Late-time decay of gravitational and electromagnetic perturbations along the event horizon

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(Received 11 May 1999; published 15 November 1999)

We study analytically, via the Newman-Penrose formalism, the late-time decay of linear electromagnetic and gravitational perturbations along the event horizon (EH) of black holes. We first analyze in detail the case of a Schwarzschild black hole. Using a straightforward local analysis near the EH, we show that, generically, the ''ingoing'' ($s>0$) component of the perturbing field dies off along the EH more rapidly than its ''outgoing'' ($s<0$) counterpart. Thus, while along $r=\text{const}>2M$ lines both components of the perturbation admit the well-known t^{-2l-3} decay rate, one finds that along the EH the $s<0$ component dies off in advanced time v as v^{-2l-3}, whereas the $s>0$ component dies off as v^{-2l-4}. We then describe the extension of this analysis to a Kerr black hole. We conclude that for axially symmetric modes the situation is analogous to the Schwarzschild case. However, for non-axially symmetric modes both $s>0$ and $s<0$ fields decay at the same rate (unlike in the Schwarzschild case). [S0556-2821(99)03422-0]

PACS number(s): 04.70.Bw, 04.25.Nx

I. INTRODUCTION

When a gravitational collapse results in the formation of a black hole (BH), the gravitational field outside the event horizon (EH) relaxes at late time to the stationary Kerr-Newman geometry. Also, when the (pure) Kerr-Newman field external to a BH is perturbed by gravitational or electromagnetic waves, the perturbing field dies off at late time everywhere outside the BH, and along its EH. In both scenarios, it is implied by the ''*no hair*'' principle that when the BH geometry settles down into its stationary state, all characteristics of the initial state (or initial perturbation) must somehow be lost, except for the conserved quantities associated with it: its total mass, electric charge, and angular momentum. (For a detailed review of the ''no hair'' theorems by Hawking, Israel, Carter, and Robinson, see [1].)

Remarkable as the ''no-hair'' principle is, it still gives no information about the mechanism through which this ''compulsory'' relaxation process occurs. For example, it tells us nothing about the rate of the decay process. Clearly, such a detailed description of the late time decay is important not only for gaining more insight into the ''no hair'' principle, but, more practically, by virtue of the recent prospects of detecting gravitational radiation from astrophysical black hole systems. Also, the characteristics of the decay along the event horizon has an impact on the nature of the singularity along the inner horizon of charged [2] and rotating [3] black holes.

A detailed description of the late time decay outside Schwarzschild black holes was first given by Price (for scalar and metric perturbations [4], and for all integer-spin fields in the Newman-Penrose formalism [5]). Price found that any radiative multipole mode l,m of an initially compact linear perturbation dies off at late time as t^{-2l-3} (where t is the Schwarzschild time coordinate). If a static multipole mode existed prior to the formation of the BH, then this mode will decay as t^{-2l-2}. Price found these power law decay tails to be the same for all kinds of perturbations, whether scalar, electromagnetic or gravitational (and in this respect, the scalar field model proved a useful toy model for the more realistic fields).

Price's results were later confirmed using several different approaches, both analytic and numerical [6–11], and where generalized to other spherically symmetric spacetimes [7,12–17]. The validity of the perturbative (linear) approach was supported by numerical analyses of the fully nonlinear dynamics [18,14], indicating virtually the same power law indices for the late-time decay.

Recently, several authors addressed the issue of the late-time decay of fields outside rotating black holes. First, a numerical simulation of the evolution of linear scalar [19] and gravitational [20] waves on the background of a Kerr black hole was carried out by Krivan *et al*. Later, an analytic treatment of this problem (in the time domain) was presented by Barack and Ori [21–23] (following a preliminary analysis by Ori [24]). Then, a study of the late-time decay in Kerr using a frequency-domain approach has been carried out by Hod, both for a scalar field [25] and for nonzero-spin Newman-Penrose fields [26] (following preliminary considerations by Andersson [9]).

The above analyses all indicate that power law tails characterize the decay in the Kerr background as well. In this case, however, the lack of spherical symmetry causes coupling between various multipoles. As a result of this coupling, the power-law indices of specific spherical-harmonics multipoles are found to be different, in general, from the ones obtained in spherically symmetric black holes. Another phenomenon caused by rotation (first observed in [24]) is the oscillatory nature of the late time tails along the null generators of the EH of the Kerr BH for nonaxially symmetric perturbation modes. (See [22,23] for details.)

As we just mentioned, power law tails are observed not only at timelike infinity, but also at future null infinity and along the (future) event horizon. Several authors have analyzed the late time behavior of a scalar field along the EH of a Schwarzschild BH [7,11] and a Kerr BH [22,25]. In both cases, the power law indices of the late-time decay along the EH were found to be the same as along any fixed-r world line outside the BH (apart from the above mentioned oscillations along the EH in the Kerr case). Thus, in the Schwarzschild case, an l,m scalar perturbation mode is found to decay along the EH as v^{-2l-3} (or v^{-2l-2} for an initially static mode). (Here, v is an advanced-time coordinate, which we define in the sequel.)

Quite surprisingly, a careful and thorough study of the behavior of realistic physical fields (electromagnetic and gravitational) along the EH has not been carried out so far (to the best of our knowledge), even in the Schwarzschild case [27]. One would expect the scalar-field model to provide, again, a reliable picture of the actual behavior of realistic physical fields; however, a careful analysis of the behavior of such realistic fields at the EH reveals several interesting new features, uncovered by the scalar-field case. These features arise already in the Schwarzschild case, and thus we find it instructive to study and explain this simpler case first. Accordingly, in this paper we first explore in detail the behavior of electromagnetic and gravitational perturbations at the EH of a Schwarzschild BH. Then we describe the extension of this analysis to the Kerr case, and derive the power-law indices at the EH. Full detail of the analysis of the Kerr case will be given in a forthcoming paper [23] (as part of a comprehensive analysis of the late time decay of perturbations in the Kerr spacetime).

We shall apply a linear perturbation analysis, based on the Newman-Penrose formalism. In this framework, a single master equation governs the (gauge-invariant) radiative parts of the linear perturbations of both the Maxwell tensor and the Weyl tensor. For both fields, our analysis reveals that the ''ingoing'' ($s>0$) part of the perturbing field dies off at late time along the EH of the Schwarzschild BH *faster* than its ''outgoing'' ($s<0$) counterpart: Whereas the $s<0$ fields admit the usual v^{-2l-3} law, the $s>0$ fields decay at the EH like v^{-2l-4}. In the Kerr case, the above difference in the behavior of the $s>0$ and $s<0$ fields occurs only for axially symmetric ($m=0$) modes; for non-axially symmetric modes, one finds the same decay rates for both $s>0$ and $s<0$. These results are summarized in Eqs. (95), (96), and (97) below, in the concluding section. We also comment there about the significance of our results to the study of the interior of spinning black holes, and discuss the relation of our analysis to previous works [27].

An important role in our analysis is played by the static solutions of the field equation. These turn out to show a peculiarity: As in the scalar field case, there is a static solution regular at the horizon, and a second, independent, solution which is irregular there. However, for $s>0$ fields, regularity of a static solution cannot be judged merely from its leading-order behavior at the EH. Rather, the distinction between the regular and irregular solutions involves the identification of a certain, sub-dominant, logarithmic term in the latter. Another peculiarity has to do with the relation between static solutions and monochromatic solutions (i.e., modes of a single Fourier frequency ω). For $s>0$ fields, unlike the scalar field (and unlike the $s<0$ case), the EH-regular static solution cannot be approached from an EH-regular monochromatic solution by naively taking the limit $\omega\to 0$. One finds that for $s>0$ this naive limit leads to a static solution *irregular* at the EH. We study these unexpected features, and then qualitatively explain them using a simple (scalar-field based) toy model.

The paper is arranged as follows: In Sec. II we give some definitions and notations, and briefly review the Newman-Penrose formalism for perturbations of the Schwarzschild geometry. In Sec. III we introduce the *late time expansion*, to be employed in our analysis. The static solutions to the field equation, central to our analysis, are obtained in Sec. IV, followed (in Sec. V) by a formulation of regularity criteria for physical fields at the EH. This puts us in position to analyze (in Sec. VI) the late time behavior of physical fields along the EH. This analysis yields the power index for both $s<0$ and $s>0$ fields. Another perspective on the subject is obtained in Sec. VII, where we consider the behavior of momochromatic modes. In Sec. VIII we then introduce a simple toy model, which yields further insight into our results. The extension of our analysis to the case of a Kerr BH is described in Sec. IX. In the concluding section (Sec. X) we summarize the results and discuss their physical significance and their relation to other works.

II. DEFINITIONS AND NOTATIONS

The line element in the Schwarzschild spacetime reads, in the standard Schwarzschild coordinates t,r,θ,φ,

$$ds^2=-(\Delta/r^2)dt^2+(r^2/\Delta)dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{1}$$

where M is the mass of the BH, and

$$\Delta(r)\equiv r^2-2Mr \tag{2}$$

is a function which vanishes at the EH, $r=2M$. Here, and throughout this paper, we use relativistic units, $c=G=1$.

As this paper is concerned with the behavior near the event horizon (EH), we shall find it convenient in the sequel to introduce a new (dimensionless) radial coordinate,

$$z\equiv\frac{r-2M}{2M}, \tag{3}$$

which vanishes at the horizon.

We shall also need the EH-regular (Kruskal) null coordinates

$$V\equiv e^{v/(4M)}, \quad U\equiv -e^{-u/(4M)}, \tag{4}$$

where $v\equiv t+r_*$ and $u\equiv t-r_*$ are the Eddington-Finkelstein null coordinates, with

$$r_*\equiv r+2M\ln z. \tag{5}$$

To discuss perturbations of the Schwarzschild BH via the Newman-Penrose formalism, we introduce the tetrad basis of null vectors $(l^\mu,n^\mu,m^\mu,m^{*\mu})$, defined in the (t,r,θ,φ) coordinate system by [5,29]

$$l^\mu=[r^2/\Delta,1,0,0]$$

$$n^\mu=[1,-\Delta/r^2,0,0]/2 \tag{6}$$

$$m^{\mu}=[0,0,1,i/\sin\theta]/(2^{1/2}r).$$

(The components of the fourth tetrad leg, $m^{*\mu}$, are obtained from the components of m^{μ} by complex conjugation.)

In the framework of the Newman–Penrose formalism [30] the gravitational field in vacuum is completely described by five complex scalars, $\Psi_0 \ldots \Psi_4$, constructed from the Weyl tensor by projecting it on the above tetrad basis. Likewise, the electromagnetic field is completely described by the three complex scalars $\varphi_0,\varphi_1,\varphi_2$, constructed by similarly projecting the Maxwell tensor. In particular,

$$\Psi_0\equiv -C_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}l^{\gamma}m^{\delta},\qquad \Psi_4\equiv -C_{\alpha\beta\gamma\delta}n^{\alpha}m^{*\beta}n^{\gamma}m^{*\delta} \tag{7}$$

represent the ingoing and outgoing radiative parts, respectively, of the Weyl tensor, and

$$\varphi_0\equiv F_{\mu\nu}l^{\mu}m^{\nu},\qquad \varphi_2\equiv F_{\mu\nu}m^{*\mu}n^{\nu} \tag{8}$$

represent the ingoing and outgoing radiative parts of the electromagnetic field.

In the Schwarzschild (unperturbed) background all Weyl scalars but Ψ_2 vanish (as directly implied by the Goldberg-Sachs theorem, in view of the Schwarzschild spacetime being of Petrov type D; see Sec. 9b,c in [31]). In the framework of a linear perturbation analysis, the symbols $\Psi_0,\Psi_1,\delta\Psi_2,\Psi_3,\Psi_4$ and $\varphi_0,\delta\varphi_1,\varphi_2$ are thus used to represent first-order perturbations of the corresponding fields (with $\delta\Psi_2\equiv\Psi_2-\Psi_2^{\text{background}}$, etc.). One can show (see Sec. 29b in [31]) that Ψ_0 and Ψ_4, and also φ_0 and φ_2, are *invariant* under gauge transformations (namely, under infinitesimal rotations of the null basis and infinitesimal coordinate transformations). The scalars Ψ_1 and Ψ_3 are not gauge invariant, and may be nullified by a suitable rotation of the null frame. The entities $\delta\Psi_2$ and $\delta\varphi_1$ represent perturbations of the ''Coulomb-like,'' non-radiative, part of the fields (in fact, one can also nullify $\delta\Psi_2$ by a suitable infinitesimal coordinate transformation.) It is therefore only the scalars defined in Eqs. (7) and (8) which carry significant information about the radiative part of the fields. (Note, however, that gauge invariance of the radiative fields is guaranteed only within the framework of linear perturbation theory.)

There is a single master equation governing linear perturbations of both the gravitational and the electromagnetic radiative fields defined in Eqs. (7), (8) [32]. In vacuum, this master perturbation equation reads

$$r^4\Delta^{-1}\Psi^s_{,tt}-\Delta^{-s}(\Delta^{s+1}\Psi^s_{,r})_{,r}-\frac{1}{\sin\theta}(\Psi^s_{,\theta}\sin\theta)_{,\theta}-\frac{1}{\sin^2\theta}\Psi^s_{,\varphi\varphi}-\frac{2is\cos\theta}{\sin^2\theta}\Psi^s_{,\varphi}-2s[Mr^2/\Delta-r]\Psi^s_{,t}+(s^2\cot^2\theta-s)\Psi^s=0, \tag{9}$$

where $\Psi^s(t,r,\theta,\varphi)$ represents the various radiative fields according to the following substitutions:

$$\varphi_0=\Psi^{s=+1},\qquad \varphi_2=r^{-2}\Psi^{s=-1},\qquad \Psi_0=\Psi^{s=+2},\qquad \Psi_4=r^{-4}\Psi^{s=-2}. \tag{10}$$

In Eq. (9), the angular dependence of Ψ^s is separable through a decomposition in *spin-weighted spherical harmonics* [33],

$$\Psi^s(r,t,\theta,\varphi)=\sum_{l=|s|}^{\infty}\sum_{m=-l}^{l}\psi^{slm}(r,t)Y^{slm}(\theta,\varphi). \tag{11}$$

The time-radial functions $\psi^{slm}(r,t)$ then satisfy the field equation

$$r^4\psi^{slm}_{,tt}-\Delta^{-s+1}(\Delta^{s+1}\psi^{slm}_{,r})_{,r}+2sr^2(r-3M)\psi^{slm}_{,t}+(l-s)(l+s+1)\Delta\psi^{slm}=0. \tag{12}$$

III. THE LATE-TIME EXPANSION

In order to analyze the power-law decay of perturbations at late time, we decompose the field in the form

$$\psi^{slm}(r,t)=\sum_{k=0}^{\infty}F_k^{slm}(r)v^{-k_0-k}, \tag{13}$$

to which we refer as the *late-time expansion* [34]. Substitution in Eq. (12) yields an ordinary equation for each function F_k^{slm}:

$$D^{sl}(F_k^{slm})=S_k^{slm}, \tag{14}$$

where D^{sl} is a differential operator defined by

$$D^{sl}\equiv\Delta d^2/dr^2+2(s+1)(r-M)d/dr-(l-s)(l+s+1), \tag{15}$$

and the source term S_k^{slm} is given by

$$S_k^{slm}\equiv 2(k_0+k-1)r\left[\frac{d(rF_{k-1}^{slm})}{dr}+2srM\Delta^{-1}F_{k-1}^{slm}\right]. \tag{16}$$

(We take $F_{k<0}^{slm}\equiv 0$.)

The dominant late-time decay at world lines of fixed r is described by the term $k=0$ in Eq. (13). To the leading order in $1/v$, we have

$$\psi^{slm}(r,t)\cong F_{k=0}^{slm}(r)v^{-k_0}. \tag{17}$$

Substituting $v=t+r_*$, we also find, to the leading order in $1/t$,

$$\psi^{slm}(r,t)\cong F_{k=0}^{slm}(r)t^{-k_0}, \tag{18}$$

which, using the well-known result by Price [5], implies $k_0=2l+3$ (or $k_0=2l+2$ if a static mode l is initially present).[1]

Since S_k^{slm} vanishes for $k=0$, the term $F_{k=0}^{slm}(r)$ satisfies the homogeneous differential equation

$$D^{sl}(F_0^{slm})=0. \tag{19}$$

This is just the field equation of a *static* mode l,m. Thus, $F_{k=0}^{slm}(r)$ must be a static solution of the field equation. In the next section we shall study the static solutions for Ψ^s, focusing attention on their asymptotic behavior at the EH.

IV. STATIC SOLUTIONS

Since Eq. (19) is a second-order differential equation, its general solution is spanned by two basis solutions. We shall primarily be interested in the asymptotic behavior of these solutions near the EH. The leading-order asymptotic behavior can be easily obtained from the asymptotic form of Eq. (19) near the EH: One finds that for both $s>0$ and $s<0$, the two asymptotic solutions behave there like

$$\psi^{slm}\cong\Delta^0 \quad \text{and} \quad \psi^{slm}\cong\Delta^{-s} \tag{20}$$

(to the leading order in Δ). However, it is also possible to write an exact, global, basis of solutions to the static equation, as we do now.

In terms of the radial variable z, Eq. (19) takes the form

$$(-z)(1+z)F_0''+[-(s+1)-2(s+1)z]F_0' +(l-s)(l+s+1)F_0=0, \tag{21}$$

where a prime denotes d/dz. This is the *hypergeometric equation* [35] for $F_0(-z)$. One solution for Eq. (21) is given by (see Sec. 2.1.1 in [35])

$$\phi_r(z)=\begin{cases} F(-l+s,l+s+1;s+1;-z)\equiv\phi_r^+ & (\text{for } s>0),\\ (4M^2z)^{-s}F(-l,l+1;-s+1;-z)\equiv\phi_r^- & (\text{for } s<0),\end{cases} \tag{22}$$

where F denotes the hypergeometric function. (Hereafter we often omit the indices slm for brevity.) Note that since in both cases the first index is a non-positive integer, F is simply a *polynomial* in z, and so is ϕ_r. (We choose this notation because, as we shall see below, ϕ_r is physically regular at the EH, whereas the other static solution, to which we shall later refer as ϕ_{ir}, is irregular there.) The normalization in Eq. (22) was chosen so as to conform with Eq. (20) [recall that at $z=0$, the hypergeometric function $F=1$, and note also the relation $\Delta=4M^2z(z+1)$]. Thus, to the leading order in Δ, ϕ_r is given by

$$\phi_r(r)\cong\begin{cases}\Delta^0 & (\text{for } s>0),\\ \Delta^{-s} & (\text{for } s<0).\end{cases} \tag{23}$$

A second, independent, static solution is given by (see Sec. 2.2.2, case 21, in [35])

$$\phi_{ir}(z)=A_{ls}\times\begin{cases}(4M^2z)^{-s}(1+z)^{-l-1}F[l-s+1,l+1;2l+2;(1+z)^{-1}]\equiv\phi_{ir}^+ & (\text{for } s>0),\\ (1+z)^{-l-s-1}F[l+s+1,l+1;2l+2;(1+z)^{-1}]\equiv\phi_{ir}^- & (\text{for } s<0),\end{cases} \tag{24}$$

where A_{sl} is a normalization factor,

$$A_{sl}=1/F(l-|s|+1,l+1;2l+2;1)=\frac{l!(l+|s|)!}{(2l+1)!(|s|-1)!} \tag{25}$$

[cf. Eq. (46) in Sec. 2.8 of [35]], chosen such that ϕ_{ir} takes the simple leading-order asymptotic form (20) at the EH, namely

$$\phi_{ir}(r)\cong\begin{cases}\Delta^{-s} & (\text{for } s>0),\\ \Delta^0 & (\text{for } s<0).\end{cases} \tag{26}$$

A careful study of the asymptotic behavior of ϕ_{ir} at the EH reveals that it includes a (sub-dominant) logarithmic term.[2] To analyze this logarithmic term, it is instructive to express the irregular solution in terms of ϕ_r via the Wronskian method. The Wronskian associated with the homogeneous equation (19) is

$$W=\Delta^{-s-1}, \tag{27}$$

and thus a static solution independent of ϕ_r may be expressed as

[1]For brevity, we hereafter consider modes without initial static multipoles.

[2]Such a logarithmic term is to be anticipated, because of the integer difference, $|s|$, between the leading powers of z in the two asymptotic solutions (20) near the regular-singular point $z=0$ [36].

$$\tilde{\phi}_{ir} = -\phi_r(r)\int^r \phi_r^{-2}(r')W(r')dr'$$
$$= -\phi_r(r)\int^r \phi_r^{-2}(r')[\Delta(r')]^{-s-1}dr'. \quad (28)$$

This solution, of course, does not necessarily coincide with ϕ_{ir}, but is, in general, a linear combination of the two basis functions ϕ_r and ϕ_{ir}. It is easy to verify that the integrand in Eq. (28) is $z^{-|s|-1}$ times a rational function which is regular (and nonvanishing) at $z=0$. The integrand can therefore be expanded as

$$z^{-|s|-1}(\gamma_0+\gamma_1 z+\gamma_2 z^2+\cdots), \quad (29)$$

where γ_i are constants, with $\gamma_0=(4M^2)^{-|s|-1}\neq 0$. By substituting the leading-order term of this expansion in Eq. (28) and comparing to Eq. (26), we find

$$\tilde{\phi}_{ir} = (2M|s|)^{-1}\phi_{ir} + \text{const}\times\phi_r. \quad (30)$$

[Here, the coefficient of ϕ_r depends on the specific choice of the lower integration limit in Eq. (28).] Substitution of the full expansion (29) in Eq. (28) yields

$$\tilde{\phi}_{ir} = -\phi_r[\gamma_{|s|}\ln z + z^{-|s|}(\hat{\gamma}_0+\hat{\gamma}_1 z+\cdots)], \quad (31)$$

where $\hat{\gamma}_i = 2M\gamma_i/(i-|s|)$ for $i\neq|s|$ (and $\hat{\gamma}_{|s|}$ is an arbitrary integration constant). We now use Eq. (30) to extract ϕ_{ir}:

$$\phi_{ir} = 2M|s|(\tilde{\phi}_{ir} - \text{const}\times\phi_r)$$
$$= -2M|s|\phi_r[\gamma_{|s|}\ln z + z^{-|s|}(\hat{\gamma}_0+\hat{\gamma}_1 z+\cdots) + \text{const}]. \quad (32)$$

It is straightforward to expand this expression about $z=0$. This expansion yields

$$\phi_{ir}(r) = \begin{cases} \Delta^{-s}(1+\alpha_1^+ z+\alpha_2^+ z^2+\cdots)+\beta_{sl}\phi_r^+\ln z & (\text{for } s>0), \\ (1+\alpha_1^- z+\alpha_2^- z^2+\cdots)+\beta_{sl}\phi_r^-\ln z & (\text{for } s<0), \end{cases} \quad (33)$$

where $\alpha_i^{\pm}$ and $\beta_{sl} = -2M|s|\gamma_{|s|}$ are constants.

The above analysis, based on Eq. (28), explains the origin of the logarithmic term and determines its exact form. The calculation of $\gamma_{|s|}$ (and hence of β_{sl}) for general l,s is tedious, however. It is easier to derive the explicit expression for β_{sl} directly from the exact expression (24) for ϕ_{ir}. The series expansion of the hypergeometric function around $(1+z)^{-1}=1$ (corresponding to $z=0$) may be obtained from a generating function through the formula

$$F(l-|s|+1,l+1,2l+2,y)$$
$$= \frac{(-1)^{|s|+1}(2l+1)!}{(l!)^2(l-|s|)!(l+|s|)!} \times\frac{d^l}{dy^l}\left[(1-y)^{l+|s|}\frac{d^l}{dy^l}\left(\frac{\ln(1-y)}{y}\right)\right] \quad (34)$$

[cf. Eq. (4) in Sec. 2.2.2 of [35]]. Note that in our case $y=1/(1+z)$, so $\ln(1-y)=\ln z-\ln(1+z)$. The logarithmic term in Eq. (33) (which comes from the first of the above two ln terms) is only obtained when none of the $2l$ derivative operators d/dy in Eq. (34) acts on $\ln(1-y)$. Thus, for the sake of calculating the logarithmic coefficient, we can replace the second factor in the right-hand side of Eq. (34) by

$$\frac{d^l}{dy^l}\left[(1-y)^{l+|s|}\frac{d^l}{dy^l}(y^{-1})\right]\ln z$$
$$= (-1)^l(l)!\frac{d^l}{dy^l}[(1-y)^{l+|s|}y^{-1-l}]\ln z. \quad (35)$$

Evaluating this expression at $y=1$, to the leading order in $z\cong 1-y$, we obtain

$$\frac{(l)!(l+|s|)!}{(|s|)!}z^{|s|}\ln z. \quad (36)$$

Substituting this in Eq. (34), and recalling Eq. (25), we obtain the desired expression for the logarithmic coefficient:

$$\beta_{sl} = \frac{(-1)^{s+1}(l+|s|)!}{(|s|-1)!(|s|)!(l-|s|)!}(4M^2)^{-|s|}\neq 0. \quad (37)$$

In summary, we have constructed a basis of solutions to the static field equation. One of the basis solutions (ϕ_r) is simply a polynomial in z, but the other (ϕ_{ir}) contains a logarithmic term. This logarithmic term will play an important role in the analysis below. Note also that for both $s>0$ and $s<0$, ϕ_r is smaller than ϕ_{ir} in the leading order by a factor $\Delta^{|s|}$.

V. REGULARITY AT THE EH

By general considerations, we expect physical perturbations to be regular and smooth at the EH. The function Ψ^s represents a perturbation in the Maxwell field tensor $F_{\alpha\beta}$ for $s=\pm 1$ and in the Weyl tensor $C_{\alpha\beta\gamma\delta}$ for $s=\pm 2$. When this perturbation is expressed in Kruskal coordinates (4) (or in any other coordinates which are regular at the EH), all components of these Maxwell or Weyl tensors must take a perfectly regular form at the EH.

To discuss the regularity of Ψ^s at the EH, it is useful to define

$$\hat{\Psi}^s \equiv \Delta^s \Psi^s, \tag{38}$$

and correspondingly,

$$\hat{\psi}^{slm} \equiv \Delta^s \psi^{slm} \quad \text{and} \quad \hat{F}_k^{slm} \equiv \Delta^s F_k^{slm}. \tag{39}$$

It is straightforward to show, via Eqs. (7) and (8), that for any s, $\hat{\Psi}^s$ directly represents a physical perturbation field which must be regular at the EH: For $s=2$, $\hat{\Psi}^s$ is a linear combination of Weyl components C_{VaVb}, where the indices a,b represent the two angular coordinates θ,φ. For $s=-2$, $\hat{\Psi}^s$ is a linear combination of Weyl components C_{UaUb}. Similarly, for $s=1$ and $s=-1$ $\hat{\Psi}^s$ is a linear combination of Maxwell components F_{Va} and F_{aU}, respectively. Therefore, a necessary condition for regularity at the EH is that $\hat{\Psi}^s$ be regular (i.e., finite and smooth). Since the spin-weighted spherical harmonics are smooth, $\hat{\psi}^{slm}$ must be smooth too.

We point out that the regularity of $\hat{\Psi}^s$ is also dictated by mathematical considerations, as follows: If one transforms the master equation (9) from Ψ^s to $\hat{\Psi}^s$, and from the original coordinates to the Kruskal coordinates (4), the field equation becomes perfectly regular at the EH (whereas with the original dependent variable Ψ^s the equation is singular at the EH, even in Kruskal coordinates). Therefore, from the hyperbolic nature of the field equation, if the initial data for $\hat{\Psi}^s$ are regular (which we assume), no irregularity may evolve at the EH.

Consider next the regularity of the static solutions. We assume that for any s and any l,m, there exists (at least) one static solution which is physically regular at the EH. For, if there is an external static source of a multipole l,m (and no incoming waves from past null infinity), the field outside the BH will be static; and we do expect this static field to be regular at the EH. The presence of two independent regular static solutions (for a given s,l,m) at the EH would violate the no-hair principle, because then *all* static solutions would be regular at the EH, including the one which is regular at infinity. We shall now show, however, that for any s, one of the static solutions (the solution ϕ_{ir}) is irregular.

For $s<0$, the irregularity of ϕ_{ir} is obvious, because the corresponding field $\hat{\phi}_{ir} \equiv \Delta^s \phi_{ir}$ diverges like Δ^s. For $s>0$ the field $\hat{\phi}_{ir}$ is finite ($\cong \Delta^0$) at the EH. Yet, the logarithmic term implies that the solution is not smooth: The derivative of order $|s|$ with respect to r (which itself is a regular coordinate) diverges. On the other hand, for both $s>0$ and $s<0$, the field $\hat{\phi}_r \equiv \Delta^s \phi_r$ is a polynomial in z [which is proportional to $(r-2M)^s$ for $s>0$ and to $(r-2M)^0$ for $s<0$], so it is perfectly smooth.

We conclude that for both positive and negative s, ϕ_{ir} is physically irregular, whereas ϕ_r is physically regular.

VI. LATE-TIME BEHAVIOR

The demand for regularity of $\hat{\psi}^{slm}$ at the EH has immediate implications to the late-time expansion (13). Since r and v form a regular coordinate system for the Schwarzschild background (the so-called ''ingoing Eddington coordinates''), $\hat{\psi}^{slm}$ must be a perfectly smooth function of r and v at the EH (recall that r and v are related to U and V by an invertible analytic transformation). Therefore, for any k (and any s), $\hat{F}_k^{slm}(r) \equiv \Delta^s F_k^{slm}(r)$ must be a smooth function of r:

$$\hat{F}_k^{slm}(r) \in C^\infty(\mathbb{R}) \quad \text{for } k. \tag{40}$$

In Sec. III we have shown that $F_{k=0}^{slm}$ must be a static solution. The regularity of $\hat{F}_{k=0}^{slm}(r)$ then implies that $F_{k=0}^{slm}$ must coincide (up to some multiplicative constant) with the *regular* static solution ϕ_r. Hence, to the leading order in $1/v$, we obtain

$$\psi^{slm}(t,r) = c_0 \phi_r(r) v^{-2l-3} + O(v^{-2l-4}), \tag{41}$$

where c_0 is constant. For the description of the late-time behavior along world lines of fixed $r>2M$, it is useful to re-write this expression in terms of powers of $1/t$:[3]

$$\psi^{slm}(r,t) = c_0 \phi_r(r) t^{-2l-3} + O(t^{-2l-4}). \tag{42}$$

From this point on we discuss the cases $s<0$ and $s>0$ separately.

A. The case $s<0$

In this case, $F_{k=0}^{slm}$ is proportional to ϕ_r^-. We shall denote the proportionality constant by c_0^-, that is,

$$F_{k=0}^{slm} = c_0^- \phi_r^- \quad (s<0). \tag{43}$$

[Recall that the parameter k_0 in Eq. (13) is so defined such that the term $F_{k=0}^{slm}$ does not vanish identically. Therefore, by definition, the constant c_0^- is non-zero.] Note that $\phi_r^- \cong \Delta^{|s|}$ near the EH.

Consider next the contribution from the terms $k>0$. From Eq. (40) it is obvious that for $s<0$ and for all $k>0$, F_k^{slm} must be a regular function of r, which vanishes at least like $\Delta^{|s|}$ at the EH (like for $k=0$). Hence, at late time the terms $k>0$ are negligible compared to the term $k=0$, due to their higher negative powers of $1/v$. Therefore, Eq. (41), which now reads

$$\psi^{slm}(r,t) \cong c_0^- \phi_r^-(r) v^{-2l-3} \quad (s<0), \tag{44}$$

provides a useful description of the late-time behavior not only at $r>2M$ but also at the EH. To the leading order in Δ, the asymptotic behavior at the EH is

$$\psi^{slm}(r,t) \cong c_0^- \Delta^{-s} v^{-2l-3} \quad (s<0). \tag{45}$$

[3]It should be stressed here that in this paper we *assume* the power index $2l+3$ derived by Price [5] for the tail at fixed $r>2M$. The new information in Eq. (42) [or in Eq. (41)] concerns the explicit form of the radial function multiplying the inverse-power factor.

B. The case $s>0$

In this case, $F^{slm}_{k=0}$ (which must coincide with the regular static solution) is proportional to ϕ^+_r, i.e., $F^{slm}_{k=0}=c^+_0\phi^+_r$ with $c^+_0\neq 0$; hence $F^{slm}_{k=0}\propto\Delta^0$ near the EH. However, for $k>0$ (for which F^{slm}_k is not the static solution), the only obvious constraint on the functions F^{slm}_k is the regularity of $\hat{F}^{slm}_k(r)$, Eq. (40). This regularity criterion allows the terms $k>0$ to be proportional to Δ^{-s} (and, as we show below, at least the term $k=1$ is indeed proportional to Δ^{-s}). Due to this Δ^{-s} factor, at the EH the $O(v^{-2l-4})$ term in Eq. (41) dominates the $O(v^{-2l-3})$ term, which is only proportional to Δ^0. Therefore, for $s>0$, Eq. (41) does not provide a useful description of the asymptotic behavior at the EH as it does for $r>2M$. A correct description of the late-time behavior there must include both terms $k=0$ and $k=1$:

$$\psi^{slm}(r,t)=c^+_0\phi^+_r(r)v^{-2l-3}+F^{slm}_{k=1}(r)v^{-2l-4}+O(v^{-2l-5})\quad(s>0). \tag{46}$$

To the leading order in Δ, the asymptotic behavior at the EH is

$$\psi^{slm}(r,t)\cong c^+_0 v^{-2l-3}+c^+_1\Delta^{-s}v^{-2l-4}\quad(s>0), \tag{47}$$

where c^+_1 is the coefficient of Δ^{-s} in $F^{slm}_{k=1}$. Note that Eqs. (41) and (42) still provide a correct and useful description of the late-time behavior along any line of fixed $r>2M$.

It is important to verify that the coefficient c^+_1 in Eq. (47) is non-vanishing. This coefficient is to be obtained from the function $F^{slm}_{k=1}(r)$ in Eq. (46). $F^{slm}_{k=1}$ satisfies the inhomogeneous equation (14), subject to the regularity condition (40). The general inhomogeneous solution takes the form

$$F^{slm}_{k=1}(r)=a^+_1\phi^+_r(r)+b^+_1\phi^+_{ir}(r)+\phi_{ih}(r), \tag{48}$$

where a^+_1 and b^+_1 are constants and ϕ_{ih} is a specific inhomogeneous solution. Using the Wronskian function $W(r)$ given in Eq. (27), we can express ϕ_{ih} as

$$\phi_{ih}(r)=\phi_r(r)\int^r\frac{\tilde{\phi}_{ir}(r')S_1(r')/\Delta(r')}{W(r')}dr'-\tilde{\phi}_{ir}(r)\int^r\frac{\phi_r(r')S_1(r')/\Delta(r')}{W(r')}dr'. \tag{49}$$

For $s>0$ it is convenient to re-express this inhomogeneous solution in the form

$$\phi_{ih}(r)=\int^r dr'\int^{r'}_{2M}dr''\frac{\phi^+_r(r)\phi^+_r(r'')}{[\phi^+_r(r')]^2}\frac{W(r')}{W(r'')}\frac{S_1(r'')}{\Delta(r'')}, \tag{50}$$

which is easily obtained from Eq. (49) by first substituting for $\tilde{\phi}_{ir}$, using Eq. (28), and then integrating the resulting expression by parts. The form (50) is advantageous as it only involves the homogeneous solution ϕ^+_r, which has a simple polynomial form.[4] The source term S_1 is to be calculated from Eq. (16) with $F_{k-1}(r)=F_0(r)=c^+_0\phi^+_r(r)$. This yields, to the leading order in Δ,

$$S_1(r)\cong 16M^3sk_0c^+_0\Delta^{-1}. \tag{51}$$

In view of Eqs. (23), (27), and (51), we find that the integrand in Eq. (50) is given, to the leading order in Δ, by $16M^3sk_0c^+_0\Delta^{s-1}(r'')/\Delta^{s+1}(r')$. Performing the double integration, we obtain (to the leading order in Δ)[5]

$$\phi_{ih}(r)\cong 4Mk_0c^+_0\ln z+O(\Delta^0). \tag{52}$$

By substitution in the inhomogeneous equation (14), one easily verifies that the term $\ln z$ in ϕ_{ih} must be multiplied by a homogeneous solution (otherwise the homogeneous operator D^{sl}, acting on the logarithmic part of ϕ_{ih}, would yield a term proportional to $\ln z$—which cannot be balanced by the logarithmic-free source term); and from Eq. (52) and Eqs. (23), (26) it follows that this homogeneous solution must be proportional to ϕ^+_r. Therefore,

$$\phi_{ih}(r)\cong 4Mk_0c^+_0\phi^+_r\ln z+O(\Delta^0) \tag{53}$$

[in which the $O(\Delta^0)$ term is logarithmic-free]. We now substitute this expression in Eq. (48), using the asymptotic forms (23) and (33), and keeping only the leading order (proportional to Δ^{-s}) of the non-logarithmic part:

$$F^{slm}_{k=1}(r)\cong b^+_1[\Delta^{-s}+\beta_{sl}\phi^+_r\ln z]+4Mk_0c^+_0\phi^+_r\ln z. \tag{54}$$

Note that the coefficient a^+_1 in Eq. (48) (which, in principle, is to be obtained by matching the solution to the late-time field at null infinity [11]) does not enter Eq. (54), as ϕ^+_r includes neither Δ^{-s} terms nor logarithmic terms.

Now, $F_1(r)$ must satisfy the regularity condition (40), so it cannot contain a logarithmic term. This dictates the value of the constant b^+_1:

$$b^+_1=-4Mk_0c^+_0\beta^{-1}_{sl}\neq 0. \tag{55}$$

One can now identify the non-vanishing coefficient b^+_1 with the above leading order coefficient c^+_1 of $F_1(r)$ at the EH. We conclude that the coefficient c^+_1 in Eq. (47) is non-vanishing. As a consequence, we find that *on the EH itself*

[4] In Eq. (50) we have not specified the lower limit of the integration over r'. Changing the value of this limit amounts to adding a regular solution $\propto\phi^+_r$, which is equivalent to re-defining the coefficient a^+_1 in Eq. (48). Note, however, that the choice $r'=2M$ as the lower integration limit is forbidden, as the integral is not defined in this case.

[5] Note that no $\ln^2 z, \ln^3 z\ldots$ terms arise from the integration in Eq. (50): For $s>0$ the integrand is actually a rational function of r'', analytic at $r''=2M$. Hence, the integration over r'' cannot produce a $\ln z'$ term. A term $\propto\ln z$ arises only from the subsequent integration over r'.

the perturbation is dominated by the second term in Eq. (47), and hence it decays there like v^{-2l-4}.

In this section we have obtained the asymptotic form (47) (and proved that the coefficient c_1^+ is non-vanishing) by a direct analysis of ψ^{slm} in the case $s>0$. There is yet another way to obtain Eq. (47), which, being somewhat outside the main course of this paper, we describe in detail in the Appendix: It is well known that each single one of the perturbation fields $s=1$ and $s=-1$ determines the full electromagnetic perturbation, i.e., the full Maxwell tensor $F_{\alpha\beta}$ (up to a trivial addition of the static Coulomb solution). Similarly, each of the perturbation fields $s=2$ and $s=-2$ determines the full gravitational perturbation, i.e., the perturbation in the Weyl tensor (up to gauge, and up to a trivial addition of the static multipoles with $l=0$ and $l=1$). In particular, φ_2 determines φ_0, and Ψ_4 determines Ψ_0, and vice versa. We use this fact in the Appendix, where we obtain the asymptotic behavior for $s>0$ from that of $s<0$ (as we showed above, the latter is relatively simple, because for $s<0$ the term $k=0$ in the late-time expansion completely describes the late-time behavior at the EH). To that end, we shall use in the Appendix the well known *Teukolsky-Starobinsky identities*.

VII. SINGLE FOURIER MODES

Consider a solution to the field equation (12), having the form

$$\psi(r,t)=\psi_\omega(r)e^{-i\omega t}. \tag{56}$$

(The indices s,l,m, which, in fact, characterize both functions ψ and ψ_ω, are omitted here and below for brevity.) Each Fourier mode $\psi_\omega(r)$ then satisfies an ordinary equation which may be written as

$$\frac{d^2}{dr_*^2}(r\Delta^{s/2}\psi_\omega)+[\omega^2+i\omega sR(r)+V_{ls}(r)](r\Delta^{s/2}\psi_\omega)=0, \tag{57}$$

in which $R(r)$ and $V_{ls}(r)$ are certain radial functions. The asymptotic form of this equation near the horizon is

$$\frac{d^2}{dr_*^2}(\Delta^{s/2}\psi_\omega)\cong(s/4M+i\omega)^2(\Delta^{s/2}\psi_\omega). \tag{58}$$

The two asymptotic solutions at the EH are

$$\psi_\omega^a\cong\Delta^0 e^{i\omega r_*} \quad \text{and} \quad \psi_\omega^b\cong\Delta^{-s}e^{-i\omega r_*} \tag{59}$$

(where use has been made of the asymptotic relation $e^{r_*/(4M)}\propto\Delta^{1/2}$). At the limit $\omega\to 0$, these two asymptotic solutions approach the two asymptotic static solutions, Eq. (20)—just as one would expect. We shall now show, however, that for $s>0$ the role of regular and irregular solutions is interchanged as the limit $\omega\to 0$ is approached.

In the case $\omega\neq 0$, too, we expect one of the two solutions to be regular and the other one to be singular (for the same reasons as in the static case). We now substitute ψ^a and ψ^b in Eq. (56), and construct the corresponding physical fields $\hat{\psi}\equiv\Delta^s\psi$ (which, as was discussed in Sec. V, should be regular functions at the EH). We denote the functions $\hat{\psi}$ obtained from ψ_ω^a and ψ_ω^b by $\hat{\psi}^a$ and $\hat{\psi}^b$, respectively, and find

$$\hat{\psi}^a\equiv\Delta^s\psi_\omega^a e^{-i\omega t}\cong\Delta^s e^{-i\omega u},$$

$$\hat{\psi}^b\equiv\Delta^s\psi_\omega^b e^{-i\omega t}\cong e^{-i\omega v}. \tag{60}$$

Recall that v is regular at the EH, but u is not (as the EH is a surface of finite v but infinite u). This implies that $\hat{\psi}^b$ is regular, but $\hat{\psi}^a$ is irregular. (For $s<0$, $\hat{\psi}^a$ diverges at the EH. For $s>0$, $\hat{\psi}^a$ is finite, but its s-order derivative with respect to U is indeterminate at the EH, and higher-order derivatives diverge there.) We conclude that for $\omega\neq 0$ (and for both $s>0$ and $s<0$), ψ_ω^b is regular and ψ_ω^a is singular. (This is a well known result; see [37].)

Let us now compare this situation to the static case, Eqs. (23) and (26). For $s<0$, the classification into regular and irregular solutions is preserved at the limit $\omega\to 0$. However, for $s>0$, the regular and irregular solutions switch role in this static limit.

VIII. SCALAR-FIELD TOY MODEL

To better understand the exchange of regular and singular solutions at the limit $\omega\to 0$ (for $s>0$), it is instructive to consider a simple scalar-field toy model. Let Φ be a minimally coupled, massless, Klein-Gordon test field on the Schwarzschild background. We make here the assumption that, in an appropriate gauge, the late-time behavior of the electromagnetic four-potential A_α and of the linear metric perturbation $h_{\alpha\beta}$ is qualitatively the same as that of a scalar field (this assumption is somewhat vague, especially because of the gauge ambiguity. Note, however, that at least for the behavior of metric perturbations along $r=\text{const}>2M$ lines, this assumption is verified in [4]). Correspondingly, we would expect that the components F_{aV} of the Maxwell tensor—which are made of terms like $A_{a,V}$—will qualitatively behave at the EH like $\Phi_{,V}$. For the same reason, we would expect F_{aU} to behave at the EH like $\Phi_{,U}$. Recalling the way $\Psi^{s=\pm 1}$ is constructed from $F_{\alpha\beta}$ by projection on the tetrad (6) [see Eq. (8)], one intuitively expects that $\Psi^{s=\pm 1}$ will qualitatively behave as follows:

$$\Psi^{s=1}\propto\Delta^{-1}\Phi_{,v}\equiv\tilde{\Psi}^{s=1},$$

$$\Psi^{s=-1}\propto\Phi_{,u}\equiv\tilde{\Psi}^{s=-1}. \tag{61}$$

Similarly, for the case $|s|=2$, one expects [in view of Eq. (7)] that

$$\Psi^{s=2}\propto\Delta^{-2}\Phi_{;vv}\equiv\tilde{\Psi}^{s=2},$$

$$\Psi^{s=-2}\propto\Phi_{;uu}\equiv\tilde{\Psi}^{s=-2}. \tag{62}$$

(For brevity, we shall focus the following discussion on the case $|s|=1$. Similar arguments apply to $|s|=2$ as well.)[6]

For any mode l,m of Φ, the two static solutions take the asymptotic forms

$$\Phi_r \cong 1+O(\Delta) \quad \text{and} \quad \Phi_{ir} \cong r_* \tag{63}$$

near the EH (cf. [4]). Clearly, Φ_r is the regular mode, while Φ_{ir} is singular (as $r_* \to -\infty$ at the EH). Let us denote the functions $\tilde{\Psi}^s$ which correspond to the regular and singular modes by $\tilde{\Psi}^s_r$ and $\tilde{\Psi}^s_{ir}$, respectively. Recalling that in the static case $\partial_v = -\partial_u = (1/2)d/dr^* = [\Delta/(2r^2)]d/dr$, we find for $|s|=1$:[7]

$$\tilde{\Psi}^s_r \propto \begin{cases} \Delta^0, & s=+1, \\ \Delta^1, & s=-1, \end{cases} \quad \tilde{\Psi}^s_{ir} \propto \begin{cases} \Delta^{-1}, & s=+1, \\ \Delta^0, & s=-1. \end{cases} \tag{64}$$

Consider next a single Fourier mode (of a given l,m,ω) $\Phi = \Phi_\omega(r)e^{-i\omega t}$. The two asymptotic solutions of the radial function at the EH are obtained by substituting $s=0$ in Eq. (59):

$$\Phi^a_\omega \cong e^{i\omega r*}[1+O(\Delta)], \quad \Phi^b_\omega \cong e^{-i\omega r*}[1+O(\Delta)]. \tag{65}$$

These two radial functions correspond to the field configurations

$$\Phi^a \equiv e^{-i\omega t}\Phi^a_\omega \cong e^{-i\omega u}[1+O(\Delta)],$$

$$\Phi^b \equiv e^{-i\omega t}\Phi^b_\omega \cong e^{-i\omega v}[1+O(\Delta)]. \tag{66}$$

Since u diverges at the EH (but v is regular), it is obvious that Φ^b is the regular solution, while Φ^a is singular. We shall denote the functions $\tilde{\Psi}^s$ which correspond to the regular and singular modes in Eq. (66) by $\tilde{\Psi}^a$ and $\tilde{\Psi}^b$, respectively. Using Eq. (61), we find for $|s|=1$:

$$\tilde{\Psi}^a \propto \begin{cases} e^{-i\omega u}[1+O(\Delta)], & s=+1, \\ e^{-i\omega u}[1+O(\Delta)], & s=-1, \end{cases}$$

$$\tilde{\Psi}^b \propto \begin{cases} \Delta^{-1}e^{-i\omega v}[1+O(\Delta)], & s=+1, \\ \Delta e^{-i\omega v}[1+O(\Delta)], & s=-1. \end{cases} \tag{67}$$

[It is assumed here that the terms $O(\Delta)$ in Eq. (66) are non-vanishing, and, moreover, that their derivatives with respect to r do not vanish at the EH.] The construction of $\tilde{\Psi}^a$ and $\tilde{\Psi}^b$ ensures that the t dependence of both functions is simply $e^{-i\omega t}$. Let us denote the radial parts of these two functions by $\tilde{\Psi}^a_\omega(r)$ and $\tilde{\Psi}^b_\omega(r)$, respectively; that is,

$$\tilde{\Psi}^a(r,t) \equiv e^{-i\omega t}\tilde{\Psi}^a_\omega(r),$$

$$\tilde{\Psi}^b(r,t) \equiv e^{-i\omega t}\tilde{\Psi}^b_\omega(r). \tag{68}$$

For both cases $s=1$ and $s=-1$, one thus finds

$$\tilde{\Psi}^a_\omega \propto e^{i\omega r}*[1+O(\Delta)],$$

$$\tilde{\Psi}^b_\omega \propto \Delta^{-s}e^{-i\omega r}*[1+O(\Delta)]. \tag{69}$$

A comparison of Eq. (64) to Eqs. (23), (26), and of Eq. (69) to Eq. (59), reveals that for both cases $\omega=0$ and $\omega \neq 0$, and for both $s=1$ and $s=-1$, the actual asymptotic form of both the regular and singular solutions agree with that obtained from the scalar-field toy model. In particular, in the case $s=-1$, at the limit $\omega \to 0$ the regular $\omega \neq 0$ solution $\tilde{\Psi}^b_\omega$ approaches the regular static solution $\tilde{\Psi}^{s=-1}_r$ (and the singular $\omega \neq 0$ solution $\tilde{\Psi}^a_\omega$ approaches the singular static static solution $\tilde{\Psi}^{s=-1}_{ir}$), whereas in the case $s=+1$ the regular and singular solutions interchange at the limit $\omega \to 0$.

Our toy model provides a simple intuitive explanation for the difference in the role of the regular and singular solutions in the static and $\omega \neq 0$ cases. The key point is the relation between the two basis solutions of the scalar field itself, i.e., Eqs. (63) and (65), (66). In the static case, there is a ''small solution'' Φ_r and a ''large solution'' Φ_{ir}. Naturally, the ''small solution'' is the regular one, and the ''large solution'' is singular. On the other hand, in the case $\omega \neq 0$ both radial solutions in Eq. (65) are of the same magnitude. In this case, the fundamental difference between the two basis solutions is that, at the leading order, one of them (Φ^a) depends solely on u, and the other one (Φ^b) depends solely on v. We can therefore refer to the two radial functions Φ^a_ω and Φ^b_ω as the ''u solution'' and the ''v solution,'' respectively. Since v is regular at the EH and u diverges, the ''v solution'' Φ^b_ω is regular and the ''u solution'' is singular.

Now, the functions $\tilde{\Psi}^s$ (which presumably represent the functions Ψ^s) are obtained in our toy model by differentiating Φ with respect to u or v (depending on the sign of s). Consider first the $\omega \neq 0$ case (in which the two basis solutions are classified as a ''v solution'' and a ''u solution''). When the operator ∂_v acts on Φ, it naturally yields a large outcome for the ''v solution,'' and a small outcome for the ''u solution.'' On the other hand, when the operator ∂_u is applied, it yields a small outcome for the ''v solution,'' and a large outcome for the ''u solution.'' Since the ''v solution'' is regular and the ''u solution'' is singular, we arrive at the following conclusion: For $\tilde{\Psi}^{s=-1}$ (which is associated

[6]A more sophisticated toy model would be obtained by replacing $F_{\alpha\beta}$ or $C_{\alpha\beta\gamma\delta}$ in the definitions of the Newman-Penrose fields by $\Phi_{;\alpha\beta}$ or $\Phi_{;\alpha\beta\gamma\delta}$, respectively. Here we adopt a simpler toy model which is easier to calculate.

[7]In deriving the asymptotic form for $\tilde{\Psi}^s_r$ it has been assumed that $d\Phi_r/dr$ does not vanish at the EH. This assumption is justified, as it is known [11,38] that the EH-regular static scalar field Φ_r is nothing but $P_l[(r-M)/M]$, the Legendre polynomial of order l (up to a multiplicative constant). At the EH we then have $d\Phi_r/dr \propto dP_l/dr = l(l+1)/2$, which does not vanish (except for $l=0$).

with $\Phi_{,u}$), the regular solution is the smaller of the two basic solutions. However, for $\tilde{\Psi}^{s=1}$ (which is associated with $\Phi_{,v}$), the regular solution is the *larger* of the two basic solutions.

On the other hand, in the static case we have a ''large solution'' and a ''small solution'' (instead of a ''v solution'' and a ''u solution''). The differentiation of the ''large solution'' with respect to either u or v yields a function $\tilde{\Psi}^s$ which is larger than that obtained from the differentiation of the ''small solution.'' Therefore, in the static case, for both $s>0$ and $s<0$ the regular solution is the smaller of the two basis solutions.

The interchange of the regular and singular $s>0$ solutions in the transition from $\omega\neq 0$ to $\omega=0$ may still look somewhat mysterious, because the limit $\omega\to 0$ is a perfectly regular limit of the differential equation (57). The mystery may again be resolved with the aid of our scalar-field toy model. Let us re-write the regular $\omega\neq 0$ solution for Φ [Eq. (66)] in a somewhat more explicit form,

$$\Phi^b \cong e^{-i\omega v}[1+c(\omega)\Delta+O(\Delta^2)]. \tag{70}$$

We assume that $c(\omega)$ is continuous and non-vanishing at the limit $\omega\to 0$. We now calculate $\tilde{\Psi}^{s=1}_\omega$ from this regular solution, via Eq. (61), keeping the leading order in Δ separately for terms proportional to ω and for terms proportional to ω^0:

$$\tilde{\Psi}^{s=1} \cong e^{-i\omega v}[-i\omega\Delta^{-1}+(4M)^{-1}c(\omega)][1+O(\Delta)]. \tag{71}$$

Restricting attention to the limit $\omega\to 0$ and to the leading order in Δ, we obtain

$$\tilde{\Psi}^{s=1} \cong e^{-i\omega v}(-i\omega\Delta^{-1}+c_0), \tag{72}$$

where $c_0=(4M)^{-1}\lim_{\omega\to 0}c(\omega)$. Equation (72) explains the change in the asymptotic form of the regular $s=1$ solution from Δ^{-1} in the case $\omega\neq 0$ to Δ^0 in the case $\omega=0$.

On the other hand, when the same calculation is carried out for $s=-1$, one obtains from Eq. (61)

$$\tilde{\Psi}^{s=-1} \cong -c_0\Delta e^{-i\omega v} \tag{73}$$

for the regular solution (for small ω). Thus, the regular solution for $s=-1$ is proportional to Δ for both $\omega\neq 0$ and $\omega=0$.

Our toy model also allows us to obtain the late-time behavior for both $s>0$ and $s<0$ directly from that of the scalar field. The late-time behavior for (a mode l,m of) Φ near the EH is known to be [7,11]

$$\Phi^l \cong \Phi^l_r(r)v^{-2l-3}, \tag{74}$$

where the radial function $\Phi^l_r(r)$ is the regular static solution for the mode l,m. Equation (61) now yields at the EH

$$\tilde{\Psi}^{s=-1} \cong -\Delta\Phi_1 v^{-2l-3} \tag{75}$$

and

$$\tilde{\Psi}^{s=1} \cong \Phi_1 v^{-2l-3}-(2l+3)\Phi_0\Delta^{-1}v^{-2l-4}, \tag{76}$$

where $\Phi_0=\Phi^l_r(r=2M)$, and $\Phi_1=(8M^2)^{-1}(d\Phi^l_r/dr)_{r=2M}$. Compare these results to Eqs. (45) and (47).

So far we have implemented the toy model for the case $|s|=1$ only. The calculations in the case $|s|=2$ are straightforward too, though they are somewhat more tedious. We shall merely point out here that all the expressions we have derived for $\tilde{\Psi}^{s=\pm 1}$ are extendible to $\tilde{\Psi}^{s=\pm 2}$, and may be used to explain the various features of $\Psi^{s=\pm 2}$—e.g., the asymptotic behavior of the regular and singular solutions for both $\omega\neq 0$ and $\omega=0$, and the late-time behavior at the EH. It should be emphasized that the late-time power index of $\tilde{\Psi}^{s=2}$ at the EH is $2l+4$ (and not $2l+5$, which might naively be anticipated due to the two v-derivatives in the definition of this function). The reason is that, the second-order covariant differentiation in Eq. (62) involves the differentiation of the affine connection. The easiest way to evaluate $\tilde{\Psi}^{s=2}$ is via the Kruskal coordinates (which at the EH minimize the connection's effect). One then finds that $\Phi_{;V}\propto v^{-2l-4}/V$, and the next differentiation with respect to V then yields, at the leading order, $\Phi_{;VV}\propto v^{-2l-4}/V^2$, i.e. $\Phi_{;vv}\propto v^{-2l-4}$.

Finally, we point out that Eq. (72), which was derived within the framework of the scalar field toy model, may also be derived for the realistic field $\Psi^{s=1}$, if Eq. (73) is assumed, using the Teukolsky-Starobinsky identities (the application of which is described in the Appendix). More explicitly, let us write the asymptotic behavior of the monochromatic $s=-1$ field at the EH, to the leading order in Δ, as

$$\Psi^{s=-1} \cong a(\omega)\Delta e^{-i\omega v}, \tag{77}$$

and assume that $a(\omega)$ is non-vanishing at the limit $\omega\to 0$. Then, applying the Teukolsky-Starobinsky identities, one can easily obtain for the corresponding $s=+1$ field

$$\Psi^{s=1}\propto e^{-i\omega v}(-i\omega\Delta^{-1}+\text{const}\times\Delta^0) \tag{78}$$

(for small ω).

IX. A KERR BLACK HOLE

The above analysis of the Schwarzschild case has immediate implications to rotating black holes as well. In a forthcoming paper [23] the late time expansion will systematically be applied to the Kerr case, in order to determine the late-time behavior of external perturbations. Here, we shall use the above methods and considerations to derive the decay rate of $s\neq 0$ fields along the Kerr EH (many of the details are left to Ref. [23]).

In the Kerr case, the Master equation is fully separable only in the frequency domain, by writing

$$\Psi^{\omega slm}(t,r,\theta,\varphi)=S^{slm}_\omega(\theta)e^{im\varphi}e^{-i\omega t}\psi^{\omega slm}(r), \tag{79}$$

where (t,r,θ,φ) are the Boyer-Lindquist coordinates, and $S^{slm}_\omega(\theta)e^{im\varphi}$ are the spin-weighted spheroidal harmonics

[37]. The behavior of the radial function $\psi^{\omega slm}(r)$ is then governed by the well known Teukolsky equation [37]. The two asymptotic solutions of this equation at the EH are

$$\psi_a^{\omega slm}(r)\cong\Delta^0 e^{i(\omega-m\Omega_+)r_*},\quad \psi_b^{\omega slm}(r)\cong\Delta^{-s}e^{-i(\omega-m\Omega_+)r_*}, \tag{80}$$

where

$$\Delta\equiv r^2-2Mr+a^2, \tag{81}$$

M and a are respectively the mass and specific angular momentum of the black hole, r_* is defined by $dr_*/dr=(r^2+a^2)/\Delta$, and

$$\Omega_+\equiv\frac{a}{2Mr_+}, \tag{82}$$

with $r_+\equiv M+(M^2-a^2)^{1/2}$ being the r value of the EH. [Compare the asymptotic solutions (80) and (59) in the case $a=0$.] We shall consider only a BH background with $0<|a|<M$ (the extremal case, $|a|=M$, requires a separate treatment).

In the Kerr case, the coordinate φ goes singular at the EH. Transforming to the regularized azimuthal coordinate

$$\tilde{\varphi}_+\equiv\varphi-\Omega_+ t \tag{83}$$

(see Sec. 58 in [31]), and substituting the solutions (80) in Eq. (79), we obtain the field configurations associated with the two asymptotic solutions:

$$\Psi_a^{\omega slm}(t,r,\theta,\tilde{\varphi}_+)\cong S_\omega^{slm}(\theta)e^{im\tilde{\varphi}_+}\Delta^0 e^{-i(\omega-m\Omega_+)u}, \tag{84a}$$

$$\Psi_b^{\omega slm}(t,r,\theta,\tilde{\varphi}_+)\cong S_\omega^{slm}(\theta)e^{im\tilde{\varphi}_+}\Delta^{-s}e^{-i(\omega-m\Omega_+)v}, \tag{84b}$$

where

$$\Psi_{a,b}^{\omega slm}\equiv S_\omega^{slm}(\theta)e^{im\varphi}e^{-i\omega t}\psi_{a,b}^{\omega slm}(r). \tag{85}$$

It is straightforward to extend the regularity criterion of Sec. V to the Kerr case: Here, too, one finds that at the EH the variable

$$\hat{\Psi}^s\equiv\Delta^s\Psi^s \tag{86}$$

must be a perfectly smooth function of the (regularized) coordinates (exactly for the same reasons described in Sec. V for the Schwarzschild case; see also [37]).

For the application of the late-time expansion we must verify which of the above two asymptotic solutions is physically regular at the EH. Teukolsky [37] asserted that the regular solution is Ψ_b. This is obvious from the oscillatory dependence of Ψ_a on u (and of Ψ_b on v)—as we have discussed in the Schwarzschild ($w\neq 0$) case. One must recall, however, that this simple classification breaks down whenever $w-m\Omega_+=0$ (in which case the above oscillatory factors in u and v degenerate to 1). In this case the classification is more involved.

We shall now restrict attention to the static[8] case $w=0$ (the case $w\neq 0$ is not required for the analysis below). In this case, the asymptotic solutions (84a), (84b) become

$$\Psi_a^{\omega=0,slm}(r,\theta,\tilde{\varphi}_+)\cong Y^{slm}(\theta,\tilde{\varphi}_+)\Delta^0 e^{im\Omega_+ u}, \tag{87a}$$

$$\Psi_b^{\omega=0,slm}(r,\theta,\tilde{\varphi}_+)\cong Y^{slm}(\theta,\tilde{\varphi}_+)\Delta^{-s}e^{im\Omega_+ v}, \tag{87b}$$

where Y^{slm} denotes the spin-weighted spherical harmonics. Teukolsky's assertion concerning the regularity of ψ_b is now valid for $m\neq 0$ only, and we still need to find out what is the regular asymptotic behavior at the EH for $m=0$.

Fortunately, at this point we can directly apply the results from the above analysis of the Schwarzschild case, for the following reason. Let us define

$$z\equiv\frac{r-r_+}{2\sqrt{M^2-a^2}}. \tag{88}$$

The relation between Δ and z is $\Delta=4(M^2-a^2)z(z+1)$ (note that as $a\to 0$ both Δ and z coincide with their above Schwarzschild definitions). One can now verify that for $m=0,\omega=0$ the master equation takes exactly the form of Eq. (21) [23]. Therefore, the two static solutions in the Kerr case are exactly ϕ_r and ϕ_{ir} defined above (viewed as functions of z). We already know that the solution ϕ_r (like $\Delta^s\phi_r$) is a perfectly regular polynomial of z (and hence of r), whereas the solution ϕ_{ir} (like $\Delta^s\phi_{ir}$) includes a term proportional to $\ln z$ and is hence irregular at the EH.

Let us summarize the above results concerning the regularity of static (i.e., $w=0$) modes: (i) *The case* $m\neq 0$: For both $s<0$ and $s>0$, the regular solution is ψ_b (just as in the $w\neq 0$ Schwarzschild case). The field associated with this regular asymptotic solution is given in Eq. (87b).

(ii) *The case* $m=0$: For both $s<0$ and $s>0$, the regular solution is ϕ_r—just as in the static Schwarzschild case. The field associated with this regular solution is

$$Y^{s,l,m=0}(\theta)\phi_r(z), \tag{89}$$

with the function $\phi_r(z)$ given in Eq. (22), and its asymptotic behavior (for both positive and negative s) given in Eq. (23). [Note that in terms of the limit $m=0$ of Eqs. (87a), (87b), Eq. (89) conforms with Ψ_a for $s>0$ and with Ψ_b for $s<0$.]

The above results (whose detailed derivation is given in [23]) are summarized in Table I. This table displays the asymptotic form of the regular and irregular static modes for the various possible values of s,m.

After we have discussed the regularity features of the static solutions, we are in a position to analyze the decay rate

[8]Throughout this section, which deals with a Kerr background, we refer to the t-independent solutions as "static," in a slight abuse of the usual terminology. (We prefer to use here the term "static" instead of "stationary" in order to simplify the terminology and preserve the semantic analogy with the Schwarzschild case.)

TABLE I. The asymptotic forms of the physically regular and physically irregular static solutions at the EH in the Kerr case. Presented are the results for the axially symmetric ($m=0$) modes of the fields, as well as for its nonaxially symmetric ($m\neq 0$) modes. [The Schwarzschild case ($a=0$) can be read from this table by referring only to the results in the first two lines (which then apply to all m).]

	Irregular static solution	Regular static solution
$am=0, \quad s>0$	Δ^{-s}	Δ^{0}
$am=0, \quad s<0$	Δ^{0}	Δ^{-s}
$am\neq 0, \quad s<0$ and $s>0$	$\Delta^{0}e^{im\Omega_{+}u}$	$\Delta^{-s}e^{im\Omega_{+}v}$

of the late time tails along the EH, using the late-time expansion. As we mentioned above, the master equation for the Kerr background is only separable in the frequency domain. Since the late-time expansion is carried out in the time domain, we cannot take advantage of the full separability of the field equation. The dependence on φ is still separable via $e^{im\varphi}$, however, and without loss of generality we shall consider a field Ψ^{sm} of a single m (the overall perturbation field will be obtained by a superposition of all m values). To deal with the dependence on θ, we proceed as follows: We first perform the late-time expansion of the full perturbation field Ψ^{sm}, and then, for each term k in this expansion, we separate the angular dependence by decomposing into spin-weighted *spherical* harmonics. The full decomposition thus takes the form

$$\Psi^{sm}=\sum_{k=0}^{\infty}\left[\sum_{l=l_0}^{\infty} Y^{slm}(\theta,\varphi)F_k^{slm}(r)\right]v^{-k_0-k}, \qquad (90)$$

where l_0 is the minimal value of l allowed for the mode m,s in question, that is, $l_0=max(|m|,|s|)$. The parameter k_0 is defined here to be the dominant late-time power index of Ψ^{sm} along lines of constant $r>r_+$; Namely, it is determined by the multipole l which has the slowest decay at $r=$const $>r_+$. Note that by this definition, k_0 is independent of l (unlike the Schwarzschild case, in which the late-time expansion was implemented for each mode l,m in separate). An investigation of the late-time decay at fixed r [23,26] indicates that generically the dominant multipole is the one with the smallest l allowed, i.e., $l=l_0$, and its decay rate (at fixed $r>r_+$) is v^{-2l_0-3}, with all other multipoles decaying faster. This means that generically $k_0=2l_0+3$, and also, the term $k=0$ includes only one multipole, $l=l_0$ (that is, $F_{k=0}^{slm}$ vanishes for all $l>l_0$).

When the expression (90) is substituted in the master equation [37], one finds that the radial functions F_k^{slm} still admit equations of the form (14). However, in the Kerr case the source term S_k^{slm} involves also contributions from other values of l. (Actually, the source term S_k^{slm} couples a multipole l to multipoles $l\pm 1, l\pm 2$.) Still, one finds that, as in the Schwarzschild case, S_k^{slm} depends only on functions $F_{k'}$ with $k'<k$ [23]. In particular, the function $F_{k=0}^{slm}$ has no source term, so it satisfies a closed homogeneous equation, which is just the *static field equation*. This structure allows one to solve for all unknowns F_k^{slm} in an inductive manner, starting with the functions $F_{k=0}^{slm}$. (The situation here is analogous to that of a scalar field in a Kerr spacetime, analyzed in Ref. [22].)

As we have just explained, the function $F_{k=0}$ must be a static solution of the master equation. Furthermore, the regularity arguments discussed in Sec. VI (for the Schwarzschild case) are applicable to the Kerr case as well, and imply that $F_{k=0}$ must be the *regular* static solution. The decay rate of the late-time tail along the Kerr EH now follows immediately from the above discussion of the regular static solutions, as we now describe.

For $m\neq 0$, the regular static solution is ψ_b. Since $\psi_b(r)$ has the maximal amplitude allowed by the regularity criterion, the terms $k\geqslant 1$ (being proportional to v^{-k_0-k}) will be negligible. Therefore, the late-time tail at the EH will be proportional to $\psi_b(r)v^{-k_0}$, and for both $s<0$ and $s>0$ we shall have (to the leading order in Δ and $1/v$)

$$\Psi^{sm}\propto Y^{s,l_0,m}(\theta,\tilde{\varphi}_+)e^{im\Omega_+ v}\Delta^{-s}[v^{-k_0}+O(v^{-k_0-1})],$$
$$\text{for } \ m\neq 0. \qquad (91)$$

(The oscillatory factor $e^{im\Omega_+ v}$ has already been observed in the scalar-field case [24,22].) Note that the angular dependence in this expression, as well as in Eqs. (92), (93) below, only includes the multipole l_0: As was mentioned above, generically $F_{k=0}^{slm}$ vanishes for all $l>l_0$.

For $m=0$, the situation is just as in the Schwarzschild case: In the case $s<0$, the regular static solution ϕ_r is proportional to Δ^{-s}. Since this is the maximal magnitude allowed by the regularity criterion, the terms $k\geqslant 1$ will be negligible in this case too (just as in the case $m\neq 0$ above). The late-time tail at the EH will therefore be proportional to $\phi_r(r)v^{-k_0}$, and we obtain

$$\Psi^{sm}\propto Y^{s,l_0,m=0}(\theta)\Delta^{-s}[v^{-k_0}+O(v^{-k_0-1})],$$
$$\text{for } \ m=0,s<0. \qquad (92)$$

However, in the case $s>0$ (yet with $m=0$), the $k=0$ term is proportional to the regular static solution $\phi_r\propto\Delta^0$, whereas the $k=1$ term diverges like Δ^{-s}. Therefore, at the EH the $k=1$ term dominates, and the late-time asymptotic behavior near the EH is given by

$$\Psi^{sm}\propto Y^{s,l_0,m=0}(\theta)[v^{-k_0}+\overline{c}\Delta^{-s}v^{-k_0-1}+O(v^{-k_0-2})],$$
$$\text{for } \ m=0,s>0. \qquad (93)$$

In Ref. [23] we verify (by calculating $F_{k=1}$) that the coefficient $\overline{c}$ is non-vanishing, and also that the term proportional to $\Delta^{-s}v^{-k_0-1}$ includes the multipole $l=l_0$ only. Note that for $m=0,s>0$, the decay rate along the EH is v^{-k_0-1}, whereas in all other cases it is v^{-k_0}. (The decay rate along lines of constant $r>r_+$ is v^{-k_0} in all cases.)

As was mentioned above, generically $k_0=2l_0+3$. Thus, the most dominant m modes are those with $|m|\leqslant|s|$. For these m modes $l_0=|s|$, so $k_0=2|s|+3$. Therefore, in the *overall* perturbation field Ψ^s, made of the superposition of all m modes, the decay rate along the EH is generically $v^{-2|s|-3}$. Recall, however, that for $s>0$ fields the axially symmetric component ($m=0$) decays faster along the EH, like $v^{-2|s|-4}$.

Note that the discussion is this section only deals with non-extremal, $|a|<M$, Kerr BHs: the extremal case needs to be considered separately. [Note, for example, that Eq. (88) is not valid in the case $a=M$.]

X. CONCLUSIONS

To summarize our results, it is most useful to refer to the physical variables $\hat{\Psi}^s=\Delta^s\Psi^s$ defined above. These variables are natural, because, as we mentioned above, at the EH they are proportional to the regular Maxwell components F_{aV} or F_{aU}, or to the regular Weyl components C_{aVbV} or C_{aUbU} (for $s=1$, $s=-1$, $s=2$, and $s=-2$, respectively; here, a,b stand for the two regular angular coordinates, i.e., θ,φ in the Schwarzschild case and $\theta,\tilde{\varphi}_+$ in the Kerr case).

For the Schwarzschild case, we find from Eqs. (45) and (47) that along the EH, each mode l,m of the physical field $\hat{\Psi}^s$ decays at late time with the leading-order tail

$$\left.\begin{aligned}\hat{\psi}^{slm}(v)&\cong\text{const}\times v^{-2l-3+p}, \quad \text{for } s<0\\ \hat{\psi}^{slm}(v)&\cong\text{const}\times v^{-2l-4+p}, \quad \text{for } s>0\end{aligned}\right\}\quad \text{Schwarzschild BH,} \tag{94}$$

where the constants are generically non-vanishing. Here, $p=1$ if there exists an initial static multipole l, or $p=0$ in the absence of such a multipole. (In the calculations above we assumed $p=0$, but the extension to the $p=1$ case is trivial.) For the comparison with the Kerr case below (in which a full angular separability of the late-time tails is not possible), it is useful to re-write Eq. (94) in terms of the field $\hat{\Psi}^{sm}$, which is the part of $\hat{\Psi}^s$ including all multipoles l for a given m. $\hat{\Psi}^{sm}$ is dominated by the minimal multipole l allowed for the values s,m, i.e., $l_0\equiv max(|m|,|s|)$. Equation (94) yields

$$\left.\begin{aligned}\hat{\Psi}^{sm}&\cong\text{const}\times Y^{s,l_0,m}(\theta,\varphi)v^{-2l_0-3+p}, \quad \text{for } s<0\\ \hat{\Psi}^{sm}&\cong\text{const}\times Y^{s,l_0,m}(\theta,\varphi)v^{-2l_0-4+p}, \quad \text{for } s>0\end{aligned}\right\}\quad \text{Schwarzschild BH.} \tag{95}$$

This holds for both axially symmetric and nonaxially symmetric modes.

For the (nonextremal) Kerr case, we find from Eqs. (91)–(93) that the physical field $\hat{\Psi}^{sm}$ for a given value of m decays along the EH according to

$$\hat{\Psi}^{sm}\cong\text{const}\times Y^{s,l_0,m}(\theta,\tilde{\varphi}_+)e^{im\Omega_+v}v^{-2l_0-3+p}, \quad \text{Kerr BH,} \quad m\neq0, \tag{96}$$

for nonaxially symmetric modes, and

$$\left.\begin{aligned}\hat{\Psi}^{sm}&\cong\text{const}\times Y^{s,l_0,m=0}(\theta)v^{-2l_0-3+p}, \quad \text{for } s<0\\ \hat{\Psi}^{sm}&\cong\text{const}\times Y^{s,l_0,m=0}(\theta)v^{-2l_0-4+p}, \quad \text{for } s>0\end{aligned}\right\}\quad \text{Kerr BH,} \quad m=0, \tag{97}$$

for axially symmetric modes.

The *overall* late-time behavior of the field $\hat{\Psi}^s$ is dominated by the m values which yield the minimal possible value of l_0, i.e., the values $-|s|\leqslant m\leqslant|s|$, for which $l_0=|s|$. Thus, the overall decay rate of $\hat{\Psi}^s$ is obtained from Eqs. (95), (96), (97) by substituting $l_0\to|s|$. Note, however, that the angular dependence of the overall late-time field (and, in the Kerr case, also the oscillations in v) will be obtained by a superposition over all m values in the range $-|s|\leqslant m\leqslant|s|$.

The late-time behavior of an $s>0$ field along the EH of the Kerr black hole displays two important differences between the axially symmetric and nonaxially symmetric modes. First, the modes $m\neq0$ oscillate along the horizon's generators according to $e^{im\Omega_+v}$, whereas the mode $m=0$ decays monotonically. Second, the modes $m\neq0$ (with $|m|\leqslant s$) decay like v^{-2s-3}, whereas the mode $m=0$ decays along the EH like v^{-2s-4}. (The first difference applies also for $s\leqslant0$ fields, but the second one is special to $s>0$.) These two differences lead to an interesting consequence: The *overall* $s>0$ late-time field oscillates along the generators of the Kerr EH (because the non-oscillatory mode $m=0$ decays faster than the oscillatory modes $m\neq0$). On the other hand, for $s<0$ fields the overall late-time tail at the Kerr EH is a superposition of oscillatory $m\neq 0$ tails and a monotonic $m=0$ tail (which decays at the same rate).

The above difference between the $s<0$ and the $s>0$ fields in the power-law indices of the tails along the EH has never been reported before (as far as we know), even in the Schwarzschild case. In Ref. [26] Hod attempts to calculate these tails in the Kerr case, but his analysis yields no difference between the $s<0$ and $s>0$ fields, even for $m=0$ (see, however, the note added). In what follows we briefly explain what seems to be the reason for the incorrect result in Ref. [26].

Hod uses the correct asymptotic radial solution for $\omega\neq 0$ modes near the EH, which for $m=0$ reads $\psi_\omega(r)\cong C(\omega)\Delta^{-s}e^{-i\omega r_*}$ [see Eq. (39) in [26][9]]. He then continues by assuming that $C(\omega)$ is ω independent. This assumption, however, is invalid in the case $am=0,s>0$, where $C(\omega)\propto\omega$. In the Schwarzschild case (for any m), this result was demonstrated in the framework of our toy model in Sec. VIII [see Eq. (72)], and can also be verified for the actual $s>0$ fields by using the Teukolsky-Starobinsky identities [see Eqs. (77), (78)]. In the Kerr case (for $m=0$), the situation seems to be the same.[10] If the correct form, $C(\omega)\propto\omega$, is used in the case $m=0,s>0$, then, Eq. (40) in [26] correctly yields a tail smaller by $1/v$ than the $s<0$ tail.

The asymptotic behavior of the various physical fields along the EH is important for understanding the dynamics of these fields *inside* the black hole: One can naturally view the field value at the EH as initial data for the black-hole's interior. (This is most naturally implemented within the framework of the characteristic initial-value formulation.) Of special importance is the case of gravitational perturbations ($s=\pm 2$) of the Kerr background. In this case, evolving the perturbation from the EH to the future, determines the gravitational field (and hence the spacetime geometry) inside the black hole, up to the inner horizon. The infinite blue-shift of the gravitational perturbations leads to a curvature singularity at the inner horizon [3]. Knowing the late-time behavior of the perturbation along the Kerr EH enables one to analyze in detail the structure of this singularity [39].

Note added. After this paper was submitted, Hod presented the correct result for $s>0$ fields (the one derived in Sec. IX above), in a recent manuscript [28].

[9]Note that in [26] a different dependent field variable is used, which is $\Delta^{s/2}$ times the Newman-Penrose fields that we use in the present paper.

[10]As was shown in Sec. IX, for $m=0,s>0$ (in Kerr) the regular solution switches from ψ_b to ψ_a as ω vanishes (just like for $s>0$ in the Schwarzschild case). This switching seems to indicate the vanishing of $C(\omega)$ as $\omega\to 0$, as discussed in Sec. VIII. [Note that for $m\neq 0,s>0$, the regular solution is ψ_b for both $\omega=0$ and $\omega\neq 0$ (we are only interested here in small ω values near $\omega=0$, so we can now assume $\omega-m\Omega_+\neq 0$), indicating that in this case $C(\omega)$ is non-vanishing as $\omega\to 0$. The same holds for $s<0$ (and any m).]

APPENDIX: DERIVATION OF $\Psi^{s>0}$ FROM $\Psi^{s<0}$ USING THE TEUKOLSKY-STAROBINSKY IDENTITIES

In this appendix we obtain the asymptotic form of $\Psi^{s>0}$ at the EH of a Schwarzschild BH from that of $\Psi^{s<0}$, using the Starobinsky-Teukolsky identities. By this we shall recover Eq. (47), and show that the leading-order coefficient c_1^+ does not vanish. (This was already verified in Sec. VI by a direct calculation of $\Psi^{s>0}$ from the field equation for $s>0$.)

The Teukolsky-Starobinsky identities [40] relate the perturbation fields $\Psi^{s>0}$ and $\Psi^{s<0}$. In the case of the Schwarzschild background, these identities take the form

$$(a_s-i\omega b_s\partial_t)\psi^s_\omega(r)=\mathcal{D}_0^{2s}[\psi^{-s}_\omega(r)]\quad (s>0),\qquad (A1)$$

where $\psi^s_\omega(r)$ is the radial Fourier mode introduced in Eq. (56), a_s is a non-vanishing constant, b_s is a constant which vanishes for $s=1$ but not for $s=2$, and $\mathcal{D}_0$ is a differential operator given by

$$\mathcal{D}_0(r)\equiv\partial_r-i\omega r^2\Delta^{-1}.\qquad (A2)$$

(There also is an analogous identity for transforming from $s>0$ to $s<0$.) If we now apply the inverse Fourier transform to Eq. (A1), we obtain for the time domain function $\psi^{slm}(r,t)$ [the one introduced in Eq. (11)]

$$(a_s+b_s\partial_t)\psi^{slm}=\hat{\mathcal{D}}_s[\psi^{-s,lm}]\quad (s>0),\qquad (A3)$$

where $\hat{\mathcal{D}}_s$ is the differential operator

$$\hat{\mathcal{D}}_s(r,t)\equiv(\partial_r+r^2\Delta^{-1}\partial_t)^{2s}=(2r^2\Delta^{-1}\partial_v)^{2s}.\qquad (A4)$$

Here, the v derivative is taken with fixed u, and the t derivative is taken with fixed r.

Before making use of this identity to study the late-time tails, let us briefly discuss its application to the static solutions. In the static case, Eq. (A3) reduces to

$$a_s\psi^{slm}=(\partial_r)^{2s}\psi^{-s,lm}\quad (s>0,\ \text{static}).\qquad (A5)$$

Consider first the application of this identity to the regular static solution ϕ_r^- (namely, we take $s>0$ and $\psi^{-s,lm}=\phi_r^-$). Since ϕ_r^- is a polynomial in r, the right-hand side must be a polynomial too. Since the outcome must be a static solution for $s>0$, it must be the polynomial static solution, i.e., ϕ_r^+ (up to some constant). This confirms our previous conclusion, namely, that for $s>0$ the regular static solution is ϕ_r^+ and not ϕ_{ir}^+ (i.e., the one proportional to Δ^0 and not to Δ^{-s}).

Next, consider the application of the identity (A5) to the other, irregular, static solution ϕ_{ir}^-. When the differential operator $(\partial_r)^{2s}$ acts on the logarithmic-free terms in the right-hand side of Eq. (33) (2nd row), it yields a regular polynomial, as before. However, when applied to the logarithmic term, it yields two types of terms: (i) logarithmic terms, proportional to Δ^0—these are obtained if the derivative operator never acts on the factor ln(z). (ii) Non-logarithmic terms proportional to negative (as well as positive) powers of Δ: These terms are obtained when one of the operators ∂_r acts on the factor ln(z). The most dominant negative power is obtained when ∂_r acts on ln(z) on its $|s|+1$ operation, which yields a contribution proportional to Δ^{-s} (all other contribution are of less negative powers of Δ). One can identify the terms (i) and (ii) with the second and first terms, respectively, in the 1st row on the right-hand side of Eq. (33). Note the crucial role played by the logarithmic term in ϕ_{ir}^{-} (despite the fact that it only appears in a sub-dominant term proportional to $\Delta^{|s|}$): Without this logarithmic term, the operation in Eq. (A5) would have yielded a perfectly smooth function of r, proportional to Δ^0.

We shall now apply the (time-domain) Teukolsky-Starobinsky identity (A3) to the $s<0$ late-time field (44). For the consistency of the notation, we shall assume that $s>0$ [as dictated by the notation of Eq. (A3)], and therefore re-write Eq. (44) as

$$\psi^{-s,lm}(t,r)\cong c_0^{-}\phi_r^{-}(r)v^{-2l-3}\quad (s>0). \tag{A6}$$

Applying the differential operator $\hat{\mathcal{D}}_s$ to the right-hand side of Eq. (A6), we obtain three types of terms:

(i) a term in which the derivative operator ∂_v never acts on v^{-2l-3},

(ii) a term in which the derivative operator ∂_v acts on v^{-2l-3} once, and

(iii) terms in which the derivative operator ∂_v acts on v^{-2l-3} more than once.

Consider first the term (i). As was discussed above, the operator $\hat{\mathcal{D}}_s$ transforms ϕ_r^{-} into ϕ_r^{+}. Hence the term (i) is nothing but

$$c_0^{-}v^{-2l-3}\hat{\mathcal{D}}_s(\phi_r^{-})=\text{const}\times v^{-2l-3}\cdot\phi_r^{+} \tag{A7}$$

(with a non-vanishing constant).

The term (ii) is proportional to v^{-2l-4}. We must keep this term, however, because its radial function will appear to diverges at the EH [whereas the radial function in the term (i) is regular]. The regular static solution ϕ_r^{-} is, at the leading order, proportional to Δ^s (recall that now $s>0$). The most divergent contribution is obtained when the differential operator ∂_v acts on v^{-2l-3} on its $|s|+1$ application (with the other contributions smaller by factors of Δ). This contribution yields

$$\begin{aligned}\partial_v(v^{-2l-3})(\partial_r)^{s-1}[2r^2\Delta^{-1}(\partial_r)^s(\Delta^s)]\\ \cong[2(2l+3)(-1)^s s!(s-1)!(2M)^{2s+1}]\\ \times\Delta^{-s}v^{-2l-4}[1+O(\Delta)].\end{aligned} \tag{A8}$$

(Recall that when acted on a purely radial function, $2r^2\Delta^{-1}\partial_v=r^2\Delta^{-1}\partial_{r*}=\partial_r$.)

The terms (iii) decay as v^{-2l-5} or faster. The radial functions involved in these terms do not diverge faster than Δ^{-s} (in fact, they diverge even slower). Therefore, these terms may be neglected. We find that

$$\begin{aligned}\hat{\mathcal{D}}_s(\psi^{-s,lm})\cong(i)+(ii)=(\tilde{c}_0v^{-2l-3}+\tilde{c}_1\Delta^{-s}v^{-2l-4})\\ \times[1+O(\Delta)+O(1/v)],\end{aligned} \tag{A9}$$

where both constants $\tilde{c}_0,\tilde{c}_1$ are nonvanishing and proportional to c_0^{-}.

So far we considered the contribution to $\hat{\mathcal{D}}_s(\psi^{-s,lm})$ from the term of $\psi^{-s,lm}$ proportional to v^{-2l-3}, which is given in Eq. (A6). This leading-order term corresponds to the term $k=0$ in Eq. (13). We now consider the contribution of the $k>0$ terms. From Eq. (40) we learn that for $s<0$ all functions F_k^{slm} are smooth functions of r, which vanish at least like $\Delta^{|s|}$ at the EH. One can now easily analyze the contribution of each term $k>0$ in the same way the dominant $k=0$ contribution was analyzed above. Again one obtains contributions which are analogous to the terms (i), (ii), or (iii) below, except that these contributions are now multiplied by an extra factor v^{-k}. The contributions from all $k>0$ terms of $\psi^{-s,lm}$ can therefore be neglected, and we are left with Eq. (A9).

Once $\hat{\mathcal{D}}_s(\psi^{-s,lm})$ is known, we can calculate ψ^{slm} via Eq. (A3). For $s=1$, the coefficient b_s vanishes, so $\psi^{slm}=\hat{\mathcal{D}}_s(\psi^{-s,lm})/a_s$. For $s=2$, the left-hand side of Eq. (A3) includes the derivative operator ∂_t. To extract ψ^{slm} in this case, we apply the differential operator $a_s+b_s\partial_t$ to the right-hand side of Eq. (13) (recalling $\partial_t\to\partial_v$), and solve for F_k^{slm} term by term by matching the powers of $1/v$ to Eq. (A9). For $k=0$, this matching yields

$$a_sF_{k=0}^{slm}=\tilde{c}_0+O(\Delta). \tag{A10}$$

For $k=1$ we obtain

$$a_sF_{k=1}^{slm}-(2l+3)b_sF_{k=0}^{slm}=\tilde{c}_1\Delta^{-s}[1+O(\Delta)], \tag{A11}$$

which, in view of Eq. (A10), we simply write as

$$a_sF_{k=1}^{slm}=\tilde{c}_1\Delta^{-s}[1+O(\Delta)]. \tag{A12}$$

We thus obtain

$$\psi^{slm}=(c_0^{+}v^{-2l-3}+c_1^{+}\Delta^{-s}v^{-2l-4})[1+O(\Delta)+O(1/v)]$$

$$(s>0), \tag{A13}$$

with $c_{0,1}^{+}\equiv\tilde{c}_{0,1}/a_s$.

Thus, relying on the late-time behavior (44) for $s<0$, and using the Teukolsky-Starobinsky identities, we have recovered Eq. (47) for $s>0$—with non-vanishing coefficients c_0^{+} and c_1^{+}.

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