Remarks on conserved quantities and entropy of BTZ black hole solutions. I. The general setting

L. Fatibene, M. Ferraris, M. Francaviglia, and M. Raiteri
Dipartimento di Matematica, Università di Torino, via C. Alberto 10, 10123 Torino, Italy

(Received 1 March 1999; published 24 November 1999)

The Bañados-Teitelboim-Zanelli (BTZ) stationary black hole solution is considered and its mass and angular momentum are calculated by means of the Nöther theorem. In particular, relative conserved quantities with respect to a suitably fixed background are discussed. Entropy is then computed in a geometric and macroscopic framework, so that it satisfies the first principle of thermodynamics. In order to compare this more general framework to the prescription of Wald and co-workers, we construct the maximal extension of the BTZ horizon by means of Kruskal-like coordinates. A discussion about the different features of the two methods for computing entropy is finally developed. [S0556-2821(99)01420-4]

PACS number(s): 04.70.−s, 03.50.−z

I. INTRODUCTION

Since its discovery in 1992, the so-called Bañados-Teitelboim-Zanelli (BTZ) black hole solution has often been used in current literature as a simple but realistic model for black hole physics (see [1–5], and references quoted therein). Specifically, it has been assumed as a *test model* for both quantum gravity and for problems related to black hole entropy. Recently, the BTZ solution has been shown to be the *effective solution* in a dimensional reduced model and many black holes ensuing from string theory are described in terms of the BTZ solution, at least near the horizon.

Certainly, many facts about entropy remain to be understood from both a statistical and a geometrical viewpoint. In recent investigations of ours we, therefore, aimed to review what is known from a geometrical macroscopic viewpoint and to add some considerations which, as far as we know, are new in the literature.

The results of our investigations will be presented in two papers, whereby the material has been divided on the basis of coherence and shortness considerations. We shall refer to them as paper I (the present paper) and paper II (forthcoming). In paper I we consider the standard BTZ solution, seen as a vacuum solution for standard $(2+1)$ general relativity with (negative) cosmological constant. We review and apply to the specific example some methods for defining conserved quantities (as is well known, there are several methods in literature and it is almost impossible to review all of them in a single paper; hereafter, we shall follow the covariant approach to conserved quantities based on the Nöther theorem—see [6–11]). Moreover a proposal for fixing a background spacetime is suggested in order to correct so-called *anomalous factors* (see [12]) and to produce the expected values of conserved quantities.

Then we calculate the entropy by relying on a geometrical and global framework presented in [11]. As it was already discussed there, our general method contains the proposal of Wald and co-workers as a particular case (see [13–15], and references quoted therein). Here we apply also the original Wald's recipe to show that it allows us to obtain the same results but by a much longer route. We believe, in fact, that a serious comparison between the two methods is important also because from a theoretical viewpoint Wald's framework requires additional hypotheses with respect to ours: basically involving the surface gravity κ which in our more general framework is not required to be nonvanishing in order to ensure the horizon to be a *bifurcate Killing horizon* (see [13,14]). In the BTZ solution these additional requirements of Wald hold true (with the exception of the extremal case $r_+=r_-$). Then the entropy calculated by using Wald's recipe necessarily agrees with our computation. Other examples in which Wald's additional hypotheses do not hold [such as the Taub-NUT (Newman-Unti-Tamburino) solutions] will be considered elsewhere (see [16]), whereby we shall show that our method works also when Wald's recipe fails for lack of properties of the concerned solution (thus providing a geometrical recipe for the correct entropy, which can be calculated on a statistical basis as in [17]).

In paper II we shall analyze a triad-affine theory with topological matter but no cosmological constant, which is called *BCEA* and describes the BTZ spacetimes. This theory has already appeared in the literature (see [5,18]). It has been shown that it exhibits an ‘‘exchange behavior’’ between conserved quantities (the total mass, i.e., the Nöther conserved quantity associated to a timelike vector is the parameter J which should correspond to the angular momentum of BTZ spacetimes when described in standard general relativity). For what concerns the entropy, an exchange of inner and outer horizons has also been noticed (see [5]). In paper II we shall first obtain the same results in a geometrical and global formalism. Then we shall explicitly build a purely metric (natural) theory, which we shall call *BCG theory*, equivalent to the triad-affine *BCEA*. The *BCG* theory can be obtained through a generalized ‘‘dual’’ Legendre transformation, which may be viewed as a generalization of the Palatini first-order variational approach to general relativity. Starting from the *BCG* Lagrangian conserved quantities and entropy will be again calculated for BTZ spacetime. The results so obtained, in our opinion, will enlighten some of the ambiguities present in the *BCEA* theory. In particular they allow one to truly isolate the ‘‘matter’’ contributions to conserved quantities from the purely gravitational contribution, so as to better explain the calculation previously performed by others in *BCEA* theory (see [5]).

II. THE BTZ SOLUTION

Let us consider a spacetime manifold M [for the moment of arbitrary dimension $n=\dim(M)$] and the bundle $\mathrm{Lor}(M)$ of Lorentzian metrics over M. Let us fix a trivialization and denote by $g_{\mu\nu}$ the coefficients of the metric field [used as coordinates on the fibers of $\mathrm{Lor}(M)$], by $\gamma^{\lambda}_{\mu\nu}$ the Christoffel symbols (i.e., the coefficients of the Levi-Civita connection of the metric g), by $r_{\mu\nu}$ the Ricci tensor, and by r the scalar curvature (of the metric g).

The Hilbert-Einstein Lagrangian with negative cosmological constant $\Lambda=-1/l^2$ is

$$L=\mathcal{L}\,\mathbf{ds}=\alpha(r-2\Lambda)\sqrt{g}\,\mathbf{ds}, \tag{2.1}$$

where $\mathbf{ds}=dx^1\wedge dx^2\wedge\ldots\wedge dx^n$ is the standard basis for n forms over M and $\alpha\neq 0$ is a coupling constant. To compare with results in [5] one has to set $\alpha=\frac{1}{2}$. Let us denote the *covariant naive momenta* by

$$\pi_{\mu\nu}=\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}=\alpha\sqrt{g}\left(r_{\mu\nu}-\frac{1}{2}rg_{\mu\nu}+\Lambda g_{\mu\nu}\right),$$
$$p^{\mu\nu}=\frac{\partial\mathcal{L}}{\partial r_{\mu\nu}}=\alpha\sqrt{g}g^{\mu\nu}, \tag{2.2}$$

so that for the variation of the Lagrangian (2.1) we have

$$\delta L=\pi_{\mu\nu}\delta g^{\mu\nu}+p^{\mu\nu}\delta r_{\mu\nu}. \tag{2.3}$$

As is well known, $\pi_{\mu\nu}=0$ are the Euler-Lagrange equations for the Lagrangian (2.1), i.e., Einstein field equations.

In dimension 3 there is a two-parameter family of black hole solutions (called BTZ black holes) given by (see [1])

$$g_{\mathrm{BTZ}}=-N^2dt^2+N^{-2}dr^2+r^2(N_\phi dt+d\phi)^2, \tag{2.4}$$

where we set

$$N^2=-\mu+\frac{r^2}{l^2}+\frac{J^2}{4r^2},\qquad \mu=\frac{r_+^2+r_-^2}{l^2},$$
$$N_\phi=-\frac{J}{2r^2},\qquad J=2\,\frac{r_+r_-}{l}. \tag{2.5}$$

We recall that it has been shown in [2] that μ and J are, respectively, the Arnowitt-Deser-Misner (ADM) mass and angular momentum at infinity. One can then apply various methods to compute the conserved quantities and the entropy via Nöther's theorem. In the present paper, we shall summarize and compare various approaches.

III. NÖTHER THEOREM

The Lagrangian (2.1) is covariant with respect to the action of diffeomorphisms of spacetime M. Infinitesimally this is expressed by the following identity which holds for any vector field ξ over M:

$$d(i_\xi L)=\pi_{\mu\nu}\pounds_\xi g^{\mu\nu}+p^{\mu\nu}\pounds_\xi r_{\mu\nu}, \tag{3.1}$$

where $\pounds_\xi$ denotes the Lie derivative operator and i_ξ is the contraction (or inner product) of forms along ξ. By expanding the Lie derivative of the Ricci tensor both field equations and the Nöther conserved current can be found. In fact, by defining

$$u^{\lambda}_{\mu\nu}:=\gamma^{\lambda}_{\mu\nu}-\delta^{\lambda}_{(\mu}\gamma_{\nu)},\qquad \gamma_\mu:=\gamma^{\epsilon}_{\epsilon\mu}, \tag{3.2}$$

the Lie derivative of the Ricci tensor can be expressed as follows:

$$\pounds_\xi r_{\mu\nu}=\nabla_\lambda\pounds_\xi(u^{\lambda}_{\mu\nu}). \tag{3.3}$$

Then we can recast Eq. (3.1) as follows:

$$\mathrm{Div}\,\mathcal{E}(L,\xi)=\mathcal{W}(L,\xi) \tag{3.4}$$

with

$$\mathcal{E}(L,\xi)=(p^{\mu\nu}\pounds_\xi u^{\lambda}_{\mu\nu}-\mathcal{L}\xi^\lambda)\mathbf{ds}_\lambda,$$
$$\mathcal{W}(L,\xi)=-(\pi_{\mu\nu}\pounds_\xi g^{\mu\nu})\mathbf{ds}. \tag{3.5}$$

Here $\mathbf{ds}_\mu=i_{\partial_\mu}\,\mathbf{ds}$ is the standard basis for $(n-1)$-forms; Div denotes the formal divergence operator which acts on forms depending on k derivatives of fields according to the general rule

$$(j^{k+1}\sigma)^*\,\mathrm{Div}(\omega)=d((j^k\sigma)^*\omega), \tag{3.6}$$

where d is the standard differential of forms over M, j^k denotes derivation up to order k and σ denotes any section of the configuration bundle [in our case the bundle is $\mathrm{Lor}(M)$, while k is usually 1 or 2, depending on how many derivatives of g enter ω; recall that g enters the Lagrangian and the theory through its second-order derivatives appearing in the curvature tensor]. For functions one has $\mathrm{Div}\,F=(d_\mu F)dx^\mu$, where the differential operators d_μ are called *total formal derivatives*.

Following the general prescription of [7] and [19], by computing Eq. (3.5) along any configuration σ, we can define the *currents* $\mathcal{E}(L,\xi,g)$ and $\mathcal{W}(L,\xi,g)$ on M. If g is a solution then $\mathcal{W}(L,\xi,g)=0$ and $\mathcal{E}(L,\xi,g)$ is conserved, i.e., it is a closed form on M. Using Bianchi identities in Eq. (3.4) and integrating by parts, we can (*algorithmically*) recast (see [7,11,19]) the current $\mathcal{E}(L,\xi)$ as

$$\mathcal{E}(L,\xi)=\tilde{\mathcal{E}}(L,\xi)+\mathrm{Div}\,\mathcal{U}(L,\xi),$$
$$\tilde{\mathcal{E}}(L,\xi)=2\alpha\sqrt{g}\left(r_{\mu\nu}-\frac{1}{2}rg_{\mu\nu}+\Lambda g_{\mu\nu}\right)g^{\mu\lambda}\xi^\nu\,\mathbf{ds}_\lambda,$$
$$\mathcal{U}(L,\xi)=\alpha\sqrt{g}\nabla^\mu\xi^\nu\,\mathbf{ds}_{\nu\mu}, \tag{3.7}$$

where $\mathbf{ds}_{\nu\mu}=i_{\partial_\mu}\,\mathbf{ds}_\nu$ is the standard basis for $(n-2)$-forms over M. Again the current $\tilde{\mathcal{E}}(L,\xi,g)=(j^2g)^*\tilde{\mathcal{E}}(L,\xi)$ vanishes along solutions of field equations, while $\mathcal{E}(L,\xi,g)-\tilde{\mathcal{E}}(L,\xi,g)=d\mathcal{U}(L,\xi,g)=(j^2g)^*\,\mathrm{Div}\,\mathcal{U}(L,\xi)$ (being exact) is *strongly conserved*, i.e., it is conserved along any configuration g (not necessarily a solution of field equations). For a generic Lagrangian current $\mathcal{U}(L,\xi)$ is known as a *superpotential*; for the Lagrangian (2.1) (with or without cosmological constant), its value (3.7) is also known as the Komar potential (see [20]). Once we specify a covariant Lagrangian L and a vector field ξ on M, a conserved current $\mathcal{E}(L,\xi)$ and a superpotential $\mathcal{U}(L,\xi)$ are defined; moreover, once we specify a configuration g, we can compute them on g obtaining $\mathcal{E}(L,\xi,g)$ and $\mathcal{U}(L,\xi,g)$. Now, let a region D be a compact submanifold with a boundary ∂D which is again a compact submanifold of M; if g is a solution of field equations, the conserved quantity $Q_D(L,\xi,g)$ is defined as the integral over a region D of codimension 1 of the current $\mathcal{E}(L,\xi,g)$, or equivalently as the integral of $\mathcal{U}(L,\xi,g)$ over the boundary ∂D of D because of Eq. (3.7). In applications one usually chooses D to be a spacelike slice of the ADM splitting such that ∂D is (a branch of) the spatial infinity. We stress that all quantities defined in Eq. (3.7) are $\mathbb{R}$-linear with respect to the vector field ξ. We also stress that ξ is by no means required to be a Killing vector of the metric g.

Reverting to the BTZ solution (2.4), notice that the integral over the "sphere" S^1_r of radius r of the superpotential (3.7) for the vector fields ∂_t and ∂_ϕ gives, respectively:

$$Q(L,\partial_t,g_{\mathrm{BTZ}})=\int_{S^1_r}\mathcal{U}(L,\partial_t,g_{\mathrm{BTZ}})=4\pi\alpha\frac{r^2}{l^2},$$
$$Q(L,\partial_\phi,g_{\mathrm{BTZ}})=\int_{S^1_r}\mathcal{U}(L,\partial_\phi,g_{\mathrm{BTZ}})=-2\pi\alpha J. \tag{3.8}$$

The *mass* $Q(L,\partial_t,g_{\mathrm{BTZ}})$ diverges as one considers the limit $r\to\infty$; this is a problem analogous to the well-known *anomalous factor problem* which the Komar potential is known to be affected by (see [12]). In other words the integral of the superpotential does not give the correct mass, though it gives the expected *angular momentum* $Q(L,\partial_\phi,g_{\mathrm{BTZ}})$. Here the anomalous factor problem is even worse than for Kerr-Newman metrics or for other asymptotically flat stationary solutions (see [8–10]). In fact, while there it was just a matter of a wrong factor to be corrected, here it is primarily a divergence to be cured. This divergence is typically due to the fact that the BTZ solution is an asymptotically anti–de Sitter spacetime and the magnitude of the timelike vector field ∂_t diverges as it approaches infinity.

There are (at least) two different possibilities for overcoming this situation. First, one may define the *total conserved quantities* without any reference to their *densities*. On the other hand, one can define the conserved quantity *in a region D* as the integral over D of its density; then the total conserved quantity is obtained by taking the limit to the whole spacelike slice, provided that the method converges to a finite result.

IV. THE TOTAL CONSERVED QUANTITIES

Let us associate to any vertical vector field $X=(\delta g^{\mu\nu})\partial/\partial g^{\mu\nu}$ over the bundle $\mathrm{Lor}(M)$, i.e., for any variation $\delta g^{\mu\nu}$ of the (inverse) metric, an $(n-1)$-form:

$$\mathbb{F}(L,g)[X]=\alpha(g^{\lambda\rho}g_{\mu\nu}-\delta^{\lambda}_{(\mu}\delta^{\rho}_{\nu)})\nabla_\rho(\delta g^{\mu\nu})\sqrt{g}\,\mathbf{ds}_\lambda\,, \tag{4.1}$$

where the section g of $\mathrm{Lor}(M)$, i.e., a Lorentzian metric, is not required to be a solution of field equations. The correspondence $\mathbb{F}(L,g)$ is called *Poincaré-Cartan morphism*. Recalling (see [21], and references quoted therein) that Lie derivatives $\pounds_\xi g$ of sections g can be interpreted as vertical vectors over $\mathrm{Lor}(M)$

$$\pounds_\xi g=(\nabla^\mu\xi^\nu+\nabla^\nu\xi^\mu)\frac{\partial}{\partial g^{\mu\nu}}, \tag{4.2}$$

one can rewrite the current (3.5) as

$$\mathcal{E}(L,\xi,g)=\mathbb{F}(L,g)[\pounds_\xi g]-i_\xi L. \tag{4.3}$$

The form $\mathbb{F}(L,g)$ is irrelevant to field equations, because it enters a divergence once one integrates by parts Eq. (2.3) and uses Eq. (3.1). However, it is tightly related to conserved quantities and the Nöther theorem. These are in fact general features of any field theory (see [7,11], and references quoted therein for the general framework).

The variation of the total conserved quantity is

$$\begin{aligned}\delta_X Q_D(L,\xi,g)&=\int_D\delta_X\mathcal{E}(L,\xi,g)\\&=\int_D\delta_X(\mathbb{F}(L,g)[\pounds_\xi g])-\int_D\pounds_\xi(\mathbb{F}(L,g)[X])\\&\quad+\int_{\partial D}i_\xi(\mathbb{F}(L,g)[X]),\end{aligned} \tag{4.4}$$

which suggests to us to define the variation of the corrected conserved quantities by the prescription

$$\delta_X\hat{Q}_D(L,\xi,g)=\int_{\partial D}\{\delta_X\mathcal{U}(L,\xi,g)-i_\xi(\mathbb{F}(L,g)[X])\}. \tag{4.5}$$

For the example under investigation, using expressions (3.1) and (4.1), we get

$$\begin{aligned}\delta_X\hat{Q}_D(L,\partial_t,g_{\mathrm{BTZ}})&=2\pi\alpha\,\delta\mu,\\ \delta_X\hat{Q}_D(L,\partial_\phi,g_{\mathrm{BTZ}})&=-2\pi\alpha\,\delta J,\end{aligned} \tag{4.6}$$

which can be integrated to give the total conserved quantities

$$\hat{Q}_D(L,\partial_t,g_{\mathrm{BTZ}})=2\pi\alpha\mu,\quad \hat{Q}_D(L,\partial_\phi,g_{\mathrm{BTZ}})=-2\pi\alpha J. \tag{4.7}$$

We see that this method provides directly the total conserved quantities (as already calculated in [1]) and no extra data are needed other than the solution and the Lagrangian.

In other words, the relevant quantity to replace the Komar potential is $\delta_X \mathcal{U}(L,\xi,g) - i_\xi(\mathbb{F}(L,g)[X])$. This quantity is uneffected by the addition of divergences to the Lagrangian. In fact, if one considers a Lagrangian $L' = \mathrm{Div}(\beta)$, which is a total divergence, one easily obtains $\delta_X \mathcal{U}(L',\xi,g) - i_\xi(\mathbb{F}(L',g)[X]) = 0$ identically.

As a second alternative approach, one can instead look for a $(n-2)$ form over M which can be integrated over the boundary of a region D to give directly the conserved quantity in that region. This approach relies on the formal integration of Eq. (4.5). To perform this task one has to specify some extra information. First, one fixes some boundary conditions, e.g., usually one requires that field variations vanish on the boundary ∂D. Then one seeks for a current $\mathcal{B}(L,\xi)$ such that, once we set, as usual, $\mathcal{B}(L,\xi,g)$ for the pullback of $\mathcal{B}(L,\xi)$ along a section g, the following holds:

$$\delta_X \mathcal{B}(L,\xi,g)|_{\partial D} = i_\xi(\mathbb{F}(L,g)[X])|_{\partial D}. \tag{4.8}$$

Usually there is no such (global and covariant) current $\mathcal{B}(L,\xi)$. For example, for standard general relativity one has

$$i_\xi(\mathbb{F}(L,g)[X]) = \delta(p^{\mu\nu} u^\lambda_{\mu\nu}\, \xi^\sigma \mathbf{ds}_{\lambda\sigma}), \tag{4.9}$$

which does not lead to a possible choice for $\mathcal{B}(L,\xi)$ because $p^{\mu\nu} u^\lambda_{\mu\nu}\, \xi^\sigma \mathbf{ds}_{\lambda\sigma}$ is not covariant, i.e., it is not a form on the bundle since $u^\lambda_{\mu\nu}$ is not a tensor. To overcome this problem one has to fix some ''coherent'' background connection $\Gamma^\lambda_{\mu\nu}$ (which is assumed to be uneffected by deformations) and define

$$\mathcal{B}(L,\xi) = p^{\mu\nu} w^\lambda_{\mu\nu} \xi^\sigma\, \mathbf{ds}_{\lambda\sigma}, \quad \begin{cases} w^\lambda_{\mu\nu} = u^\lambda_{\mu\nu} - U^\lambda_{\mu\nu}, \\ U^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \delta^\lambda_{(\mu}\Gamma_{\nu)}, \\ \Gamma_\nu = \Gamma^\lambda_{\nu\lambda}. \end{cases} \tag{4.10}$$

In this way the correction term $\mathcal{B}(L,\xi)$ is covariant; one can recast Eq. (4.5) as

$$\begin{aligned} \delta_X \hat{Q}_D(L,\xi,g) &= \int_{\partial D} \{\delta_X \mathcal{U}(L,\xi,g) - i_\xi(\mathbb{F}(L,g)[X])\} \\ &= \int_{\partial D} \delta_X[\mathcal{U}(L,\xi,g) - \mathcal{B}(L,\xi,g)], \end{aligned} \tag{4.11}$$

that can be formally integrated giving

$$\hat{Q}_D(L,\xi,g) = \int_{\partial D} [\mathcal{U}(L,\xi,g) - \mathcal{B}(L,\xi,g)]. \tag{4.12}$$

We stress that in order to construct a formula like Eq. (4.12) a background connection is needed (or some other ''globalizing'' tool). The new conserved quantities $\hat{Q}_D(L,\xi,g)$ depend both on the solution $g_{\mu\nu}$ *and* on the background connection $\Gamma^\lambda_{\mu\nu}$. Then, they have to be interpreted as the *relative* conserved quantities with respect to $\Gamma^\lambda_{\mu\nu}$. The physical importance of some background for the theory of conserved quantities was already recognized in the literature; see, for example, [3,17,22]. As an example, it has been proved (see [10,23]) that if one analyzes the (charged) Kerr-Newmann solutions, there exist suitable backgrounds which provide *reasonable* mass densities which when integrated on the boundary of a spacelike slice (i.e., on spatial infinity) give the correct total mass.

Of course the choice of $\mathcal{B}(L,\xi)$ is not unique. In particular, and for the sake of simplicity, the background $\Gamma^\lambda_{\mu\nu}$ can be assumed to be the Levi-Civita connection of some background metric $h_{\mu\nu}$ (also considered to remain unchanged under deformations). Then one can add a term which depends just on the background $(h_{\mu\nu}, \Gamma^\lambda_{\mu\nu})$, which, being unchanged under deformations, does not effect Eq. (4.8). To fix such a term one can reasonably require that, if $h_{\mu\nu}$ is also a solution of field equations (as it seems physically and mathematically reasonable to require), the *relative* conserved quantities of the background with respect to itself vanish. This amounts to redefining the correction as follows:

$$\tilde{\mathcal{B}}(L,\xi) = [p^{\mu\nu} w^\lambda_{\mu\nu} \xi^\rho + \alpha \sqrt{h} h^{\alpha\rho} \overset{(h)}{\nabla}_\alpha \xi^\lambda] \mathbf{ds}_{\lambda\rho}, \tag{4.13}$$

where $\nabla^{(h)}$ denotes the covariant derivative induced by the background *h*. Generally speaking, the correction (4.13) may be used also in asymptotically flat solutions when, however, there is a preferred vacuum ($h = \eta =$ Minkowski metric) which, being flat, reduces Eq. (4.13) to the simpler correction (4.10).

One can also *derive* both the corrected superpotentials $\mathcal{U} - \mathcal{B}$ and $\mathcal{U} - \tilde{\mathcal{B}}$ as (uncorrected) superpotentials of some suitable Lagrangian. In particular $\mathcal{U} - \mathcal{B}$ is the superpotential for the *first-order Lagrangian* for standard general relativity (see [10])

$$L_1 = [\alpha(r - 2\Lambda)\sqrt{g} - d_\lambda(p^{\mu\nu} w^\lambda_{\mu\nu})]\mathbf{ds}, \tag{4.14}$$

where d_λ denotes the formal divergence while $\mathcal{U} - \tilde{\mathcal{B}}$ is the superpotential for the equivalent Lagrangian

$$\tilde{L}_1 = [\alpha(r - 2\Lambda)\sqrt{g} - d_\lambda(p^{\mu\nu} w^\lambda_{\mu\nu}) - \alpha(R - 2\Lambda)\sqrt{h}]\mathbf{ds}, \tag{4.15}$$

where R is the scalar curvature of the background *h*. In both cases the background has to be considered as a *parameter* so that ξ has to be a Killing vector of the background (see [10]). We remark that both Lagrangians (4.14) and (4.15) induce a well-defined action functional for a variational principle based on fixing the metric on the boundary. We stress that, in general, these actions differ for surface terms from that used in [3].

For the Lagrangian (2.1) and the solution (2.4) one can obviously choose as a background another metric of the same type (2.4) with fixed values (μ_0, J_0) as parameters. By a direct computation we find for the corrected superpotential $\mathcal{U} - \tilde{\mathcal{B}}$ the conserved quantity

$$\hat{Q}(L,\partial_t,g_{\mathrm{BTZ}})=2\pi\alpha(\mu-\mu_0),$$
$$\hat{Q}(L,\partial_\phi,g_{\mathrm{BTZ}})=-2\pi\alpha(J-J_0). \tag{4.16}$$

Here the Komar potential of the background in Eq. (4.13) is essential in order to *cure* the quadratic divergence of the Komar potential of the solution. This background fixing contains, as particular cases, the backgrounds usually adopted in the literature (see, for example, [1–3]). In particular, the limit ($\mu_0\to 0$, $J_0\to 0$) corresponds to the vacuum state in which the black hole disappears. Another allowed choice analyzed in the literature (see [3]) is the anti–de Sitter spacetime which corresponds to the different limit ($\mu_0\to -1$, $J_0\to 0$).

V. ENTROPY

The entropy of a (black hole) solution is defined to be a quantity that satisfies the first principle of thermodynamics:

$$\delta_X\mu=T\,\delta_X\mathcal{S}+\Omega\,\delta_X\mathcal{J} \tag{5.1}$$

for any variation X tangent to the space of solutions, i.e., X has to satisfy linearized field equations. Here T and Ω are constant quantities with respect to variations δ_X (namely they are related to the unperturbed solution). Usually they are assumed to represent the temperature and angular velocity of the horizon of the black hole, so that $\mathcal{S}$ can be interpreted as the physical entropy of the system. The physical value of these parameters has to be provided by physical arguments, since they are almost undetermined in the present context (see [1,4,5,24,25]). Of course, one can compute one of them out of the others by requiring that Eq. (5.1) is integrable so that there exists a state function $\mathcal{S}$ to fulfill the first principle of thermodynamics. However, other parameters have to be provided by physical arguments (e.g., T has to be the temperature of Hawking radiation). The ultimate meaning of the work terms in Eq. (5.1) is that, for example, $\Omega\delta\mathcal{J}$ is the change in the total mass along an isoentropic transformation. It has been shown elsewhere (see [1,4,5]) that, in order to make this true, Ω has to be the angular velocity of the horizon. Further terms may in general appear in Eq. (5.1) due to gauge charges. Since the example which is here under consideration has no further gauge symmetries, we do not consider these further contributions.

Of course, entropy should also satisfy further requirements (e.g., the second principle of thermodynamics). However, these additional requirements are generally out of control so that the first principle is what one usually requires with the hope of checking the second principle afterwards.

By solving Eq. (5.1) with respect to $\delta_X\mathcal{S}$ and by setting $\xi=\partial_t+\Omega\,\partial_\phi$ one finds (see [11])

$$\delta_X\mathcal{S}=\frac{1}{T}[\,\delta_X\mu-\Omega\,\delta_X\mathcal{J}]$$
$$=\frac{1}{T}\int_\infty(\,\delta_X\mathcal{U}(L,\xi,g)-i_\xi(\mathbb{F}(L,g)[X])), \tag{5.2}$$

where ∞ denotes the spatial infinity of a spacelike slice. Now one can prove under quite general hypotheses (basically just requiring ξ to be a Killing vector of the solution g) that the integrand quantity $\delta_X\mathcal{U}(L,\xi,g)-i_\xi(\mathbb{F}(L,g)[X])$ is a closed form, so that its integral does not depend on the integration domain but just on its homology.

Then one can define the following quantity:

$$\delta_X\mathcal{S}=\frac{1}{T}\int_\Sigma(\,\delta_X\mathcal{U}(L,\xi,g)-i_\xi(\mathbb{F}(L,g)[X])), \tag{5.3}$$

where Σ is any spatial surface such that $\infty-\Sigma$ does not enclose any singularity (in homological notation it is a boundary). Then Eq. (5.3) can be integrated to give a quantity $\mathcal{S}$ which [because of Eq. (5.2)] satisfies the first principle of thermodynamics and which is then a natural candidate to be interpreted as the entropy. We remark that we do not need anything but a one-parameter family of solutions $g^{\epsilon}_{\mu\nu}$ and a Killing vector ξ for the unperturbed solution $g^0_{\mu\nu}$ (here and everywhere we denote by X the infinitesimal generator of the family, which is a solution of the linearized field equations). In particular, differently from [13,14], and [15], we do not require anything about the maximality of the solution under consideration, anything about horizons, and anything about the vanishing of ξ on horizons (see [11,13]). This latter remark is particularly important for the actual calculations because, as we shall see, it simplifies considerably (both conceptually and computationally) the expressions involved.

Let now κ denote the surface gravity so that $T=\kappa/(2\pi)$ is the temperature of the Hawking radiation of BTZ, as shown in [4,24] by means of Euclidean path integrals. Let us set $\Omega=-N_\phi(r_+)$ which can be shown to be the angular velocity of the BTZ horizon. We remark that because of the value of Ω the Killing vector ξ becomes null on the horizon $\mathcal{N}:(r=r_+)$. We then easily get

$$\delta_X\mathcal{S}=8\pi^2\alpha\,\delta r_+\,, \tag{5.4}$$

which in turn gives

$$\mathcal{S}=8\pi^2\alpha r_+ \tag{5.5}$$

for the entropy.

The original Wald's recipe for the entropy needs to choose a particular $\bar{\Sigma}$ (the bifurcation surface) on which the Killing vector ξ, and then the whole term $i_\xi(\mathbb{F}(L,g)[X])$ in Eq. (5.3), vanishes. Of course the existence of such a surface is ensured just in the maximal extension of the solution. [For example, for the Schwarzschild solution this surface $\bar{\Sigma}$ corresponds to the surface $U=0$, $V=0$ in Kruskal coordinates. Notice that the two-sphere $U=V=0$ is not covered by spherical coordinates (t,r,θ,ϕ) or outgoing (nor ingoing) coordinates. To be more precise, if one considers any cross section $t=t_0$ and $r=2m$ of the horizon, then ξ vanishes on none of it for any value of t_0. Thus Kruskal coordinates are needed in a somehow ''essential'' way to apply Wald's recipe.]

In order to apply the original Wald's recipe to the BTZ solution, one should first build Kruskal-type coordinates as shown in [26]. Once one has verified that κ is a nonvanishing constant on the horizon (which is false in the extreme case), this can be done in two steps. First of all we define Eddington-Finkelstein coordinates (u,ρ,φ) which are the parameters along the flows of the vector fields (ξ,ζ,X) we are going to define. The vector field $\xi=\partial_t+\Omega\partial_\phi$ is the Killing vector whose Killing horizon $\mathcal{N}$ has to be extended; u is the parameter along its flow; X is a vector field tangent to $\mathcal{N}$ such that $X^\lambda\nabla_\lambda u=0$. Finally, ζ is a vector field such that the following conditions have to be satisfied:

$$\begin{aligned}&\xi^\lambda\nabla_\lambda u=1,\\ &g(\zeta,\xi)=1 \quad |\zeta|^2=0 \ \text{ on } \ \mathcal{N},\\ &g(\zeta,X)=0.\end{aligned} \tag{5.6}$$

In these new coordinates the vector fields (ξ,ζ,X) read as

$$\xi=\frac{\partial}{\partial u},\qquad \zeta=\frac{\partial}{\partial\rho},\qquad X=\frac{\partial}{\partial\varphi}. \tag{5.7}$$

Notice that even in Eddington-Finkelstein coordinates the vector field ξ does not vanish anywhere in the chart domain. The BTZ metric reads as

$$\begin{aligned}g_{\mathrm{BTZ}}=&-f(r(\rho))du^2+2\,du\,d\rho+\Phi(r(\rho))du\,d\varphi\\ &+\Psi(r(\rho))d\varphi^2,\end{aligned} \tag{5.8}$$

where we set

$$\begin{aligned}f(r)&=N^2-r^2(N_\phi+\Omega)^2,\\ \Phi(r)&=\frac{N^2-r^2(N_\phi^2-\Omega^2)}{\Omega},\\ \Psi(r)&=-\frac{N^2-r^2(N_\phi-\Omega)^2}{4\Omega^2}.\end{aligned} \tag{5.9}$$

Here $r(\rho)$ is obtained by inverting the change of coordinates, namely

$$\rho(r)=\int_{r_+}^r\frac{dr}{F(r)},\qquad F(r)=\sqrt{\frac{\Psi(r)}{r^2}}. \tag{5.10}$$

We remark that $F(r)$ is well defined in a neighborhood of the horizon $\mathcal{N}$. Now we can define Kruskal-type coordinates (U,V,φ) as

$$\begin{aligned}U&=e^{\kappa u},\\ V&=-\rho e^{-\kappa u}\exp\left(2\kappa\int_0^\rho H(\rho)d\rho\right),\end{aligned} \tag{5.11}$$

$$H(\rho)=\frac{1}{f(r(\rho))}-\frac{1}{2\kappa\rho}.$$

The product UV depends just on ρ and can be regarded as an implicit definition of ρ as a function $\tilde\rho(UV)$. Then the BTZ metric reads as

$$\begin{aligned}g_{\mathrm{BTZ}}=&G(UV)dU\,dV+\frac{\Phi(r(\tilde\rho))}{\kappa U}dU\,d\varphi\\ &+\Psi(r(\tilde\rho))d\varphi^2,\\ G(UV)=&\frac{f(r(\tilde\rho))}{\kappa^2UV}.\end{aligned} \tag{5.12}$$

In these coordinates the Killing vector ξ reads as

$$\xi=-\kappa\left(V\frac{\partial}{\partial V}-U\frac{\partial}{\partial U}\right), \tag{5.13}$$

which finally vanishes for $\bar\Sigma:(U=V=0)$.

Thus we have extended the Killing horizon $\mathcal{N}$ to a bifurcate Killing horizon and we have identified the bifurcate surface $\bar\Sigma$. Now one can ''easily'' compute the entropy:

$$\begin{aligned}&\delta_X\mathcal{S}=\frac{1}{T}\int\delta_X\mathcal{U}[\xi]_3d\varphi\underset{U=V=0}{\overset{\rho=0}{\Rightarrow}}\delta_X\mathcal{S}=8\pi^2\alpha\,\delta r_+\\ &\Rightarrow\mathcal{S}=8\pi^2\alpha r_+,\end{aligned} \tag{5.14}$$

where $\mathcal{U}[\xi]_3\,d\varphi$ is the angular part of the superpotential 1 form $\mathcal{U}[\xi]$.

We stress that the first expression in Eq. (5.14) for $\delta_X\mathcal{S}$ is meaningful just on the bifurcate surface where we can ignore the contribution of the term $i_\xi(\mathbb{F}(L,g)[X])$. We stress moreover that the above method fails in the extreme case. In fact, in this case $\kappa=0$ and the derivation of the Kruskal-type coordinates fails at Eqs. (5.11) and (5.12) [notice that in this case $f(r)$ identically vanishes]. On the contrary, as we proved [see Eq. (5.4)] the second expression in Eq. (5.14) for $\delta_X S$ is correct on any surface Σ. In this way the entropy is not related directly to a quantity computed on the horizon (see [17] for a discussion of entropy of Taub-NUT solutions). The computations of this section have been carried out by using the MAPLE V and TENSOR package.

VI. CONCLUSION AND PERSPECTIVES

We have determined and discussed the entropy of BTZ solutions. In our framework the entropy (5.2) is clearly related, by its own definition, to Nöther charges by the first principle of thermodynamics; actually our proposal is to determine *a priori* exactly the quantity that satisfies the first principle of thermodynamics, so that it can be, *a posteriori*, physically interpreted as the entropy.

At first, entropy is a quantity computed at spatial infinity (5.2), as all conserved quantities are. Then one can compute it *also* by an integral on the finite regions, provided that the Nöther generator ξ is a Killing vector of the solution under consideration [see Eq. (5.3)]. Finally, if the surface gravity κ does not vanish on the horizon, one can extend such a horizon to a bifurcate Killing horizon and compute integrals on the bifurcate Killing surface on which ξ vanishes [see Eq. (5.14)]. This latter step is completely useless and computationally boring in applications as well as in the theoretical framework, as we hope to have shown in the present paper. Furthermore, our framework, being intrinsically and geometrically formulated in a global setting, is in fact valid for a much larger class of theories, namely all field theories with a gauge invariance, the so-called *gauge-natural theories* (see [11]). For the same reasons, no requirements on signature and/or dimension of spacetime are needed, as our framework relies only on globality and covariance of the Lagrangian. We remark that in the original Wald's procedure, $\kappa \neq 0$ is used for two different purposes, namely to compute the temperature and to prove the existence of the bifurcation surfaces. Nonextremality is essential to the second issue. Since, in the extreme case, the construction of Kruskal-like coordinates and bifurcation surfaces break down (as Wald himself noticed), we believe that because of this there is little hope to treat the extreme case through any approaches based on bifurcation surfaces. We instead believe that our approach, which does not use a bifurcation surface, is in a good position to treat the extremal cases, too. Clearly, the extreme cases have to be discussed separately and we hope to address the problem in a forthcoming paper. As we already noted in the Introduction, paper II will revisit the above results in the light of *BCEA theories* (see [5] and [18]).

Future investigations will be devoted to those cases (e.g., Taub-NUT solutions) in which Wald's prescription for entropy cannot apply at all, as noticed by Wald himself (see [26]) and other authors (see also [17]). In these cases, Wald's prescription fails because of various reasons, e.g., because the orbits of timelike vectors are closed and extra contributions to the entropy are due to singularities other than those enclosed in Killing horizons (in particular the *Misner string*). Both these reasons prevent the application of the latter prescription. Since our general prescription does not require the existence of a bifurcate Killing surface, it allows us to overcome these problems, in Taub-NUT solutions. Results will be published in [16], where they will be shown to agree with those found in [17] by another formalism.

We finally remark that our formalism is hopefully in a good position to be generalized to nonstationary black holes, since extra contributions due to nonstationarity seem to be under control.

ACKNOWLEDGMENTS

We are grateful to I. Volovich for having long ago drawn our attention to the entropy formula of Wald, as well as to R. Mann for having directed our attention to the BTZ solution and for useful discussions about it.

[1] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992); M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D **48**, 1506 (1993).
[2] D. Cangemi, M. Leblanc, and R. B. Mann, Phys. Rev. D **48**, 3606 (1993).
[3] J. D. Brown, J. Creighton, and R. B. Mann, Phys. Rev. D **50**, 6394 (1994).
[4] S. Carlip and C. Teitelboim, Phys. Rev. D **51**, 622 (1995).
[5] S. Carlip, J. Gegenberg, and R. B. Mann, Phys. Rev. D **51**, 6854 (1995).
[6] A. Trautman, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), p. 168.
[7] M. Ferraris and M. Francaviglia, in *Mechanics, Analysis and Geometry: 200 Years after Lagrange*, edited by M. Francaviglia (Elsevier, Amsterdam, 1991).
[8] M. Ferraris and M. Francaviglia, Atti Semin. Mat. Fis. Univ. Modena **37**, 61 (1989).
[9] M. Ferraris, M. Francaviglia, and I. Sinicco, Nuovo Cimento B **107**, 1303 (1992).
[10] M. Ferraris and M. Francaviglia, Gen. Relativ. Gravit. **22**, 965 (1990).
[11] L. Fatibene, M. Ferraris, M. Francaviglia, and M. Raiteri, Ann. Phys. (N.Y.) **275**, 27 (1999).
[12] J. Katz, Class. Quantum Grav. **2**, 423 (1985).
[13] V. Iyer and R. Wald, Phys. Rev. D **50**, 846 (1994).
[14] R. M. Wald, J. Math. Phys. **31**, 2378 (1993).
[15] B. S. Kay and R. M. Wald, Phys. Rep. **207**, 49 (1991).
[16] L. Fatibene, M. Ferraris, M. Francaviglia, and M. Raiteri, ''On the Entropy of Taub-NUT Black Hole Solutions,'' Ann. Phys. (N.Y.) (to be published).
[17] S. W. Hawking, C. J. Hunter, and D. N. Page, Phys. Rev. D **59**, 044033 (1999); S. W. Hawking and C. J. Hunter, *ibid.* **59**, 044025 (1999); C. J. Hunter, *ibid.* **59**, 024009 (1999); S. W. Hawking and C. J. Hunter, Class. Quantum Grav. **13**, 2735 (1996); S. W. Hawking and G. T. Horowitz, *ibid.* **13**, 1487 (1996); D. N. Page, Phys. Lett. **78B**, 249 (1978).
[18] S. Carlip and J. Gegenberg, Phys. Rev. D **44**, 424 (1991).
[19] M. Ferraris, M. Francaviglia, and O. Robutti, in *Géométrie et Physique*, Proceedings of the Journées Relativistes, 1985 (Marseille, 1985), edited by Y. Choquet-Bruhat, B. Coll, R. Kerner, and A. Lichnerowicz (Hermann, Paris, 1987), pp. 112–125.
[20] A. Komar, Phys. Rev. **113**, 934 (1959).
[21] I. Kolář, P. W. Michor, and J. Slovák, *Natural Operations in Differential Geometry* (Springer-Verlag, New York, 1993).
[22] J. D. Brown and J. W. York, Phys. Rev. D **47**, 1407 (1993).
[23] M. Ferraris and M. Francaviglia, in *8th Italian Conference on General Relativity and Gravitational Physics*, Cavalese (Trento) (World Scientific, Singapore, 1988).
[24] R. B. Mann, hep-th/9903229; hep-th/9904148.
[25] J. D. Brown, Phys. Rev. D **52**, 7011 (1995).
[26] I. Racz and R. M. Wald, Class. Quantum Grav. **9**, 2643 (1992).