Casimir energy of a compact cylinder under the condition $\varepsilon\mu=c^{-2}$

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The Casimir energy of an infinite compact cylinder placed in a uniform unbounded medium is investigated under the continuity condition for the light velocity when crossing the interface. As a characteristic parameter in the problem the ratio $\xi^2=(\varepsilon_1-\varepsilon_2)^2/(\varepsilon_1+\varepsilon_2)^2=(\mu_1-\mu_2)^2/(\mu_1+\mu_2)^2\leqslant 1$ is used, where ε_1 and μ_1 are, respectively, the permittivity and permeability of the material making up the cylinder and ε_2 and μ_2 are those for the surrounding medium. It is shown that the expansion of the Casimir energy in powers of this parameter begins with the term proportional to ξ^4. The explicit formulas permitting us to find numerically the Casimir energy for any fixed value of ξ^2 are obtained. Unlike a compact ball with the same properties of the materials, the Casimir forces in the problem under consideration are attractive. The implication of the calculated Casimir energy in the flux tube model of confinement is briefly discussed. [S0556-2821(99)06622-9]

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I. INTRODUCTION

The calculation of the Casimir energy for boundary conditions given on the surface of an infinite cylinder has turned out to be the most complicated problem in this field [1–6]. In Ref. [3] an attempt was undertaken to predict the Casimir energy of a conducting cylindrical shell treating the cylinder as an intermediate configuration between a sphere and two parallel plates. Taking into account that the vacuum energies of a conducting sphere and conducting plates have the opposite signs, the authors hypothesized that the Casimir energy of a cylindrical perfectly conducting shell should be zero. However, a direct calculation [1,2] showed that this energy is negative as in the case of parallel conducting plates. This calculation was repeated only in recent papers [4–6] by making use of comprehensive methods, more simple but more formal at the same time.

Thus in spite of its half-century history the Casimir effect still remains a problem where physical intuition does not work, and in order to reveal even the sign of the Casimir energy (i.e. the direction of the Casimir forces) it is necessary to carry out a consistent detailed calculation.

The account for dielectric and magnetic properties of the media in the case of nonplanar interface proved to be a very complicated problem in calculation of the Casimir energy [7]. However if the light velocity is constant when crossing the interface, then the calculation of the Casimir energy of a compact ball [8] or cylinder [4] is the same as that for conducting spherical or cylindrical shells, respectively. In such calculations the expansion of the Casimir energy in terms of the parameter $\xi^2=(\varepsilon_1-\varepsilon_2)^2/(\varepsilon_1+\varepsilon_2)^2=(\mu_1-\mu_2)^2/(\mu_1+\mu_2)^2\leqslant 1$ is usually constructed, where ε_1 and μ_1 are, respectively, the permittivity and permeability of the material making up the ball or cylinder, and ε_2, μ_2 are those for the surrounding medium. The same velocity of light, c, in both the media implies that the condition $\varepsilon_1\mu_1=\varepsilon_2\mu_2=c^{-2}$ is satisfied.

The Casimir energy of a compact ball with the same speed of light inside and outside [8] and the Casimir energy of a pure dielectric ball [9–11] turned out to be of the same sign: they are positive, and consequently the Casimir forces are repulsive.[1] Moreover, the extrapolation of the result obtained under the condition $\varepsilon\mu=c^{-2}$ to a pure dielectric ball gives a fairly good prediction [8,9].

For a compact cylinder under the condition $\varepsilon\mu=c^{-2}$ it has been found [4] that the linear term in the Casimir energy expansion in powers of ξ^2 vanishes. Keeping in mind the situation with a compact ball possessing the same speed of light inside and outside and a pure dielectric ball, it is tempting to check whether the Casimir energy of a compact cylinder under the condition $\varepsilon\mu=c^{-2}$ is close to the Casimir energy of a pure dielectric cylinder. However, in the case of a dielectric cylinder a principal difficulty arises, namely, in the integral representation for the corresponding spectral ζ-function (or, in other words, for the sum of eigenfrequencies) it is impossible to carry out the integration over the longitudinal momentum k_z. On the other hand, in Ref. [4] the Casimir energy of a compact dielectric cylinder was evaluated by a direct summation of the van der Waals interaction between individual fragments (molecules) of the cylinder. By making use of the dimensional regularization, a vanishing value for this energy was obtained. It is worth noting that this procedure, having been applied to a pure dielectric ball [12], gives the same result as the quantum field theory approach [10]. In view of all this, it is undoubtedly interesting to elucidate whether the vacuum energy of the electromagnetic field for a compact cylinder with the condition $\varepsilon\mu=c^{-2}$ vanishes exactly. Therefore the main goal of the present paper is, namely, to extend the analysis made in [4] up to the fourth order in ξ. To this accuracy the Casimir energy in question turns out to be nonvanishing. Our consideration is concerned with zero temperature theory

[1] We use the terms ‘‘pure dielectric ball’’ and ‘‘pure dielectric cylinder’’ for the corresponding nonmagnetic configurations with $\mu_1=\mu_2=1$ and $\varepsilon_1\neq\varepsilon_2$.

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only, and the main calculation ignores dispersion.

The layout of the paper is as follows. In Sec. II the first nonvanishing term proportional to ξ^4 is calculated in the expansion of the Casimir energy of a compact cylinder in powers of ξ^2 under the condition $\varepsilon\mu=c^{-2}$. This term proves to be negative, and the Casimir forces seek to contract the cylinder reducing its radius, unlike the repulsive forces acting on a compact ball under the same conditions. In Sec. III the Casimir energy in the problem at hand is calculated numerically for several fixed values of the parameter ξ^2 without assuming the smallness of ξ^2, and the corresponding plot is presented. In Sec. IV the implication of the obtained results in the flux tube model (hadronic string) describing the quark dynamics inside the hadrons is considered. In the Conclusion (Sec. V) some general properties of the Casimir effect are briefly discussed.

II. EXPANSION OF THE CASIMIR ENERGY IN POWERS OF ξ^2

We start with the formulas which allow us to construct the expansion of the Casimir energy of a compact infinite cylinder, possessing the same speed of light inside and outside, in powers of the parameter ξ^2. The derivation of these formulas can be found in the papers cited below.

When using the mode-by-mode summation method [4] or the zeta function technique [5] the Casimir energy per unit length of a cylinder is represented as a sum of partial energies

$$E=\sum_{n=-\infty}^{+\infty}E_n, \tag{2.1}$$

where

$$E_n=\frac{c}{4\pi a^2}\int_0^\infty dy\, y\ln\{1-\xi^2[y(I_n(y)K_n(y))']^2\}. \tag{2.2}$$

Here the condition

$$\varepsilon_1\mu_1=\varepsilon_2\mu_2=c^{-2} \tag{2.3}$$

is assumed to hold, with c being the velocity of light inside and outside the cylinder (in units of that velocity in vacuum). The parameter ξ^2 in Eq. (2.2) is defined by the dielectric and magnetic characteristics of the material of a cylinder and a surrounding medium

$$\xi^2=\frac{(\varepsilon_1-\varepsilon_2)^2}{(\varepsilon_1+\varepsilon_2)^2}=\frac{(\mu_1-\mu_2)^2}{(\mu_1+\mu_2)^2}. \tag{2.4}$$

The representation (2.1), (2.2) for the Casimir energy is formal because the integral in Eq. (2.2) diverges logarithmically at the upper limit, and the sum over n in Eq. (2.1) is also divergent. These difficulties are removed by the following transformation of the sum (2.1):

$$\begin{aligned}E&=\sum_{n=-\infty}^{+\infty}(E_n-E_\infty+E_\infty)=\sum_{n=-\infty}^{+\infty}(E_n-E_\infty)+\sum_{n=-\infty}^{+\infty}E_\infty\\&=\sum_{n=-\infty}^{\infty}\overline{E}_n+E_\infty\sum_{n=-\infty}^{\infty}n^0,\end{aligned} \tag{2.5}$$

where

$$\overline{E}_n=E_n-E_\infty,\quad n=0,\pm1,\pm2\ldots, \tag{2.6}$$

$$E_\infty=E_n|_{n\to\infty}=-\frac{c\xi^2}{16\pi a^2}\int_0^\infty\frac{z^5dz}{(1+z^2)^3}. \tag{2.7}$$

A consistent treatment of the product of two infinities $E_\infty\times\Sigma_{n=-\infty}^{\infty}n^0$ leads to a finite result (see [4] and, especially, [5])

$$E_\infty\cdot\sum_{n=-\infty}^{+\infty}n^0=\frac{c\xi^2}{16\pi a^2}\ln(2\pi). \tag{2.8}$$

Thus

$$E=\sum_{n=-\infty}^{\infty}\overline{E}_n+\frac{c\xi^2}{16\pi a^2}, \tag{2.9}$$

where

$$\overline{E}_n=\overline{E}_{-n}=\frac{c}{4\pi a^2}\int_0^\infty dy\,y\left\{\ln[1-\xi^2\sigma_n^2(y)]+\frac{\xi^2}{4}\frac{y^4}{(n^2+y^2)^3}\right\},\quad n=1,2,\ldots, \tag{2.10}$$

$$\overline{E}_0=\frac{c}{4\pi a^2}\int_0^\infty dy\,y\left\{\ln[1-\xi^2\sigma_0^2(y)]+\frac{\xi^2}{4}\frac{y^4}{(1+y^2)^3}\right\},\quad \sigma_n(y)=y(I_n(y)K_n(y))'. \tag{2.11}$$

The Casimir energy (2.9) is defined correctly because the integrals in Eqs. (2.10) and (2.11) exist and the sum in Eq. (2.9) converges [4]. It is this formula that should be expanded in powers of ξ^2. We confine ourselves with the first two terms in this expansion

$$E\equiv E(\xi^2)=E^{(2)}\xi^2+E^{(4)}\xi^4+O(\xi^6). \tag{2.12}$$

In the same way we have for $\overline{E}_n$

$$\overline{E}_n \equiv \overline{E}_n(\xi^2) = E_n^{(2)}\xi^2 + E_n^{(4)}\xi^4 + O(\xi^6), \quad n = 0,1,2,\ldots, \tag{2.13}$$

where

$$E_0^{(2)} = -\frac{c}{4\pi a^2}\int_0^\infty dy\, y\left[\sigma_0^2(y) - \frac{y^2}{4(1+y^2)^3}\right] = \frac{c}{4\pi a^2}(-0.490878), \tag{2.14}$$

$$E_n^{(2)} = -\frac{c}{4\pi a^2}\int_0^\infty dy\, y\left[\sigma_n^2(y) - \frac{y^2}{4(n+y^2)^3}\right], \quad n = 1,2,\ldots, \tag{2.15}$$

$$E_0^{(4)} = -\frac{c}{8\pi a^2}\int_0^\infty dy\, y\,\sigma_0^4(y) = \frac{c}{4\pi a^2}(-0.0860808), \tag{2.16}$$

$$E_n^{(4)} = -\frac{c}{8\pi a^2}\int_0^\infty dy\, y\,\sigma_n^4(y), \quad n = 1,2,\ldots. \tag{2.17}$$

The integrals in Eqs. (2.15) and (2.17) containing Bessel functions can be calculated numerically only for $n < n_0$ with a certain fixed value of n_0. For all the rest partial energies with $n \geqslant n_0$ one needs an analytic expression. We derive such a formula using the uniform asymptotic expansion (UAE) for the product of the modified Bessel functions [13]. Taking into account all the terms up to the n^{-6} order we can write

$$\ln\left\{1 - \xi^2\left[y\frac{d}{dy}(I_n(ny)K_n(ny))\right]^2\right\} = -\xi^2\frac{y^4t^6}{4n^2}\left[1 + \frac{t^2}{4n^2}(3 - 30t^2 + 35t^4) + \frac{t^4}{4n^4}(9 - 256t^2 + 1290t^4 - 2037t^6 + 1015t^8)\right] - \xi^4\frac{y^8t^{12}}{32n^4}\left[1 + \frac{t^2}{2n^2}(3 - 30t^2 + 35t^4)\right] - \xi^6\frac{y^{12}t^{18}}{192n^6} + O\left(\frac{1}{n^8}\right), \tag{2.18}$$

where $t = 1/\sqrt{1+y^2}$.

Substituting this expression into Eq. (2.10) and integrating with the use of the formula [14]

$$\int_0^\infty dy\, y^\alpha t^\beta = \frac{1}{2}\,\frac{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta-\alpha-1}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)}, \tag{2.19}$$

$$\mathrm{Re}(\alpha+1) > 0, \quad \mathrm{Re}\left(\frac{\alpha-\beta+3}{2}\right) < 1$$

one obtains [4]

$$\overline{E}_n = \overline{E}_n^{asymp} + O\left(\frac{1}{n^6}\right), \tag{2.20}$$

$$\overline{E}_n^{asymp} = \frac{c\,\xi^2}{4\pi a^2}\left(\frac{10 - 3\,\xi^2}{960\,n^2} - \frac{28224 - 7344\,\xi^2 + 720\,\xi^4}{15482880\,n^4}\right). \tag{2.21}$$

From here we find the coefficients $E_n^{(2)}$ and $E_n^{(4)}$ entering Eq. (2.13)

$$E_n^{(2)asymp} = \frac{c}{4\pi a^2}\left(\frac{1}{96\,n^2} - \frac{7}{38040\,n^4}\right), \tag{2.22}$$

$$E_n^{(4)asymp} = -\frac{c}{4\pi a^2}\left(\frac{1}{320\,n^2} - \frac{17}{560\cdot 64\,n^4}\right). \tag{2.23}$$

Now by a direct numerical calculation it is necessary to estimate the value $n = n_0$ starting from which the exact formulas (2.15) and (2.17) can be substituted by the approximate ones (2.22) and (2.23). In Ref. [4] it was shown that when calculating $E^{(2)}$ one can begin to use the approximate formula from $n_0 = 6$

$$E^{(2)}=E_0^{(2)}+2\sum_{n=-1}^{5} E_n^{(2)}+2\sum_{n=6}^{\infty} E_n^{(2)asymp}+\frac{c}{16\pi a^2}\ln(2\pi)=E_0^{(2)}+2\sum_{n=-1}^{5} E_n^{(2)}+\frac{c}{4\pi a^2}\left(\frac{1}{48}\sum_{n=6}^{\infty}\frac{1}{n^2}-\frac{7}{19020}\sum_{n=6}^{\infty}\frac{1}{n^4}\right)$$
$$+\frac{c}{16\pi a^2}\ln(2\pi)=\frac{c}{4\pi a^2}(-0.490878+0.027638+0.003778-0.000007+0.459469)=\frac{c}{4\pi a^2}(0.000000). \quad (2.24)$$

This result obtained in [4] was interpreted there as the vanishing of the Casimir energy of a compact cylinder under the condition (2.3). However, as it will be shown below this is not the case.

Table I shows that when calculating the coefficient $E^{(4)}$ in Eq. (2.13), one can also take $n_0=6$. As a result we obtain for this coefficient

$$E^{(4)}=E_0^{(4)}+2\sum_{n=1}^{5} E_n^{(4)}+2\sum_{n=6}^{\infty} E_n^{(4)asymp}=E_0^{(4)}+2\sum_{n=1}^{5} E_n^{(4)}-\frac{c}{4\pi a^2}\left(\frac{1}{160}\sum_{n=6}^{\infty}\frac{1}{n^2}-\frac{17}{560\cdot 32}\sum_{n=6}^{\infty}\frac{1}{n^4}\right)$$
$$=\frac{c}{4\pi a^2}(-0.0860808-0.008315-0.0011334+0.0000018)=-\frac{c}{4\pi a^2}0.095528. \quad (2.25)$$

Thus, the Casimir energy of a compact cylinder possessing the same speed of light inside and outside does not vanish and is defined up to the ξ^4 term by the formula

$$E(\xi^2)=-\frac{c\xi^4}{4\pi a^2}0.0955275=-0.007602\frac{c\xi^4}{a^2}. \quad (2.26)$$

In contrast to the Casimir energy of a compact ball [8] with the same properties

$$E_{ball}\simeq\frac{3}{64a}c\xi^2=0.046875\frac{c\xi^2}{a} \quad (2.27)$$

the Casimir energy of a cylinder under consideration turned out to be negative. Consequentially, the Casimir forces strive to contract the cylinder. The numerical coefficient in Eq. (2.26) proved really to be small, for example, in comparison with the analogous coefficient in Eq. (2.27). Probably it is a manifestation of the vanishing of the Casimir energy of a pure dielectric cylinder noted in the Introduction.

TABLE I. The dimensionless coefficients $\mathcal{E}_n^{(4)}=(4\pi a^2/c)E_n^{(4)}$ and $\mathcal{E}_n^{(4)asymp}=(4\pi a^2/c)E_n^{(4)asymp}$ calculated according to Eqs. (2.17) and (2.23), respectively.

n	$\mathcal{E}_n^{(4)}$	$\mathcal{E}_n^{(4)asymp}$
1	0.002747	0.003599
2	0.000752	0.000811
3	0.000341	0.000353
4	0.000193	0.000197
5	0.000124	0.000125
6	0.000086	0.000087

III. NUMERICAL CALCULATION OF THE CASIMIR ENERGY FOR ARBITRARY ξ^2

Equations (2.9)–(2.11) obtained in the preceding section enable one to calculate the Casimir energy $E(\xi^2)$ numerically, without making any assumptions concerning the smallness of the parameter ξ^2. Comparing the results obtained by the exact formula (2.10) and by the approximate one (2.21) we again find the value $n=n_0$ starting from which $\overline{E}_n^{asymp}$ reproduces $\overline{E}_n$ precisely enough. In the general case there is its own n_0 for each value of ξ^2. Obviously, one should expect a substantial deviation from Eq. (2.26) only for $\xi^2\simeq 1$. Moreover the main contribution into the Casimir energy determined by the sum (2.9) is given by the term $\overline{E}_0$ which is evaluated now exactly using Eq. (2.11) without expanding in powers of ξ^2 as it has been done in the preceding section.

The results of the calculations accomplished in this way for $E(\xi^2)$ are presented in Fig. 1 (solid curve). Here the Casimir energy defined by Eq. (2.26) as a function of ξ^2 is

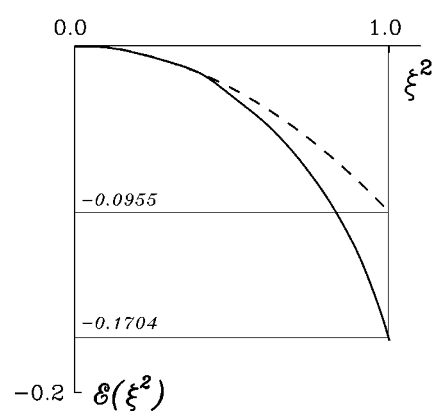

FIG. 1. The dimensionless Casimir energy $\mathcal{E}(\xi^2)=(4\pi a^2/c)E(\xi^2)$ as a function of the parameter ξ^2. The solid curve is obtained without assuming the smallness of ξ^2 (the exact result); the dashed curve presents the approximate equation (2.26).

also plotted (dashed curve). When $\xi^2=1$ we get the Casimir energy of a perfectly conducting cylindrical shell [4]. If we used for its calculation the approximate formula (2.26), we should obtain for the dimensionless energy $\mathcal{E}=(4\pi a^2/c)E$ the value -0.0955 instead of -0.1704. Thereby, the approximate formula (2.26) at this point gives a considerable error of $\sim 70\%$. At the same time the analogous formula (2.27) for a compact ball at $\xi^2=1$ gives the Casimir energy of a perfectly conducting spherical shell with a few percent error [8,15].

IV. IMPLICATION OF THE CALCULATED CASIMIR ENERGY IN THE FLUX TUBE MODEL OF CONFINEMENT

The constancy condition for the velocity of gluonic field when crossing the interface between two media is used, for example, in a dielectric vacuum model (DVM) of quark confinement [16–18]. This model has many elements in common with the bag models [19], but among the other differences, in DVM there is no explicit condition of the field vanishing outside the bag. It proves to be important for calculation of the Casimir energy contribution to the hadronic mass in DVM. The point is that in the case of boundaries with nonvanishing curvature there happens a considerable (not full, however) mutual cancellation of the divergences from the contributions of internal and external (with respect to the boundary) regions. If only the field confined inside the cavity is considered, as in the bag models [20–22], then there is no such a cancellation, and one has to remove some divergences by means of renormalization of the phenomenological parameter in the model defining the QCD vacuum energy density.

From a physical point of view the vanishing of the field or its normal derivative precisely on the boundary is an unsatisfactory condition, because due to quantum fluctuations it is impossible to measure the field as accurately as desired at a certain point of the space [23].

In the DVM there is also considered a cavity that appears in the QCD vacuum when the invariant $F_{\mu\nu}F^{\mu\nu}\sim \mathbf{E}^2-\mathbf{B}^2$ exceeds a certain critical value ($\mathbf{E}$ and $\mathbf{B}$ are the color fields). Inside the cavity the gluonic field can be treated as an Abelian field in view of the asymptotic freedom in QCD. In this approach it is assumed that in the QCD vacuum (outside the cavity) the dielectric constant tends to zero $\varepsilon_2\to 0$ while the magnetic permeability tends to infinity $\mu_2\to\infty$ in such a way that the relativistic condition $\epsilon_2\mu_2=1$ holds. Inside the cavity $\epsilon_1=\mu_1=1$. As it was shown in the present paper for a compact cylinder and in Ref. [8] for a compact ball, in calculation of the Casimir energy the condition $\varepsilon_1\mu_1=\varepsilon_2\mu_2$ proves to be essential, and it is possible to take the limit $\varepsilon_2\to 0$, $\mu_2\to\infty$ in the resulting formula putting $\xi^2=(\varepsilon_1-\varepsilon_2)^2/(\varepsilon_1+\varepsilon_2)^2=(\mu_1-\mu_2)^2/(\mu_1+\mu_2)^2=1$.

Hence, in the DVM as a vacuum energy of gluonic field one should take the Casimir energy of a perfectly conducting infinitely thin shell having the shape either of a sphere, or expanded ellipsoid, or cylinder. In the last case we deal with the flux tube model of confinement [17,24,25] in which a heavy quark and antiquark are considered to be coupled through a cylindrical cavity (flux tube) in the QCD vacuum. Obviously, in the flux tube model of confinement the Casimir energy of a compact cylinder calculated under the condition $\epsilon_1\mu_1=\epsilon_2\mu_2=1$ should be regarded as a quantum correction to the classical string tension. To estimate this correction, it is necessary to define the value of the radius a of the flux tube. Without pretending at high accuracy we shall take a of the same order as the critical radius R_c in the hadronic string model.[2] At the distances between the quarks smaller than R_c the flux tube model has no sense. In the Nambu-Goto string model R_c is determined by the string tension M_0^2 [26,27]

$$R_c^2=\frac{\pi}{6M_0^2}. \tag{4.1}$$

Hence, we obtain the following estimation for the Casimir energy contribution into the string tension in the flux tube model

$$\left|\frac{8E}{M_0^2}\right|=8\frac{7.6\cdot 10^{-3}}{a^2M_0^2}=8\frac{7.6\cdot 10^{-3}}{R_c^2M_0^2}\simeq 0.1. \tag{4.2}$$

The multiplier 8 makes an account for the contribution of the eight gluonic field components into the string tension. Thus, unlike the conclusion made in (1988) [17], the quantum correction to the classical string tension, determined by the gluonic field confined in the flux tube, turned out to be essential ($\sim 10\%$). This fact should be taken into account in detailed examination of this model.

V. CONCLUSION

The Casimir energy of a compact cylinder under the condition $\varepsilon\mu=c^{-2}$ does not vanish, but it is negative with the absolute value increasing as ξ^4 for small ξ^2. The Casimir forces seek to contract the cylinder.

The calculation of the vacuum energy for the boundary conditions of different geometries both with the account for the properties of the materials and without such accounting enables one to make the following general conclusion. In a concrete problem the direction of the Casimir forces is determined only by the geometry of the boundaries. Dielectric and magnetic properties of the media cannot change the direction of these forces.

This conclusion is confirmed by the calculation of the Casimir effect for parallel conducting plates, for a sphere and cylinder, these boundaries being considered in the vacuum or dividing the materials with different dielectric and magnetic properties. Even a dilute dielectric cylinder mentioned above

[2] In principle the radius of the gluonic tube may be deduced by minimizing the linear density of a total energy in this model, the QCD vacuum energy being considered to be negative [25]. However in this case a is expressed through the phenomenological parameter, the flux of the gluonic field, that in its turn requires a definition.

does not violate this pattern. Maybe the Casimir forces in this case vanish in fact, but there are no indications that they can become repulsive.

The account for the dispersion probably does not change this inference. The calculation of the Casimir energy carried out in [28] for a compact ball with ε and μ dependent on the frequencies of electromagnetic oscillations ω confirms this. In Ref. [29] the Casimir forces affecting a compact cylinder when $\varepsilon(\omega)\mu(\omega)=c^{-2}$ were investigated. To remove the divergences the authors introduced a double cutoff over the frequency ω_0 and over the angular momentum n_0. The finite answer proved to be very involved and depended on the cutoff parameters, but the Casimir forces are attractive as in our consideration. However there are other points of view concerning the role of dispersion in the Casimir effect [30–33].

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