle equation (5.2) it becomes clear that the quantity $g_R(k)\equiv e\hat{G}(k)$, defined in Eq. (2.10), plays indeed the role of a ''running coupling.'' The situation is analogous, but not identical, to that of [11], where a running coupling has also been obtained from the gap equation in the context of a large $N$ analysis. However, in that case, one assumed an ansatz for the vertex, satisfying a truncated form of the Ward identities. Instead, in our approach there was no necessity for a vertex ansatz, since the non-perturbative vertex function was determined self-consistently from the SD equations. As a consequence, the running coupling (2.10) is constructed out of the amputated vertex function alone; the latter is a manifestly gauge-independent quantity, at least perturbatively.

Passing into dimensionless variables, in units of $\alpha=e^2/8,\tilde{M}\equiv M(k)/\alpha,x\equiv p/\alpha,y\equiv k/\alpha$, and working in the regime of low momenta $x\ll 1$ as usual, one obtains after some straightforward algebra, involving a truncated expansion of the logarithmic kernel:

$$\tilde{M}(x)=\frac{12}{\pi^2}\left[\frac{1}{x^2}\int_0^x dy\frac{y^2\tilde{M}(y)}{y^2+\tilde{M}^2(y)}\hat{G}^2(y)+\int_x^\infty dy\frac{\tilde{M}(y)}{y^2+\tilde{M}^2(y)}\hat{G}^2(y)\right]. \tag{5.3}$$

Differentiating twice with respect to $x$ one arrives at

$$x\frac{d^2}{dx^2}\tilde{M}(x)+3\frac{d}{dx}\tilde{M}(x)+\frac{24}{\pi^2}\frac{\tilde{M}(x)}{x^2+\tilde{M}^2(x)}\hat{G}(x)^2=0. \tag{5.4}$$

Given that dynamical mass generation is expected to be an infrared phenomenon, we now restrict ourselves to the region[5]

$$x^2\ll\tilde{M}^2\ll 1. \tag{5.5}$$

In this region we neglect $x^2$ next to $\tilde{M}^2$ in Eq. (5.4) and use the solution (3.11) for $\hat{G}(x)\simeq\tilde{M}\sqrt{\tfrac{3}{8}}\pi c$ as $x\to 0$. The result is

$$x\frac{d^2}{dx^2}\tilde{M}(x)+3\frac{d}{dx}\tilde{M}(x)+9c^2\tilde{M}(x)=0,\quad x\to 0. \tag{5.6}$$

Changing variables $x\to\xi=x^{-1}$, the equation reduces to a Bessel equation [17]

$$\xi^2\frac{d^2}{d\xi^2}\tilde{M}(\xi)-\xi\frac{d}{d\xi}\tilde{M}(\xi)+9c^2\xi^{-1}\tilde{M}(\xi)=0,\quad \xi\to\infty \tag{5.7}$$

with (formally) the general solution

$$\tilde{M}(\xi)=C_1\xi J_{-2}(-6c\xi^{-1/2})+C_2\xi Y_{-2}(-6c\xi^{-1/2}), \tag{5.8}$$

where $C_i,i=1,2$ are arbitrary constants, $J_{-n}(x)=(-1)^nJ_n(x)$, $n=1,2,3,\ldots$, is a Bessel function of the first kind, and $Y_n(x)$ is a generalized Bessel function of the second kind [17]. The latter is only defined for positive $n$ and this imposes the choice $C_2=0$ in Eq. (5.8). Expressing the solution in terms of $x,x\to 0$, one has the following power series expression for the dynamical mass:

[5]Note that this is the opposite limit than the one usually considered in the framework of large-$N$ analysis of dynamical mass generation, where one arrives at a gap equation by making the assumption $\alpha\gg p\gg m$ for the momenta $p$ of the pertinent excitations.

$$\tilde{M}(x)=C_1x^{-1}\sum_{n=0}^{\infty}(-1)^n(3c)^{2n+2}x^{1+n}\frac{1}{n!\Gamma(3+n)}$$
$$\simeq\frac{9}{2}C_1c^2+\mathcal{O}(x),\quad x\to 0. \tag{5.9}$$

From this one obtains the following relation between $\hat{G}(0)$ and $\tilde{M}(0)\equiv m_f/\alpha$:

$$\tilde{M}(0)\equiv m_f/\alpha=\left(\frac{12}{\pi^2}\right)^{1/3}C_1^{1/3}\hat{G}^{2/3}(0). \tag{5.10}$$

This result is to be compared to the result (3.16) within the context of a large-$N$ analysis. In particular, at first sight it seems that the relation (5.10) does not have a critical coupling, above which dynamical mass generation occurs. However, because the result (5.10) has been derived in the context of the solution (3.11), one should bear in mind the restrictions characterizing that situation, in particular Eq. (4.17). This implies an appropriate restriction for $C_1$,[6]

$$\hat{G}^{5/3}(0)-\frac{12\sqrt{6}}{5}\hat{G}^{2/3}(0)-\frac{8}{3(12\pi C_1)^{1/3}}\left(1-\frac{1}{\pi}\right)<0,$$
$$\hat{G}(0)>\frac{12\sqrt{6}}{5}. \tag{5.11}$$

This restriction implies a critical coupling, $\hat{G}_c=12\sqrt{6}/5\simeq 5.88$ but it is derived in a way independent of any large-$N$ analysis. The way to understand Eq. (5.11) is the following: one should first fix a range of $\hat{G}(0)$, with $\hat{G}(0)>5.88$, and then use a $C_1$ that will be such that, within this range of the couplings, Eq. (5.11) is satisfied for masses $\tilde{m}\ll 1$. As can be readily seen, the bound for $C_1$ obtained from the requirement that $m\ll 1$ is far less restrictive than the one associated with Eq. (5.11), provided $\hat{G}(0)$ is not too close to the critical $\hat{G}_c$, where the mass $m$ vanishes. For instance, for $\hat{G}(0)=\mathcal{O}(8)$, the upper bound on $C_1$ from Eq. (5.11) is of order $\mathcal{O}(10^{-4})$, while for $\hat{G}(0)=6$ the upper bound is $C_1<4$. Notice that the bound is very sensitive to small changes in $\hat{G}(0)$.

A typical situation is depicted in Fig. 8 for two values of $C_1=10^{-5},10^{-2}$. We observe that the case $C_1=10^{-2}$ yields an upper bound in the mass which is of order 0.8 and hence should be discarded on the basis that it is not small enough. On the other hand, the value $C_1=10^{-5}$ yields an acceptable upper bound $m\sim 0.1$. In that case, from Fig. 8, we observe that the allowed region of $m$ is

$$0.08\lesssim m\lesssim 0.12 \tag{5.12}$$

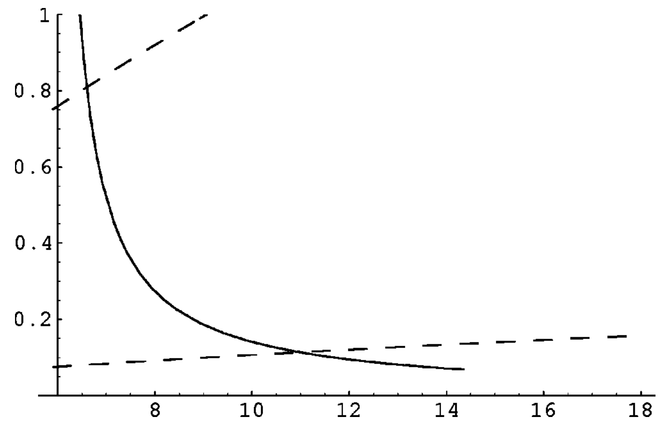


FIG. 8. Fermion mass versus the infrared-value of the coupling $\hat{G}(0)$ using Eq. (5.10) (dashed curves), for two values of $C_1=10^{-5}$ (lower dashed curve) and $C_1=10^{-2}$ (upper dashed curve). The continuous curve is Eq. (4.18), viewed as a boundary condition. The value $C_1=10^{-2}$ should be excluded on grounds of yielding too high values of the mass $\tilde{m}$.

to be compared with Eq. (4.20), derived using the result for the mass in the context of a large-$N$ analysis [5]. The corresponding regime of the couplings $\hat{G}(0)$ is

$$5.88<\hat{G}(0)<11. \tag{5.13}$$

Before closing this section we would like to discuss possible applications of the above behavior; specifically, the restriction (5.13) of the allowed values of $\hat{G}(0)$ may have interesting applications in case the above model turns out to describe the physics of high-temperature superconductors. We remind the reader that in such models dynamical mass generation coincides with superconductivity [1–3]. In that case, the fermion fields (''electrons'') of $QED_3$ represent ''holons,'' i.e., electrically charged excitations of fermionic statistics, which are constituents of the physical electron. The latter is believed to exhibit an effective spin-charge separation [26] in the complicated ground state of high-temperature superconductors. The photon of $QED_3$ then represents an effective Heisenberg spin-spin antiferromagnetic interaction, responsible for binding the holons in Cooper-like pairs. In some models [1,3] the effective (gauge-invariant) coupling $\hat{G}(0)$ may be expressed in terms of the parameters of the microscopic condensed-matter lattice systems, whose long-wavelength limit is equivalent to the above $QED_3$ model, as

$$\hat{G}(0)^2\sim\frac{J}{e^2}(1-\eta), \tag{5.14}$$

where $\eta$ expresses the concentration of impurities in the system (doping), and $J$ denotes the Heisenberg (antiferromagnetic) exchange energy. Hence, on account of Eq. (5.13), Eq. (5.14) implies that $6\lesssim(J/e^2)(1-\eta)\lesssim 11$ for superconductivity to occur. In phenomenologically acceptable models [1] $e^2/J\sim 0.1$, which implies an upper bound on $\eta\sim 0.4$. However, the reader should bear in mind that the above-described limiting values are rather indicative at present, given that a

[6]The restriction (4.17) is to be viewed as a boundary condition. We remind the reader that the requirement of finiteness of $\tilde{M}$ had already fixed the other constant $C_2$ to zero.

complete quantitative understanding of the underlying dynamics of high-temperature superconductivity from an effective gauge-theory point of view is still lacking.

## VI. EQUATION FOR THE VERTEX IN THE CASE OF NON-ZERO PHOTON MASS

In this section we study the case where a (small) covariant photon mass $\delta$ is added to the photon propagator (2.6):

$$\Delta_{\mu\nu}(k) \sim \frac{\delta_{\mu\nu}}{k^2+\delta^2}. \tag{6.1}$$

For the purposes of our analysis in this work, such a mass term will simply be added by hand, without further discussions about its origin. However, we point out for completeness that a small photon mass may be the result of non-perturbative configurations (instantons) in compact U(1) three-dimensional electrodynamics [27]. Moreover, a term acting like a photon mass term arises in real-time finite-temperature considerations [28]; in the latter case, of course, one loses manifest Lorentz covariance.

It is straightforward to see that, in the presence of a photon mass, small compared to the scale $\alpha = e^2/8$, the resulting integral equation for the amputated vertex reads

$$\hat{G}(p) = \frac{1}{24\pi^2}\frac{e^2}{p}\int \frac{dk}{k}\hat{G}^3(k)\ln\left|\frac{(k+p)^2+\delta^2}{(k-p)^2+\delta^2}\right|. \tag{6.2}$$

We are interested in the limit $p \ll \delta$, which would allow us to study the effects of a photon mass on the infrared regime of the theory. Expanding the logarithm in Eq. (6.2) appropriately, we obtain

$$\hat{G}(x) = \frac{4}{3\pi^2}\frac{1}{\hat{\delta}^2}\int_0^x dy\,\hat{G}^3(y) + \frac{4}{3\pi^2}\int_x^\infty \frac{dy}{y^2+\hat{\delta}^2}\hat{G}^3(y), \tag{6.3}$$

where $\hat{\delta} \equiv \delta/\alpha, x = p/\alpha$.

Differentiating once with respect to $x$ one obtains

$$\frac{d}{dx}\hat{G}(x) = \frac{4}{3\pi^2\hat{\delta}^2}\hat{G}^3(x)\frac{x^2}{x^2+\hat{\delta}^2}, \tag{6.4}$$

which can be integrated to yield

$$\hat{G}(x) = \frac{1}{\sqrt{c + \frac{8}{3\pi^2\hat{\delta}^2}\left[\hat{\delta}\,\mathrm{arctg}\left(\frac{x}{\hat{\delta}}\right) - x\right]}}, \tag{6.5}$$

where $c$ is an integration constant to be fixed by the boundary condition imposed by the integral equation (see below).

The running coupling $G(x)$ tends to $\hat{G}(0) = 1/\sqrt{c}$ as $x \to 0$. There is a non-trivial fixed point at $x=0$, given that the renormalization-group $\beta$-function (2.14) vanishes:

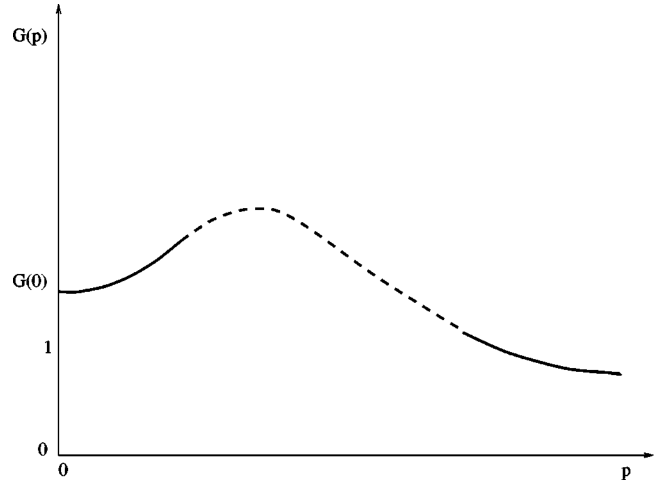


FIG. 9. Running coupling (versus momentum) in the case of a small photon mass. Note the bump at low momenta, which is absent in the case of non-zero fermionic masses. The dashed part of the curve is conjectural at present, and can only be derived numerically. Continuous curves are the result of analytic studies.

$$x\frac{d}{dx}\hat{G}(x) \to \frac{4}{3\pi^2}\hat{G}^3(x)\frac{x^3}{x^2+\hat{\delta}^2} = 0, \quad \text{as } x \to 0. \tag{6.6}$$

Notice that the function $\hat{G}(x)$ increases monotonically with increasing $x$, for small $x$. We next note that by choosing $c$ appropriately one can satisfy the condition (4.15) $\hat{G}(0) \gg \sqrt{3/2}$, required for self-consistency of the approximation of ignoring the effects of the inhomogeneous term $Z(p)$ in the infrared. In such a case one expects that at larger $x$ the increase of the coupling stops at a certain (small) $x = x_0$, and then the coupling starts decreasing to reach asymptotically the perturbative result $\hat{G} \to 1$ (see Fig. 9). The existence of a local maximum is characteristic of the effects of the photon mass on the infrared behavior, to be contrasted with the monotonically decreasing situation in the case of a non-zero fermion mass (see Fig. 6). We expect that in the general case, when both photon and fermion masses are present, the running coupling will exhibit a local maximum at low $x$ when the photon mass is larger than the fermion mass, while this behavior will be replaced by a monotonic decrease, in the case when the fermion mass is considerably larger than the photon one. This situation should be compared with the corresponding one in four-dimensional QCD in the presence of non-vanishing gluon and quark masses [14].

Before closing we discuss briefly a physical situation which could be qualitatively similar to the case of a covariant infrared cutoff in the form of a non-zero photon mass, discussed in this section. This situation has been argued in [11] to simulate finite-temperature effects, given that in such a case the photon propagator acquires a longitudinal plasmon mass term:

$$\Delta_{\mu\nu}(p_0=0, P\to 0, T) = \frac{\delta_{\mu 0}\delta_{0\nu}}{P^2+M_{00}^2} + \text{longitudinal parts}, \tag{6.7}$$

where, for simplicity, we restricted ourselves to the instantaneous approximation [1], $p_0=0$, and the plasmon mass term

$$M_{00}(T)=\sqrt{\frac{2\alpha \ln 2T}{\pi}}. \qquad (6.8)$$

We observe from the form (6.7) that the presence of the mass term $M_{00}$ behaves for low $T$ somewhat analogously (but not identical to) a small photon cutoff mass $\delta$.

Prompted by this observation we would like now to make some speculations regarding the low temperature effects of the $\delta$ cutoff term in Eq. (6.5). We hasten to emphasize that a discussion could at best only qualitatively correct, given that a quantitative understanding would require a finite-temperature extension of the above analysis. Due to the loss of Lorentz covariance in the respective propagators the analysis is far more complicated than the zero-temperature one. However, when the temperature $T$ is sufficiently low, its effects on the pole structure of the photon propagator may be simulated by a covariant photon-mass cutoff, at least qualitatively [11,28]. For this reason, we are motivated to examine the effects of $\hat{\delta}\ll 1$ on the low-temperature scaling of the resistivity [4] $\rho$; the latter is defined as the inverse of the conductivity $\sigma_f$, which expresses the response of the system to a change in an externally applied electromagnetic potential $A_\mu$. Ohm's law, in a gauge with $A_0=0$, then, gives [29,11,4]:

$$\sigma_f \equiv \rho^{-1}=\left.\frac{1}{(P^2+p_0^2)[1+\Pi(P,p_0,T)]}\right|_{P=0}, \qquad (6.9)$$

where $\Pi(P,p_0,T)$ denotes the dimensionless photon polarization used in this work. Following the analysis of [28] we may assume for our discussion below that[7]

$$p_0^2/\alpha \rightarrow M_{00}^2/\alpha \sim \hat{\delta}^2 \sim \frac{2\ln 2}{\pi}T. \qquad (6.10)$$

From the discussion following Eq. (2.10) one immediately sees that the resistivity, defined for $P=0, x^2=\hat{\delta}^2$, would be given in terms of the running coupling $\hat{G}(p)$ by

$$\rho=\hat{\delta}^2/\hat{G}^2(0,T). \qquad (6.11)$$

Using Eq. (6.5) in the limit $x^2=\hat{\delta}^2$, one has

[7]The analysis of [28] is based on a real-time finite temperature approach. In such an approach the resistivity is defined in the infrared limit $p_0\rightarrow 0, P=0$ in the presence of a plasmon mass term $M_{00}$ Eq. (6.8) in the photon propagator; as noted in [28], the way in which the two mass scales $(p_0,P)$ approach 0 suffers from ambiguities, related to physical Landau-damping processes; it is not our purpose here to resolve such issues.

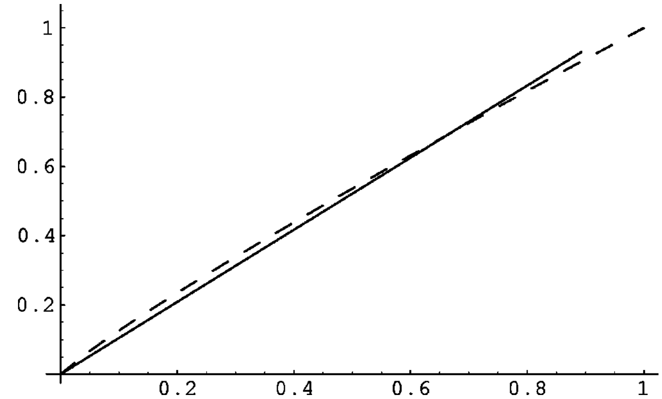


FIG. 10. Plot of the resistivity (6.12) versus temperature $T$ (in units of $\alpha$), for a low-temperature region (continuous curve), and comparison with the experimentally observed single-scaling (basically linear) behavior $T^{1-\chi}, \chi=0.1$ (dashed curve). The agreement is very good for a regime of temperatures accessible to experiment.

$$\rho \sim \frac{2cT}{\pi}\ln 2+\frac{8}{3\pi^2}\left(\frac{\pi}{2}-1\right)\sqrt{\frac{2T}{\pi}\ln 2}. \qquad (6.12)$$

The function $\rho$ is plotted in Fig. 10, versus a single-power scaling behavior of the form: $\rho \sim T^{1-\chi}$, with $\chi$ small; the latter is characteristic of certain theoretical condensed-matter models [30], and has been used in order to bound possible deviations from the experimentally observed linear-$T$ behavior of the resistivity of high-temperature superconductors at optimal doping [4].

We observe that, upon the appropriate choice of the integration constant $c$, the temperature dependence of Eq. (6.12) for a low-temperature regime, accessible to experiment, is hardly distinguishable from the single power scaling behavior.

## VII. CONCLUSIONS AND OUTLOOK

In this work we have presented a study of chiral-symmetry breaking in $QED_3$, based on a system of coupled non-linear SD equations. The novel ingredient is the introduction of a semi-amputated vertex, whose dynamics is governed by a suitably truncated SD equation of cubic order. This allows for a self-consistent determination of the infrared value of the effective charge, in the presence of infrared cutoffs provided by either the fermion mass or a covariant photon mass. The theory is characterized by a (gauge-invariant) non-trivial infrared fixed-point, suggestive of non-Fermi-liquid behavior [15,11].

The non-linear vertex equation furnishes highly non-trivial constraints among the infrared value of the effective coupling and the mass of the fermions. When these constraints are combined with the fermion mass-gap equation, they select specific regions of the coupling space for which dynamical mass generation occurs. It will be interesting to study how these constraints are affected if one goes beyond the one-loop dressed approximation considered here.

From the physical point of view, we remark that dynamical fermion mass generation in $QED_3$ is associated with superconductivity [1,3]. In the relevant statistical models, whose low-energy effective theories are of the $QED_3$ type, the couplings depend on the doping concentration. Therefore, the restrictions in the parameter space we have found above might impose restrictions in the allowed models.

An interesting feature of our approach is the fact that the infrared structure of the photon propagator, derived as a consistent solution of a SD equation, implied formally a confining effective potential. The fact that this behavior occurs in the chiral-symmetry broken phase of the theory, appears to be in agreement with generic expectations that confinement is a sufficient condition for chiral symmetry breaking.

An obvious next step in this program is the attempt to solve the coupled system of SD equations numerically, without the approximation of ignoring the effects of the inhomogeneous term $Z(p)$ in the infrared. We remind the reader that in the present work we have restricted the allowed values of the coupling such that this approximation was self-consistent. This, however, does not imply that solutions with $Z(p)$ not meeting these requirements are impossible; to establish or exclude their existence one should solve the whole system of equations, which at present can only be done numerically. Such a numerical treatment will have the additional advantage of not resorting to the approximations made to the kernel of the integral equations in order to convert them into differential ones.

Furthermore, an analysis at finite temperatures will reveal whether our conjecture in Sec. VI, on the scaling behavior of the resistivity, is correct. As we have discussed there, this issue acquires great importance in view of very accurate experiments in high-temperature superconductors indicating a basically linear scaling with temperatures of the resistivity in the normal phase (no fermion mass gap) of the system.

Finally, an extension of these ideas to the non-Abelian case appears as a challenge for the future, especially in view of recent claims [3,31] that non-Abelian gauge symmetries might describe the dynamics of spin-charge separation in realistic antiferromagnetic condensed-matter systems [26].

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