Binary black-hole problem at the third post-Newtonian approximation in the orbital motion: Static part

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Post-Newtonian expansions of the Brill-Lindquist and Misner-Lindquist solutions of the time-symmetric two-black-hole initial value problem are derived. The static Hamiltonians related to the expanded solutions, after identifying the bare masses in both solutions, are found to differ from each other at the third post-Newtonian approximation. By shifting the position variables of the black holes the post-Newtonian expansions of the three metrics can be made to coincide up to the fifth post-Newtonian order resulting in identical static Hamiltonians up the third post-Newtonian approximation. The calculations shed light on previously performed binary point-mass calculations at the third post-Newtonian approximation. [S0556-2821(99)01922-0]

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I. INTRODUCTION AND SUMMARY

In a recent paper by the authors [1] the claim was put forward that at the third post-Newtonian (3PN) approximation of general relativity point-mass models for binary systems have to be replaced by black-hole models to become unique. To confirm the claim that binary point-mass models are incomplete at the 3PN approximation, in the present paper the Brill-Lindquist (BL) [2] and Misner-Lindquist (ML) [3,4] solutions of the time-symmetric initial value problem for binary black holes are expanded into post-Newtonian series and post-Newtonian Hamiltonians related to these expansions are calculated.

The BL and ML solutions are known to differ from each other topologically as well as geometrically [5,6]. The interesting question therefore arises if at the 3PN order of approximation in the relative motion differences show up. The remarkable outcome of our calculations from Sec. II is that the two solutions have different Hamiltonians starting at the 3PN order, but also that these Hamiltonians can be made to coincide by shifting the centers of the black holes. Especially interesting shifts are those where the two black holes of the ML solution obtain vanishing dipole moments so as to possibly coincide with the monopolar black-hole potentials of the BL solution to higher orders.

It is evident that at the 3PN order of approximation point-mass models in many-body systems are no longer applicable. In Sec. III we perform static (i.e., with linear momenta of the bodies and transverse-traceless part of the three metric set equal to zero) binary point-mass calculations using different regularization methods which lead to different metric coefficients and Hamiltonians, i.e., we end up with a static point-mass ambiguity. Section IV is devoted to some consistency calculations for the regularization procedures of Sec. III and to the comparison with previous results. Some details of the calculations are given in two appendixes.

Although it holds that through the postulate of vanishing black-hole dipole moments in isotropic coordinates a unique static Hamiltonian can be obtained at the 3PN order of approximation, the problem of finding a similar unique total Hamiltonian is more complicated because the point-mass ambiguity detected in the previous paper [1] is a dynamical one which includes the momenta of the objects as well as the radiation degrees of freedom of the gravitational field (transverse-traceless part of the three metric). This ambiguity lies far beyond any known exact or approximate solutions of the Einstein field equations and thus, it cannot be resolved in a way the BL and ML solutions for the constraint equations allow for a clear identification of the ambiguous static contributions. The static ambiguity has not been mentioned in the paper [1].

We use units in which $16\pi G=c=1$, where G is the Newtonian gravitational constant and c the velocity of light. We employ the following notation: $\mathbf{x}=(x^i)(i=1,2,3)$ denotes a point in the three-dimensional Euclidean space endowed with a standard Euclidean metric and a scalar product (denoted by a dot). Letters a and b are body labels ($a,b=1,2$), so $\mathbf{x}_a$ denotes the position of the ath body, and m_a denotes its mass parameter. We also define $\mathbf{r}_a:=\mathbf{x}-\mathbf{x}_a$, $r_a:=|\mathbf{r}_a|$, $\mathbf{n}_a:=\mathbf{r}_a/r_a$; and for $a\neq b$, $\mathbf{r}_{ab}:=\mathbf{x}_a-\mathbf{x}_b$, $r_{ab}:=|\mathbf{r}_{ab}|$, $\mathbf{n}_{ab}:=\mathbf{r}_{ab}/r_{ab}$; $|\cdot|$ stands here for the Euclidean length of a vector. Indices with round brackets, like in $\phi_{(2)}$, give the order of the object in inverse powers of the velocity of light, in this case, $1/c^2$. We abbreviate $\delta(\mathbf{x}-\mathbf{x}_a)$ by δ_a.

II. PN EXPANDED BRILL-LINDQUIST AND MISNER-LINDQUIST SOLUTIONS

In the static case, as defined above, the three metric can be put into conformally flat form [cf. Eq. (3) in [1]]

$$g_{ij}=\left(1+\frac{1}{8}\phi\right)^4\delta_{ij}. \tag{1}$$

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We use the representation of the BL and ML solutions as given in Ref. [6]. For the BL solution the function ϕ from Eq. (1) equals [cf. Eq. (3) in [6]]

$$\phi^{\mathrm{BL}}=8\left(\frac{\alpha_1}{r_1}+\frac{\alpha_2}{r_2}\right). \tag{2}$$

Here the positive parameters α_1 and α_2 can be expressed in terms of the bare masses m_1 and m_2 of the black holes and the coordinate distance r_{12} between them. The relations are given in Appendix A. The ML solution is described by the function [cf. Eq. (4) in [6]]

$$\phi^{\mathrm{ML}}=8\left(\frac{a}{r_1}+\frac{b}{r_2}\right)+8\sum_{n=2}^{\infty}\left(\frac{a_n}{|\mathbf{x}-\mathbf{d}_n|}+\frac{b_n}{|\mathbf{x}-\mathbf{e}_n|}\right), \tag{3}$$

where $\mathbf{x}_1$ in $r_1=|\mathbf{x}-\mathbf{x}_1|$ is the position of the center of the black hole 1 of radius a and $\mathbf{x}_2$ in $r_2=|\mathbf{x}-\mathbf{x}_2|$ is the position of the center of the black hole 2 of radius b, relative to a given origin in the flat space; $\mathbf{d}_n(n\geqslant 2)$ are the positions of the image poles of black hole 1, $\mathbf{e}_n(n\geqslant 2)$ are the positions of the image poles of black hole 2, a_n and $b_n(n\geqslant 2)$ are the corresponding weights. For the ML solution the choice of the bare masses is not as obvious as in the case of the BL solution [6]. We use the definition of the bare masses introduced by Lindquist in [4]. Then the solution (3) can be iteratively expressed in terms of the bare masses m_1, m_2 and the vector $\mathbf{r}_{12}$ connecting the centers of the black holes, cf. Appendix B. In the following we identify the bare masses of both solutions.

The post-Newtonian expansions of the functions ϕ^{BL} and ϕ^{ML} can be written as follows

$$\phi^{\mathrm{BL}}=\phi^{\mathrm{BL}}_{(2)}+\phi^{\mathrm{BL}}_{(4)}+\phi^{\mathrm{BL}}_{(6)}+\phi^{\mathrm{BL}}_{(8)}+\phi^{\mathrm{BL}}_{(10)}+\mathcal{O}\left(\frac{1}{c^{12}}\right), \tag{4}$$

$$\phi^{\mathrm{ML}}=\phi^{\mathrm{ML}}_{(2)}+\phi^{\mathrm{ML}}_{(4)}+\phi^{\mathrm{ML}}_{(6)}+\phi^{\mathrm{ML}}_{(8)}+\phi^{\mathrm{ML}}_{(10)}+\mathcal{O}\left(\frac{1}{c^{12}}\right), \tag{5}$$

where $\phi_{(n)}$ are functions of $(n/2)$PN order, as they belong to the three metric. The details of the expansions are given in Appendixes A and B. The results read

$$\phi^{\mathrm{BL}}_{(2)}=\phi^{\mathrm{ML}}_{(2)}=\frac{1}{4\pi}\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right), \tag{6}$$

$$\phi^{\mathrm{BL}}_{(4)}=\phi^{\mathrm{ML}}_{(4)}=-\frac{2}{(16\pi)^2}\frac{m_1m_2}{r_{12}}\left(\frac{1}{r_1}+\frac{1}{r_2}\right), \tag{7}$$

$$\phi^{\mathrm{BL}}_{(6)}=\phi^{\mathrm{ML}}_{(6)}=\frac{1}{(16\pi)^3}m_1m_2(m_1+m_2)\frac{1}{r_{12}^2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right), \tag{8}$$

$$\phi^{\mathrm{BL}}_{(8)}=-\frac{1}{2(16\pi)^4}m_1m_2(m_1^2+3m_1m_2+m_2^2)\times\frac{1}{r_{12}^3}\left(\frac{1}{r_1}+\frac{1}{r_2}\right), \tag{9}$$

$$\phi^{\mathrm{BL}}_{(10)}=\frac{1}{4(16\pi)^5}m_1m_2(m_1+m_2)(m_1^2+5m_1m_2+m_2^2)\times\frac{1}{r_{12}^4}\left(\frac{1}{r_1}+\frac{1}{r_2}\right), \tag{10}$$

$$\phi^{\mathrm{ML}}_{(8)}=\phi^{\mathrm{BL}}_{(8)}+\frac{1}{2(16\pi)^4}\frac{m_1m_2}{r_{12}^2}\left(m_2^2\frac{\mathbf{n}_2\cdot\mathbf{n}_{12}}{r_2^2}-m_1^2\frac{\mathbf{n}_1\cdot\mathbf{n}_{12}}{r_1^2}\right), \tag{11}$$

$$\phi^{\mathrm{ML}}_{(10)}=\phi^{\mathrm{BL}}_{(10)}+\frac{1}{4(16\pi)^5}\frac{m_1m_2}{r_{12}^3}\left[m_1^2(m_1+6m_2)\frac{\mathbf{n}_1\cdot\mathbf{n}_{12}}{r_1^2}-m_2^2(m_2+6m_1)\frac{\mathbf{n}_2\cdot\mathbf{n}_{12}}{r_2^2}-m_1m_2(m_1+m_2)\times\frac{1}{r_{12}}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right]. \tag{12}$$

The equations (6)–(12) show that the ML solution at the 4PN order of approximation attributes a dipole moment to each black hole whereas the BL solution, as it is already evident from the exact expression (2), shows monopoles only. In shifting the centers of the black holes one can arrange that also in case of the ML solution the dipole moments do vanish. To show this let us introduce

$$\phi^{\mathrm{BL}}_{\leqslant 5\mathrm{PN}}(\mathbf{x};\mathbf{x}_1,\mathbf{x}_2):=\phi^{\mathrm{BL}}_{(2)}+\phi^{\mathrm{BL}}_{(4)}+\phi^{\mathrm{BL}}_{(6)}+\phi^{\mathrm{BL}}_{(8)}+\phi^{\mathrm{BL}}_{(10)}, \tag{13}$$

$$\phi^{\mathrm{ML}}_{\leqslant 5\mathrm{PN}}(\mathbf{x};\mathbf{x}_1,\mathbf{x}_2):=\phi^{\mathrm{ML}}_{(2)}+\phi^{\mathrm{ML}}_{(4)}+\phi^{\mathrm{ML}}_{(6)}+\phi^{\mathrm{ML}}_{(8)}+\phi^{\mathrm{ML}}_{(10)}. \tag{14}$$

Then the shifted ML solution can be defined as

$$\phi^{\mathrm{ML\ shifted}}_{\leqslant 5\mathrm{PN}}(\mathbf{x};\mathbf{x}_1,\mathbf{x}_2):=\phi^{\mathrm{ML}}_{\leqslant 5\mathrm{PN}}(\mathbf{x};\mathbf{x}_1+\alpha\mathbf{r}_{12},\mathbf{x}_2+\beta\mathbf{r}_{21}), \tag{15}$$

where α and β are some dimensionless parameters. We have found that for

$$\alpha=\frac{1}{(16\pi)^3}\frac{m_1^2m_2}{8r_{12}^3}-\frac{1}{(16\pi)^4}\frac{m_1^2(m_1m_2+5m_2^2)}{16r_{12}^4}, \tag{16}$$

$$\beta=\frac{1}{(16\pi)^3}\frac{m_1m_2^2}{8r_{12}^3}-\frac{1}{(16\pi)^4}\frac{m_2^2(m_1m_2+5m_1^2)}{16r_{12}^4}, \tag{17}$$

the shifted ML solution coincides with the BL solution up to the 5PN order of approximation:

$$\phi^{\text{ML shifted}}_{\leqslant 5\text{PN}}(\mathbf{x};\mathbf{x}_1,\mathbf{x}_2)=\phi^{\text{BL}}_{\leqslant 5\text{PN}}(\mathbf{x};\mathbf{x}_1,\mathbf{x}_2)+\mathcal{O}\left(\frac{1}{c^{12}}\right). \quad (18)$$

The Hamiltonian we calculate by means of formula

$$H=-\lim_{R\to\infty}\oint_{S(0,R)} d\sigma_i \phi_{,i}\,, \quad (19)$$

where $S(0,R)$ is a sphere of radius R centered at the origin of the coordinate system. Making use of Eqs. (2) and (3) we obtain that the BL and ML solutions lead to the Hamiltonians

$$H^{\text{BL}}=32\pi(\alpha_1+\alpha_2), \quad (20)$$

$$H^{\text{ML}}=32\pi(a+b)+32\pi\sum_{n=2}^{\infty}(a_n+b_n). \quad (21)$$

Using Eqs. (20) and (21) [or Eqs. (6)–(12) together with Eq. (19)] we calculate the static Hamiltonian up to the 3PN order of approximation [notice: the nPN Hamiltonian is determined by the $(n+2)$PN three metric]. We obtain, dropping the total mass m_1+m_2 contribution (in the reduced variables [7])

$$\hat{H}^{\text{BL}}_{\leqslant 3\text{PN}}=-\lim_{R\to\infty}\oint_{S(0,R)} d\sigma_i(\phi^{\text{BL}}_{\leqslant 5\text{PN}})_{,i}$$
$$=-\frac{1}{r}+\frac{1}{2r^2}-\frac{1}{4}(1+\nu)\frac{1}{r^3}+\frac{1}{8}(1+3\nu)\frac{1}{r^4}, \quad (22)$$

$$\hat{H}^{\text{ML}}_{\leqslant 3\text{PN}}=-\lim_{R\to\infty}\oint_{S(0,R)} d\sigma_i(\phi^{\text{ML}}_{\leqslant 5\text{PN}})_{,i}$$
$$=-\frac{1}{r}+\frac{1}{2r^2}-\frac{1}{4}(1+\nu)\frac{1}{r^3}+\frac{1}{8}(1+2\nu)\frac{1}{r^4}. \quad (23)$$

Obviously, both 3PN Hamiltonians are different. The difference vanishes, however, if in the case of the ML solution the shifted solution for the potential function ϕ is used in the calculation of the Hamiltonian. Therefore, the postulate of vanishing black-hole dipole moments in isotropic coordinates yields a unique static 3PN Hamiltonian for binary black holes. The condition of vanishing dipole moments needs the metric coefficients; it cannot be formulated on the Hamiltonian level alone.

III. BINARY POINT-MASS CALCULATIONS

In this section we show the results of the static binary point-mass calculations. In the static case the Hamiltonian constraint equation for the two-body point-mass system in the canonical formalism of Arnowitt-Deser-Misner (ADM) reads

$$\left(1+\frac{1}{8}\phi\right)\Delta\phi=-\sum_a m_a\delta_a \quad (24)$$

(for binary systems all sums run over $a=1,2$). Equation (24) yields the following formal expansion

$$\Delta\phi=-\left(1+\frac{1}{8}\phi\right)^{-1}\sum_a m_a\delta_a$$
$$=-\left(\sum_a m_a\delta_a\right)\sum_{n=0}^{\infty}\left(-\frac{1}{8}\phi\right)^n. \quad (25)$$

Using Eq. (25) we obtain the Hamiltonian constraint equations valid at individual orders in $1/c$. They read

$$\Delta\phi_{(2)}=-\sum_a m_a\delta_a\,, \quad (26)$$

$$\Delta\phi_{(4)}=\frac{1}{8}\phi_{(2)}\sum_a m_a\delta_a\,, \quad (27)$$

$$\Delta\phi_{(6)}=\left(-\frac{1}{64}\phi_{(2)}^2+\frac{1}{8}\phi_{(4)}\right)\sum_a m_a\delta_a\,, \quad (28)$$

$$\Delta\phi_{(8)}=\left(\frac{1}{512}\phi_{(2)}^3-\frac{1}{32}\phi_{(2)}\phi_{(4)}+\frac{1}{8}\phi_{(6)}\right)\sum_a m_a\delta_a\,, \quad (29)$$

$$\Delta\phi_{(10)}=\left(-\frac{1}{4096}\phi_{(2)}^4+\frac{3}{512}\phi_{(2)}^2\phi_{(4)}-\frac{1}{64}\phi_{(4)}^2\right.$$
$$\left.-\frac{1}{32}\phi_{(2)}\phi_{(6)}+\frac{1}{8}\phi_{(8)}\right)\sum_a m_a\delta_a\,. \quad (30)$$

All Poisson equations (26)–(30) are of the form

$$\Delta\phi=\sum_a f(\mathbf{x})\delta_a\,, \quad (31)$$

where the function f is usually singular at $\mathbf{x}=\mathbf{x}_a$. One can propose three different ways of solving equations of type (31). The first two ways are based on the following sequence of equalities:

$$\phi=\Delta^{-1}\left(\sum_a f(\mathbf{x})\delta_a\right)=\Delta^{-1}\left(\sum_a f_{\text{reg}}(\mathbf{x}_a)\delta_a\right)$$
$$=\sum_a f_{\text{reg}}(\mathbf{x}_a)\Delta^{-1}\delta_a=-\frac{1}{4\pi}\sum_a f_{\text{reg}}(\mathbf{x}_a)\frac{1}{r_a}, \quad (32)$$

where $f_{\text{reg}}(\mathbf{x}_a)$ is the regularized value of the function f at $\mathbf{x}=\mathbf{x}_a$, defined by means of the Hadamard's ''partie finie'' procedure [8]. The difference between the two first methods relies on different evaluating the regular value of the products of singular functions. In the first method we use

$$(f_1(\mathbf{x})f_2(\mathbf{x}))\delta_a=f_{1\text{reg}}(\mathbf{x}_a)f_{2\text{reg}}(\mathbf{x}_a)\delta_a\,, \quad (33)$$

whereas in the second method instead of the rule (33) we apply

$$(f_1(\mathbf{x}) f_2(\mathbf{x}))\delta_a = (f_1 f_2)_{\rm reg}(\mathbf{x}_a)\delta_a. \tag{34}$$

For the developments in the book by Infeld and Plebański [9], it was crucial that the both regularization procedures coincided, i.e., $(f_1 f_2)_{\rm reg} = f_{1\,\rm reg} f_{2\,\rm reg}$ (''tweedling of products''). In the third method we regularize the Poisson integral rather than the source function:

$$\phi = -\frac{1}{4\pi}\sum_a \int d^3x' \frac{f(\mathbf{x}')\,\delta(\mathbf{x}'-\mathbf{x}_a)}{|\mathbf{x}'-\mathbf{x}|} = -\frac{1}{4\pi}\sum_a \left(\frac{f(\mathbf{x}')}{|\mathbf{x}'-\mathbf{x}|}\right)_{\rm reg} (\mathbf{x}'=\mathbf{x}_a). \tag{35}$$

Let us denote the results of applying the regularization method based on Eqs. (32) and (33) by primes, the results of the method based on Eqs. (32) and (34) by double primes, and the results of the method based on Eq. (35) by triple primes. It turns out that up to $1/c^6$ the results of the three methods coincide and, moreover, they are identical with the results of the expansions of the BL and ML solutions:

$$\phi'_{(n)} = \phi''_{(n)} = \phi'''_{(n)} = \phi^{\rm BL}_{(n)} = \phi^{\rm ML}_{(n)}, \quad n=2,4,6. \tag{36}$$

The results of the first method coincide with the expansion of the BL solution, what we have checked up to the 5PN order:

$$\phi'_{(n)} = \phi^{\rm BL}_{(n)}, \quad n=2,4,6,8,10. \tag{37}$$

The function $\phi''_{(8)}$ calculated by means of the second method coincides with $\phi'_{(8)}$ (and $\phi^{\rm BL}_{(8)}$), and the function $\phi''_{(10)}$ reads

$$\phi''_{(10)} = \phi^{\rm BL}_{(10)} + \frac{1}{2(16\pi)^5}\frac{m_1^2 m_2^2}{r_{12}^4}\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right). \tag{38}$$

The functions $\phi'''_{(8)}$ and $\phi'''_{(10)}$ are equal to

$$\phi'''_{(8)} = \phi^{\rm BL}_{(8)} + \frac{1}{2(16\pi)^4}\frac{m_1 m_2}{r_{12}^2}\left(m_1^2\frac{\mathbf{n}_1\cdot\mathbf{n}_{12}}{r_1^2} - m_2^2\frac{\mathbf{n}_2\cdot\mathbf{n}_{12}}{r_2^2}\right), \tag{39}$$

$$\phi'''_{(10)} = \phi^{\rm BL}_{(10)} + \frac{1}{4(16\pi)^5}\frac{m_1 m_2}{r_{12}^3}\left\{ m_2^2(m_2+6m_1)\frac{\mathbf{n}_2\cdot\mathbf{n}_{12}}{r_2^2} - m_1^2(m_1+6m_2)\frac{\mathbf{n}_1\cdot\mathbf{n}_{12}}{r_1^2} + \frac{m_1 m_2}{r_{12}}\left[(2m_1+m_2)\frac{1}{r_1} + (2m_2+m_1)\frac{1}{r_2}\right]\right\}. \tag{40}$$

Notice, the mass parameters m_1 and m_2 in Eqs. (36)–(40) denote total rest masses of infinitely separated bodies whereas the mass parameters in Eqs. (24)–(30) denote some formal rest masses only; the former results from the latter by our regularization procedures. The total rest-mass parameters we identify with the bare masses of the BL and ML solutions.

By means of Eq. (19) we calculate the static Hamiltonian up to the 3PN order applying the functions $\phi'_{(n)}$, $\phi''_{(n)}$, and $\phi'''_{(n)}$. The results are (dropping the total mass m_1+m_2 contribution)

$$\hat{H}'^{\rm static}_{\leqslant 3\rm PN} = -\lim_{R\to\infty}\oint_{S(0,R)} d\sigma_i (\phi'_{\leqslant 5\rm PN})_{,i} = -\frac{1}{r} + \frac{1}{2r^2} - \frac{1}{4}(1+\nu)\frac{1}{r^3} + \frac{1}{8}(1+3\nu)\frac{1}{r^4}, \tag{41}$$

$$\hat{H}''^{\rm static}_{\leqslant 3\rm PN} = -\lim_{R\to\infty}\oint_{S(0,R)} d\sigma_i (\phi''_{\leqslant 5\rm PN})_{,i} = -\frac{1}{r} + \frac{1}{2r^2} - \frac{1}{4}(1+\nu)\frac{1}{r^3} + \frac{1}{8}(1+4\nu)\frac{1}{r^4}, \tag{42}$$

$$\hat{H}'''^{\rm static}_{\leqslant 3\rm PN} = -\lim_{R\to\infty}\oint_{S(0,R)} d\sigma_i (\phi'''_{\leqslant 5\rm PN})_{,i} = -\frac{1}{r} + \frac{1}{2r^2} - \frac{1}{4}(1+\nu)\frac{1}{r^3} + \frac{1}{8}\left(1+\frac{9}{2}\nu\right)\frac{1}{r^4}. \tag{43}$$

Obviously, at the 3PN order of approximation, the three Hamiltonians differ from each other and from the Hamiltonian (23) obtained from the expanded ML solution. The Hamiltonian (41) coincides with the Hamiltonian (22) obtained using the PN expansion of the BL solution, what obviously follows from Eq. (37).

IV. CONSISTENCY CALCULATIONS AND COMPARISON WITH THE PREVIOUS RESULTS

In the region $\Omega := B(0,R)\backslash[B(\mathbf{x}_1,\varepsilon_1)\cup B(\mathbf{x}_2,\varepsilon_2)]$ [where $B(\mathbf{x}_a,\varepsilon_a)(a=1,2)$ is a ball of radius ε_a around the position $\mathbf{x}_a$ of the ath body and $B(0,R)$ is a ball of radius R centered at the origin of the coordinate system] the right-hand sides of the Eqs. (26)–(30) vanish, so the functions $\phi_{(n)}$ fulfil the Laplace equation in this region. Applying Gauss's theorem we thus obtain

$$0 = \int_\Omega d^3x\,\Delta\phi_{(n)} = \oint_{\partial B(\mathbf{x}_1,\varepsilon_1)} d\sigma_i \phi_{(n),i} + \oint_{\partial B(\mathbf{x}_2,\varepsilon_2)} d\sigma_i \phi_{(n),i} + \oint_{\partial B(0,R)} d\sigma_i \phi_{(n),i}, \tag{44}$$

with the normal vectors pointing inwards the spheres $\partial B(\mathbf{x}_a,\varepsilon_a)$ and outwards the sphere $\partial B(0,R)$. From Eq. (44) it follows that

$$\lim_{R\to\infty}\oint_{\partial B(0,R)} d\sigma_i \phi_{(n),i} = -\lim_{\varepsilon_1\to 0}\oint_{\partial B(\mathbf{x}_1,\varepsilon_1)} d\sigma_i \phi_{(n),i} - \lim_{\varepsilon_2\to 0}\oint_{\partial B(\mathbf{x}_2,\varepsilon_2)} d\sigma_i \phi_{(n),i}, \tag{45}$$

so the Hamiltonian $H_{n\mathrm{PN}}$ at the nPN order, according to Eq. (19), can be calulated as

$$H_{n\mathrm{PN}} = -\lim_{\varepsilon_1\to 0}\oint_{\partial B(\mathbf{x}_1,\varepsilon_1)} d\sigma_i \phi_{(2n+4),i} - \lim_{\varepsilon_2\to 0}\oint_{\partial B(\mathbf{x}_2,\varepsilon_2)} d\sigma_i \phi_{(2n+4),i}. \tag{46}$$

We have used Eq. (46) and the functions $\phi'_{(n)}$, $\phi''_{(n)}$, and $\phi'''_{(n)}$, to calculate the static Hamiltonian up to the 3PN order. The integrals over the spheres $\partial B(\mathbf{x}_a,\varepsilon_a)$ diverge as $\varepsilon_a \to 0$, so to calculate them we have used the Hadamard's procedure [8]. The results coincide with those given by Eqs. (41)–(43).

The nPN Hamiltonian can also be written in the form of a volume integral

$$H_{n\mathrm{PN}} = -\int d^3x\, \Delta\phi_{(2n+4)}, \tag{47}$$

so still another way of calculating it relies on direct integration of the (minus) right-hand sides of Eqs. (26)–(30). To do this one must use the Hadamard's regularization [8] together with the rule (33) or (34). For the functions $\phi'_{(n)}$ we have used the rule (33) (as in deriving the functions $\phi'_{(n)}$), whereas for the functions $\phi''_{(n)}$ and $\phi'''_{(n)}$ we have applied the rule (34). The results coincide again with those given by Eqs. (41)–(43).

In the paper [1] we applied the ''double prime'' regularization procedure and had thus obtained the Hamiltonian (42); see Sec. VI in Ref. [1]. The same result we had obtained from the static n-body Hamiltonian of Ref. [10] by applying some expansion-and-limiting procedure. If we take the n-body static Hamiltonian of [10] but specialize it to the two-body case adapting only those terms which are directly finite, we get the BL result of Eq. (22). It corresponds to our ''single prime'' regularization procedure described above.

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APPENDIX A: BRILL-LINDQUIST SOLUTION

We use the representation of the Brill-Lindquist (BL) solution taken from Appendix A of [6]. The BL solution can be written in the form [cf. Eq. (3) in [6]]

$$\phi^{\mathrm{BL}} = 8\left(\frac{\alpha_1}{r_1} + \frac{\alpha_2}{r_2}\right). \tag{A1}$$

The bare masses m_1 and m_2 of the black holes depend on the parameters α_1, α_2 and the coordinate distance r_{12} between the black holes [cf. Eqs. (A7) and (A8) in [6]]:

$$m_1 = 32\pi\alpha_1\left(1+\frac{\alpha_2}{r_{12}}\right), \quad m_2 = 32\pi\alpha_2\left(1+\frac{\alpha_1}{r_{12}}\right). \tag{A2}$$

The unique positive solutions of Eqs. (A2) for α_1 and α_2 read

$$\alpha_1 = -\frac{1}{4}\left(2r_{12} + \frac{m_2-m_1}{16\pi}\right) + \frac{1}{4}r_{12}\sqrt{4+\frac{m_1+m_2}{4\pi r_{12}}+\left(\frac{m_1-m_2}{16\pi r_{12}}\right)^2}, \tag{A3}$$

$$\alpha_2 = -\frac{1}{4}\left(2r_{12} + \frac{m_1-m_2}{16\pi}\right) + \frac{1}{4}r_{12}\sqrt{4+\frac{m_1+m_2}{4\pi r_{12}}+\left(\frac{m_1-m_2}{16\pi r_{12}}\right)^2}. \tag{A4}$$

We expand the right-hand sides of Eqs. (A3) and (A4) in powers of $1/c$ taking into account that the masses m_1 and m_2 can be regarded as being of order $1/c^2$. The results we substitute into Eq. (A1) to obtain the post-Newtonian expansion of the function ϕ^{BL}. Such obtained functions $\phi^{\mathrm{BL}}_{(n)}$ for $n = 2,4,6,8,10$ are given in Eqs. (6), (7), (8), (9), and (10), respectively.

APPENDIX B: MISNER-LINDQUIST SOLUTION

The form of the Misner-Lindquist (ML) solution we use is taken from Appendix B of [6]. The ML solution we write in the form [cf. Eq. (4) in [6]]

$$\phi^{\mathrm{ML}} = 8\left(\frac{a}{r_1}+\frac{b}{r_2}\right) + 8\sum_{n=2}^{\infty}\left(\frac{a_n}{|\mathbf{x}-\mathbf{d}_n|}+\frac{b_n}{|\mathbf{x}-\mathbf{e}_n|}\right), \tag{B1}$$

where $r_a = |\mathbf{x}-\mathbf{x}_a|$ $(a=1,2)$ and $\mathbf{x}_a$ is the position of the center of the ath black hole relative to a given origin in the flat space, a and b are the radii of the black hole 1 and 2, respectively; $\mathbf{d}_n(n\geqslant 2)$ are the positions of the image poles of black hole 1, $\mathbf{e}_n(n\geqslant 2)$ are the positions of the image poles of black hole 2, a_n and $b_n(n\geqslant 2)$ are the corresponding weights.

The black hole 1 together with its odd images and the even images of the black hole 2 are located on the positive z axis, whereas the black hole 2 together with its odd images and the even images of the black hole 1 lie on the negative z axis. The relative distances $|\mathbf{x}-\mathbf{d}_n|$ and $|\mathbf{x}-\mathbf{e}_n|$ entering Eq. (B1) can be expressed by radii a and b of the black holes and their relative position vector $\mathbf{r}_{12}$ as follows (here $n\geqslant 2$)

$$
\begin{aligned}
|\mathbf{x}-\mathbf{d}_n| &= \sqrt{r_1^2+D_n^2+2r_1D_n(\mathbf{n}_1\cdot\mathbf{n}_{12})} \quad \text{for } n \text{ odd},\\
|\mathbf{x}-\mathbf{d}_n| &= \sqrt{r_2^2+D_n'^2-2r_2D_n'(\mathbf{n}_2\cdot\mathbf{n}_{12})} \quad \text{for } n \text{ even},\\
|\mathbf{x}-\mathbf{e}_n| &= \sqrt{r_2^2+E_n'^2-2r_2E_n'(\mathbf{n}_2\cdot\mathbf{n}_{12})} \quad \text{for } n \text{ odd},\\
|\mathbf{x}-\mathbf{e}_n| &= \sqrt{r_1^2+E_n^2+2r_1E_n(\mathbf{n}_1\cdot\mathbf{n}_{12})} \quad \text{for } n \text{ even},
\end{aligned}
\tag{B2}
$$

where

$$
\begin{aligned}
D_n &= r_{12}\left(1-\frac{\sinh[(n+1)\mu_0]}{\sinh[(n+1)\mu_0]+(a/b)\sinh[(n-1)\mu_0]}\right) \quad \text{for } n \text{ odd},\\
D_n' &= r_{12}\left(1-\frac{ab\sinh[(n+2)\mu_0]+a^2\sinh n\mu_0}{r_{12}^2\sinh n\mu_0}\right) \quad \text{for } n \text{ even},\\
E_n &= r_{12}\left(1-\frac{ab\sinh[(n+2)\mu_0]+b^2\sinh n\mu_0}{r_{12}^2\sinh n\mu_0}\right) \quad \text{for } n \text{ even},\\
E_n' &= r_{12}\left(1-\frac{\sinh[(n+1)\mu_0]}{\sinh[(n+1)\mu_0]+(b/a)\sinh[(n-1)\mu_0]}\right) \quad \text{for } n \text{ odd}.
\end{aligned}
\tag{B3}
$$

The quantity μ_0 entering Eqs. (B3) is given by

$$
\cosh 2\mu_0=\frac{r_{12}^2-a^2-b^2}{2ab}. \tag{B4}
$$

The weights a_n and $b_n(n\geqslant 2)$ of the image poles depend on the radii a and b of the black holes and the distance r_{12} between them:

$$
\begin{aligned}
a_n &= \frac{ab\sinh 2\mu_0}{r_{12}\sinh n\mu_0} \quad \text{for } n \text{ even},\\
a_n &= \frac{ab\sinh 2\mu_0}{b\sinh[(n+1)\mu_0]+a\sinh[(n-1)\mu_0]} \quad \text{for } n \text{ odd},\\
b_n &= \frac{ab\sinh 2\mu_0}{r_{12}\sinh n\mu_0} \quad \text{for } n \text{ even},\\
b_n &= \frac{ab\sinh 2\mu_0}{a\sinh[(n+1)\mu_0]+b\sinh[(n-1)\mu_0]} \quad \text{for } n \text{ odd}.
\end{aligned}
\tag{B5}
$$

To obtain the post-Newtonian expansion of the ML solution we expand the right-hand side of Eq. (B1) in powers of $1/c$ taking into account that the radii a and b are of order $1/c^2$. To do this we use Eqs. (B2)–(B5). We have found that only the first five terms from the sum on the right-hand side of Eq. (B1) contributes to the 5PN order. The result of the expansion reads

$$
\begin{aligned}
\frac{1}{8}\phi^{\mathrm{ML}} &= \frac{a}{r_1}+\frac{b}{r_2}+\frac{ab}{r_{12}}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)+\frac{ab}{r_{12}^2}\left(\frac{a}{r_1}+\frac{b}{r_2}\right)+\frac{a^2b^2}{r_{12}^3}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)+\frac{ab}{r_{12}^2}\left(\frac{b^2(\mathbf{n}_2\cdot\mathbf{n}_{12})}{r_2^2}-\frac{a^2(\mathbf{n}_1\cdot\mathbf{n}_{12})}{r_1^2}\right)\\
&\quad+\frac{a^2b^2(a+b)}{r_{12}^4}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)+\frac{ab}{r_{12}^3}\left(\frac{b^3(\mathbf{n}_2\cdot\mathbf{n}_{12})}{r_2^2}-\frac{a^3(\mathbf{n}_1\cdot\mathbf{n}_{12})}{r_1^2}\right)+\mathcal{O}\left(\frac{1}{c^{12}}\right).
\end{aligned}
\tag{B6}
$$

For the ML solution we use the definition of the bare masses m_1 and m_2 introduced by Lindquist in [4]. The masses m_1 and m_2 depend on the radii a and b of the black holes and the distance r_{12} between the centers of the black holes [cf. Eqs. (B19) and (B20) in Ref. [6]]:

$$
m_1=\frac{32\pi ab}{r_{12}}\sinh 2\mu_0\sum_{n=1}^{\infty} n\left\{\frac{2}{\sinh 2n\mu_0}+\frac{1}{\sinh(2n\mu_0-\mu_2)}+\frac{1}{\sinh(2n\mu_0+\mu_2)}\right\}, \tag{B7}
$$

$$
m_2=\frac{32\pi ab}{r_{12}}\sinh 2\mu_0\sum_{n=1}^{\infty} n\left\{\frac{2}{\sinh 2n\mu_0}+\frac{1}{\sinh[2(n+1)\mu_0-\mu_2]}+\frac{1}{\sinh[2(n-1)\mu_0-\mu_2]}\right\}, \tag{B8}
$$

where μ_2 is given by

$$
\sinh\mu_2=\frac{a}{r_{12}}\sinh 2\mu_0. \tag{B9}
$$

We have iteratively solved Eqs. (B7) and (B8) with respect to a and b. The result, valid up to the 5PN order of approximation, reads

$$a = \frac{1}{16\pi}\frac{m_1}{2} - \frac{1}{(16\pi)^2}\frac{m_1 m_2}{2r_{12}} + \frac{1}{(16\pi)^3}\frac{m_1 m_2(2m_1+3m_2)}{8r_{12}^2} - \frac{1}{(16\pi)^4}\frac{m_1 m_2(m_1^2+4m_1m_2+2m_2^2)}{8r_{12}^3} + \frac{1}{(16\pi)^5}\frac{m_1 m_2(2m_1^3+14m_1^2m_2+18m_1m_2^2+5m_2^3)}{32r_{12}^4} + \mathcal{O}\left(\frac{1}{c^{12}}\right), \tag{B10}$$

the equation for b can be obtained from the above one by replacements $m_1 \to m_2$ and $m_2 \to m_1$.

To obtain the PN expansion of the ML solution we substitute Eqs. (B10) into Eq. (B6). Such obtained functions $\phi_{(n)}^{\mathrm{ML}}$ for $n=2,4,6,8,10$ are given in Eqs. (6), (7), (8), (11), and (12), respectively.

[1] P. Jaranowski and G. Schäfer, Phys. Rev. D **57**, 7274 (1998).

[2] D. R. Brill and R. W. Lindquist, Phys. Rev. **131**, 471 (1963).

[3] C. W. Misner, Ann. Phys. (N.Y.) **24**, 102 (1963).

[4] R. W. Lindquist, J. Math. Phys. **4**, 938 (1963).

[5] M. Cantor and A. D. Kulkarni, Phys. Rev. D **25**, 2521 (1982).

[6] Z. Andrade and R. H. Price, Phys. Rev. D **56**, 6336 (1997).

[7] The reduced relative position vector $\mathbf{r}$ is defined as $\mathbf{r} := 16\pi(\mathbf{x}_1 - \mathbf{x}_2)/M$, together with $r := |\mathbf{r}|$ and $\mathbf{n} := \mathbf{r}/r$; here $M = m_1 + m_2$. We also introduce the parameter $\nu := \mu/M$, where $\mu := m_1 m_2/M$ is the reduced mass of the system. As the reduced Hamilton function $\hat{H}$ we define $\hat{H} := H/\mu$. $\hat{H}$ depends on masses of the binary system only through the parameter ν. From now on the hat will indicate division by the reduced mass μ.

[8] The regularized value of the function f at its singular point $\mathbf{x} = \mathbf{x}_0$ is based on the Hadamard's ''partie finie'' regularization. We expand $f(\mathbf{x}_0 + \varepsilon\mathbf{n})$ (where $\mathbf{n}$ is a unit vector) into a Laurent series around $\varepsilon = 0$: $f(\mathbf{x}_0 + \varepsilon\mathbf{n}) = \Sigma_{m=-N}^{\infty} a_m(\mathbf{n})\varepsilon^m$. The coefficients a_m in this expansion depend on the unit vector $\mathbf{n}$. The regularized value of the function f at $\mathbf{x}_0$ is the coefficient a_0 averaged over all directions, i.e., $f_{\mathrm{reg}}(\mathbf{x}_0) := (1/4\pi)\oint d\Omega\, a_0(\mathbf{n})$. We also always define $\int d^3x\, f(\mathbf{x})\,\delta(\mathbf{x} - \mathbf{x}_a) := f_{\mathrm{reg}}(\mathbf{x}_a)$. More details on the applications of the Hadamard's regularization can be found in P. Jaranowski, in *Mathematics of Gravitation. Part II. Gravitational Wave Detection,* edited by A. Królak (Institute of Mathematics, Polish Academy of Sciences, Warszawa, 1997), Vol. 41, Part II, p. 55.

[9] L. Infeld and J. Plebański, *Motion and Relativity* (Pergamon Press, Oxford, 1960), p. 217.

[10] T. Kimura and T. Toiya, Prog. Theor. Phys. **48**, 316 (1972).