

## Perturbative solutions of Bohmian quantum gravity

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(Received 24 May 1999; published 29 October 1999)

In this paper we have solved the Bohmian equations of quantum gravity, perturbatively. Solutions up to second order are derived explicitly, but in principle the method can be used in any order. Some consequences of the solution are discussed. [S0556-2821(99)01222-9]

PACS number(s): 04.60.Ds, 03.65.Bz, 98.80.Hw

### I. INTRODUCTION

Recently [1–3], a perturbative method for solving classical Einstein's equations in its Hamilton-Jacobi form was presented. The method rests on expanding the Hamilton-Jacobi generating functional in terms of the powers of spatial gradients of the metric and matter fields, and then solving the equations order by order. This expansion is valid when the characteristic scale of spatial variation of physical quantities is larger than the characteristic length of the theory, e.g., the Hubble's radius. In fact it can be shown that the solution can be calculated at any order. The form of the Hamilton-Jacobi generating functional in each order is chosen such that it will be 3-diffeomorphic invariant. This method is used and examined for many physical cases. Salopek *et al.* [1] have solved the Hamilton-Jacobi and the momentum constraint equations in the presence of matter fields up to second order in spatial gradients. Parry *et al.* [2] have used a specific conformal transformation of 3-metric to simplify the Hamiltonian and solve the problem in higher orders of spatial gradients. Then they have compared their results with exact solutions for some specific cases and obtained a recursion relation for different orders and so they have presented the solution up to any order. In addition, similar calculations are made for Brans-Dicke theory. [3]

An essential question would be whether the method can be applied to quantum gravity realm. Unfortunately there are different approaches to quantum gravity, none of them completely acceptable and self-consistent. These include the standard Wheeler-DeWitt (WDW) canonical approach [4], the Hawking path integral approach [5], the Narlikar-Padmanabhan quantization of conformal degree of freedom of the space-time metric [6], the Bohmian approach to quantum gravity [7], and the approach presented by author *et al.* as geometrization of quantum theory [8]. Among these approaches, Bohmian quantum gravity is of our concern here, because it highly relates to Hamilton-Jacobi theory. In fact, as we shall review in the next section, in Bohmian quantum gravity one encounters a modified Hamilton-Jacobi equation.

We shall apply the above-mentioned perturbative method for solving Bohmian quantum gravity equations. We shall do this up to the second order, but in principle the method can be applied to any order.

### II. BOHMIAN QUANTUM GRAVITY

Bohm's theory is a causal version of quantum mechanics [9]. According to this theory, any particle is accompanied with an objectively real field ( $\Psi$ ) satisfying Schrödinger equation. This field exerts a quantum force derivable from a quantum potential given by

$$Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 |\Psi|}{|\Psi|}. \quad (1)$$

This theory is motivated from the fact that when one sets  $\Psi = |\Psi| \exp[iS/\hbar]$  in the Schrödinger equation, one arrives at a continuity equation

$$\frac{\partial |\Psi|^2}{\partial t} + \vec{\nabla} \cdot \left( |\Psi|^2 \frac{\vec{\nabla} S}{m} \right) = 0, \quad (2)$$

and a modified Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{|\vec{\nabla} S|^2}{2m} + V + Q = 0, \quad (3)$$

in which  $V$  represents the classical potential and  $Q$ , the quantum potential, is defined as above. The main positive point of Bohm's theory, which uses only quantum potential, is that it is able to explain all enigmatic aspects of quantum theory. These include presentation of a causal description for wavefunction collapse during a measurement, description of uncertainty relations, and also presentation of particle trajectories [9]. Particle trajectory can be obtained through the modified Hamilton-Jacobi equation (3) and by using the guidance relation  $\vec{p} = \vec{\nabla} S$ , or using Newton's law of motion including the quantum potential. It is worth noting that the trajectories explain many nonordinary behaviors in quantum mechanics. For example, particle trajectories in a two-slit experiment can be calculated [7] and it can be seen how quantum potential forces particles to move in such a way to make the interference pattern.

Bohm's theory can be applied to any system. Application of this theory to gravity leads to Bohmian quantum gravity. Its properties and positive points are expressed in the literature [7,9]. Application of Bohm's theory to quantum gravity has several advantages. First of all, in this approach different quantities such as the 3-space geometry, intrinsic and extrinsic curvatures of the space-like surfaces, and so on have physical reality without any dependence upon the measure-

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ment process. Second, the metric has a definite time evolution in this theory. Third, in this approach the wave function has two roles. One role is in generating the quantum potential and the other is in acting as the probabilistic interpretation. When one deals with a single system (as is the case for quantum cosmology) for which the probability is not defined, the first role of the wave function is important. Note that in the standard quantum theory only the second role is highlighted, and thus the meaning of the wave function is questionable in quantum gravity. Finally, the classical limit is

well defined in Bohm's theory. When both quantum potential and its gradient are small compared to the classical potential and its gradient, then we are in the classical limit. This allows one, in specific cases, to have quantum effects at large scales and classical limit in small scales.

Here we use Bohmian quantum gravity, not only because of the above-mentioned advantages, but also because it highly relates to the Hamilton-Jacobi equation. Before proceeding, we present the Bohmian equations for quantum gravity [7] which we shall refer to later. These equations are<sup>1</sup> (setting  $\hbar = c = 8\pi G = 1$ )

$$\frac{\delta}{\delta h_{ij}} \left( 2h^q G_{ijkl} \frac{\delta \mathcal{S}}{\delta h_{kl}} \mathcal{A}^2 \right) + \frac{\delta}{\delta \phi} \left( \frac{h^q}{\sqrt{h}} \frac{\delta \mathcal{S}}{\delta \phi} \mathcal{A}^2 \right) = 0, \quad (4)$$

$$G_{ijkl} \frac{\delta \mathcal{S}}{\delta h_{ij}} \frac{\delta \mathcal{S}}{\delta h_{kl}} + \frac{1}{2\sqrt{h}} \left( \frac{\delta \mathcal{S}}{\delta \phi} \right)^2 - \sqrt{h} (\mathcal{R}^{(3)} + 2\Lambda - Q_G) + \frac{1}{2} \sqrt{h} h^{ij} \partial_i \phi \partial_j \phi + \frac{1}{2} \sqrt{h} (V + Q_M) = 0, \quad (5)$$

$$Q_G = - \frac{1}{\sqrt{h} \mathcal{A}} \left( G_{ijkl} \frac{\delta^2 \mathcal{A}}{\delta h_{ij} \delta h_{kl}} + h^{-q} \frac{\delta h^q G_{ijkl}}{\delta h_{ij}} \frac{\delta \mathcal{A}}{\delta h_{kl}} \right), \quad (6)$$

$$Q_M = - \frac{1}{h \mathcal{A}} \frac{\delta^2 \mathcal{A}}{\delta \phi^2}, \quad (7)$$

$$2\nabla_j \frac{\delta \mathcal{A}}{\delta h_{ij}} - h^{ij} \partial_j \phi \frac{\delta \mathcal{A}}{\delta \phi} = 0, \quad (8)$$

$$2\nabla_j \frac{\delta \mathcal{S}}{\delta h_{ij}} - h^{ij} \partial_j \phi \frac{\delta \mathcal{S}}{\delta \phi} = 0, \quad (9)$$

in which  $\mathcal{A}$  is the norm of the wave function,  $\mathcal{S}$  is its phase times  $\hbar$  and is in fact the quantum Einstein-Hamilton-Jacobi function,  $q$  is an ordering parameter,  $h_{ij}$  is the spatial metric in Arnowitt-Deser-Misner (ADM) decomposition of the space-time metric,  $G_{ijkl}$  is super metric on 3-space,  $\phi$  denotes the matter field, and  $Q_G$  and  $Q_M$  are gravity and matter quantum potentials, respectively.

Equation (4) is the continuity equation representing the conservation law of probability in the super space, and Eq. (5) is the quantum Einstein-Hamilton-Jacobi equation, which shows that the difference between quantum and classical worlds is only the presence of the quantum potential consisting of two terms, gravity and matter quantum potentials. Equations (8) and (9) are 3-diffeomorphism invariance conditions for  $\mathcal{A}$  and  $\mathcal{S}$ . Time evolution of metric and the matter field can be derived from the canonical relations

$$\pi^{kl} = \frac{\delta \mathcal{S}}{\delta h_{kl}} = \frac{\sqrt{h}}{2} (K^{kl} - h^{kl} K), \quad (10)$$

$$\pi_\phi = \frac{\delta \mathcal{S}}{\delta \phi} = \frac{\sqrt{h}}{N} \dot{\phi} - \sqrt{h} \frac{N^i}{N} \partial_i \phi, \quad (11)$$

$$K_{ij} = \frac{1}{2N} (\nabla_i N_j + \nabla_j N_i - \dot{h}_{ij}), \quad (12)$$

in which  $N$  and  $N^i$  are the lapse and shift functions, respectively, and  $K_{ij}$  is the extrinsic curvature of the 3-space. It can be seen that in Bohmian quantum gravity, there is no time problem. Time emerges naturally from the equations of motion. Bohmian trajectories can be obtained from the above equations. For example, the Bohmian trajectories for Robertson-Walker universe are derived by Horiguchi in [7] and other references cited in [7]. Another example is Bohmian trajectories for black holes. They are obtained in [10]. In this reference it is shown that the quantum black hole geom-

<sup>1</sup>They can be obtained by setting  $\Psi = \mathcal{A} \exp[i\mathcal{S}/\hbar]$  in the WDW equation and the 3-diffeomorphism invariance condition.

etry is highly sensible to the ordering parameter. For some specific ordering parameter, Bohmian quantum gravity presents a good framework for understanding Hawking radiation. Some other aspects of Bohmian quantum gravity can be found in [11]. For a complete review of the theory see [7].

### III. SOLVING THE EQUATIONS

It was noted in the previous section that the complete set of equations of quantum gravity are Eqs. (4), (5), (8), and (9). The first is the continuity equation, while the second is the quantum Einstein-Hamilton-Jacobi equation. The third and fourth equations guarantee that  $\mathcal{A}$  and  $\mathcal{S}$  be 3-diffeomorphic invariants. A perturbative solution can be achieved via expansion of  $\mathcal{S}$  and  $\mathcal{A}$  in terms of powers of spatial gradients. In the long-wavelength approximation a few terms of the expansion are sufficient. Therefore one should set

$$\Omega = \sum_{n=0}^{\infty} \Omega^{(2n)}; \quad \mathcal{A} = e^{\Omega}, \quad (13)$$

$$\mathcal{S} = \sum_{n=0}^{\infty} \mathcal{S}^{(2n)}. \quad (14)$$

Note that introducing the new functional  $\Omega$  will simplify the equations. In each order, the two coupled equations, quantum Einstein-Hamilton-Jacobi equation and continuity equation, should be solved. The two other equations only show that the functionals  $\mathcal{S}^{(2n)}$  and  $\Omega^{(2n)}$  must be 3-diffeomorphic invariants. By considering special forms for  $\mathcal{S}^{(2n)}$  and  $\Omega^{(2n)}$ , these equations would be satisfied automatically.

#### A. Zeroth order solution

In this order, the continuity equation reads as

$$\begin{aligned} & - \left( q + \frac{3}{2} \right) h_{ij} \frac{\delta \mathcal{S}^{(0)}}{\delta h_{ij}} + 4 \sqrt{h} G_{ijkl} \frac{\delta \Omega^{(0)}}{\delta h_{ij}} \frac{\delta \mathcal{S}^{(0)}}{\delta h_{kl}} \\ & + 2 \sqrt{h} G_{ijkl} \frac{\delta^2 \mathcal{S}^{(0)}}{\delta h_{ij} \delta h_{kl}} + 2 \frac{\delta \Omega^{(0)}}{\delta \phi} \frac{\delta \mathcal{S}^{(0)}}{\delta \phi} + \frac{\delta^2 \mathcal{S}^{(0)}}{\delta \phi^2} = 0, \end{aligned} \quad (15)$$

while the zeroth order quantum Einstein-Hamilton-Jacobi equation is

$$\begin{aligned} & 2 \sqrt{h} G_{ijkl} \frac{\delta \mathcal{S}^{(0)}}{\delta h_{ij}} \frac{\delta \mathcal{S}^{(0)}}{\delta h_{kl}} + \left( \frac{\delta \mathcal{S}^{(0)}}{\delta \phi} \right)^2 - 2 \sqrt{h} G_{ijkl} \frac{\delta^2 \Omega^{(0)}}{\delta h_{ij} \delta h_{kl}} \\ & - 2 \sqrt{h} G_{ijkl} \frac{\delta \Omega^{(0)}}{\delta h_{ij}} \frac{\delta \Omega^{(0)}}{\delta h_{kl}} + \left( q + \frac{3}{2} \right) h_{ij} \frac{\delta \Omega^{(0)}}{\delta h_{ij}} - \frac{\delta^2 \Omega^{(0)}}{\delta \phi^2} \\ & - \left( \frac{\delta \Omega^{(0)}}{\delta \phi} \right)^2 = 0, \end{aligned} \quad (16)$$

in which for simplicity of calculations, we have assumed that the scalar field has no self-interaction, i.e., we have set  $V(\phi) = 0$ .

In order for  $\mathcal{S}^0$  and  $\Omega^0$  to be 3-diffeomorphic invariants and thus satisfy Eqs. (8) and (9) automatically, one should set

$$\mathcal{S}^{(0)} = \int d^3x \sqrt{h} H(\phi), \quad (17)$$

$$\Omega^{(0)} = \int d^3x \sqrt{h} K(\phi), \quad (18)$$

in which  $H$  and  $K$  are functions of the scalar field and contain no spatial derivatives. Since  $d^3x \sqrt{h}$  is 3-diffeomorphic invariant measure, the above expressions are also 3-diffeomorphic invariant. By substituting these relations for  $\Omega^0$  and  $\mathcal{S}^0$  in Eqs. (15) and (16), we have the following equations for  $H$  and  $K$ :

$$\frac{d^2 H}{d\phi^2} - \frac{3}{2}(q+5)H + 2\sqrt{h} \left( \frac{dH}{d\phi} \frac{dK}{d\phi} - \frac{3}{4}KH \right) = 0 \quad (19)$$

$$\begin{aligned} & \frac{d^2 K}{d\phi^2} - \frac{3}{2}(q+5)K - \sqrt{h} \left( \left( \frac{dH}{d\phi} \right)^2 - \left( \frac{dK}{d\phi} \right)^2 + \frac{3}{4}K^2 - \frac{3}{4}H^2 \right) \\ & = 0. \end{aligned} \quad (20)$$

Setting both metric-dependent (terms containing  $\sqrt{h}$ ) and metric-independent terms equal to zero, we have four equations with the simultaneous solution:

$$H = A e^{\alpha\phi}; \quad \alpha = \pm \frac{\sqrt{3}}{2}, \quad (21)$$

$$K = BH, \quad (22)$$

$$q = -\frac{9}{2}, \quad (23)$$

in which  $A$  and  $B$  are constants of integration. It must be noted here that when using this solution it is a simple task to show that quantum potential is zero at this order. So the solution at this order is in fact classical.

#### B. Second order solution

In the second order, the continuity and quantum Einstein-Hamilton-Jacobi equations are, respectively,

$$\begin{aligned}
& - \left( q + \frac{3}{2} \right) h_{ij} \frac{\delta \mathcal{S}^{(2)}}{\delta h_{ij}} + 4 \sqrt{h} G_{ijkl} \frac{\delta \Omega^{(0)}}{\delta h_{ij}} \frac{\delta \mathcal{S}^{(2)}}{\delta h_{kl}} + 4 \sqrt{h} G_{ijkl} \frac{\delta \Omega^{(2)}}{\delta h_{ij}} \frac{\delta \mathcal{S}^{(0)}}{\delta h_{kl}} + 2 \sqrt{h} G_{ijkl} \frac{\delta^2 \mathcal{S}^{(2)}}{\delta h_{ij} \delta h_{kl}} + 2 \frac{\delta \Omega^{(0)}}{\delta \phi} \frac{\delta \mathcal{S}^{(2)}}{\delta \phi} \\
& + 2 \frac{\delta \Omega^{(2)}}{\delta \phi} \frac{\delta \mathcal{S}^{(0)}}{\delta \phi} + \frac{\delta^2 \mathcal{S}^{(2)}}{\delta \phi^2} = 0,
\end{aligned} \tag{24}$$

$$\begin{aligned}
& 2 \sqrt{h} G_{ijkl} \frac{\delta \mathcal{S}^{(0)}}{\delta h_{ij}} \frac{\delta \mathcal{S}^{(2)}}{\delta h_{kl}} + \frac{\delta \mathcal{S}^{(0)}}{\delta \phi} \frac{\delta \mathcal{S}^{(2)}}{\delta \phi} - \sqrt{h} G_{ijkl} \frac{\delta^2 \Omega^{(2)}}{\delta h_{ij} \delta h_{kl}} - 2 \sqrt{h} G_{ijkl} \frac{\delta \Omega^{(0)}}{\delta h_{ij}} \frac{\delta \Omega^{(2)}}{\delta h_{kl}} + \frac{1}{2} (q+3) h_{ij} \frac{\delta \Omega^{(2)}}{\delta h_{ij}} - \frac{1}{2} \frac{\delta^2 \Omega^{(2)}}{\delta \phi^2} \\
& - \frac{\delta \Omega^{(0)}}{\delta \phi} \frac{\delta \Omega^{(2)}}{\delta \phi} - \sqrt{h} (\mathcal{R}^{(3)} - \frac{1}{2} \nabla_i \phi \nabla^i \phi) = 0.
\end{aligned} \tag{25}$$

On using the zeroth order solution and again setting both terms with and without  $\sqrt{h}$  equal to zero, one arrives at the following four equations:

$$I[\mathcal{S}^{(2)}] \equiv 2 \sqrt{h} G_{ijkl} \frac{\delta^2 \mathcal{S}^{(2)}}{\delta h_{ij} \delta h_{kl}} + 3 h_{ij} \frac{\delta \mathcal{S}^{(2)}}{\delta h_{ij}} + \frac{\delta^2 \mathcal{S}^{(2)}}{\delta \phi^2} = 0, \tag{26}$$

$$\begin{aligned}
II[\mathcal{S}^{(2)}, \Omega^{(2)}] & \equiv 2 \alpha B \frac{\delta \mathcal{S}^{(2)}}{\delta \phi} - B h_{ij} \frac{\delta \mathcal{S}^{(2)}}{\delta h_{ij}} + 2 \alpha \frac{\delta \Omega^{(2)}}{\delta \phi} \\
& - h_{ij} \frac{\delta \Omega^{(2)}}{\delta \phi} = 0,
\end{aligned} \tag{27}$$

$$\begin{aligned}
III[\mathcal{S}^{(2)}, \Omega^{(2)}] & \equiv H h_{ij} \frac{\delta \mathcal{S}^{(2)}}{\delta h_{ij}} - 2 \alpha H \frac{\delta \mathcal{S}^{(2)}}{\delta \phi} - B H h_{ij} \frac{\delta \Omega^{(2)}}{\delta h_{ij}} \\
& + 2 \alpha H B \frac{\delta \Omega^{(2)}}{\delta \phi} + 2 \sqrt{h} (\mathcal{R}^{(3)} - \frac{1}{2} \nabla_i \phi \nabla^i \phi) \\
& = 0,
\end{aligned} \tag{28}$$

$$IV[\Omega^{(2)}] \equiv 2 \sqrt{h} G_{ijkl} \frac{\delta^2 \Omega^{(2)}}{\delta h_{ij} \delta h_{kl}} + 3 h_{ij} \frac{\delta \Omega^{(2)}}{\delta h_{ij}} + \frac{\delta^2 \Omega^{(2)}}{\delta \phi^2} = 0. \tag{29}$$

In order to solve the above equations, we use a different method with respect to the zeroth order. Our goal is to find the quantum corrections on the Hamilton-Jacobi functional at second order, to the classical functional  $\mathcal{S}_c^{(2)}$ . It must be noted that  $III[\mathcal{S}^{(2)}, \Omega^{(2)}] = 0$  is just the classical Einstein-Hamilton-Jacobi equation except for its third and fourth terms. So, with a glance at the form of the third and fourth terms, one easily can solve  $III = 0$  as

$$\mathcal{S}^{(2)} - B \Omega^{(2)} = \mathcal{S}_c^{(2)}. \tag{30}$$

Therefore for finding  $\mathcal{S}^{(2)}$ , it is sufficient to solve  $IV[\Omega^{(2)}] = 0$  to find  $\Omega^{(2)}$  and use the above equation. On the other hand, since for the classical limit the  $\sqrt{h}$ -independent terms of the continuity equation lead to  $I[\mathcal{S}_c^{(2)}] = 0$  and since  $I$  is linear, we have

$$\begin{aligned}
I[\mathcal{S}^{(2)}] & = I[\mathcal{S}_c^{(2)} + B \Omega^{(2)}] = I[\mathcal{S}_c^{(2)}] + B I[\Omega^{(2)}] = B I[\Omega^{(2)}] \\
& = B IV[\Omega^{(2)}]
\end{aligned} \tag{31}$$

so

$$I[\mathcal{S}^{(2)}] = 0 \Leftrightarrow IV[\Omega^{(2)}] = 0. \tag{32}$$

It remains for the second equation  $II[\mathcal{S}^{(2)}, \Omega^{(2)}] = 0$ . On using the relation (30) and linearity of  $II$ , one arrives at

$$2 \alpha \frac{\delta \Omega^{(2)}}{\delta \phi} - h_{ij} \frac{\delta \Omega^{(2)}}{\delta h_{ij}} = \frac{B}{1+B^2} \left( -2 \alpha \frac{\delta \mathcal{S}_c^{(2)}}{\delta \phi} + h_{ij} \frac{\delta \mathcal{S}_c^{(2)}}{\delta h_{ij}} \right), \tag{33}$$

which has the solution

$$\Omega^{(2)} = - \frac{B}{1+B^2} \mathcal{S}_c^{(2)} + \Lambda, \tag{34}$$

where the functional  $\Lambda$  satisfies the equation

$$2 \alpha \frac{\delta \Lambda}{\delta \phi} = h_{ij} \frac{\delta \Lambda}{\delta h_{ij}}. \tag{35}$$

In addition, using the relation (34) and  $IV[\Omega^{(2)}] = 0$  and the linearity of  $IV$ , one has  $IV[\Lambda] = 0$ . So it is sufficient to find the simultaneous solution of the relations (35) and  $IV[\Lambda] = 0$ . In finding the solution, we use the techniques of [2]. Making the conformal transformation

$$f_{ij}(x) = F^{-2}[\phi(x)] h_{ij}(x), \tag{36}$$

one can see that Eq. (35) requires  $F$  to satisfy the relation

$$-4 \alpha \frac{dF}{d\phi} = F \tag{37}$$

with the solution  $F = \text{const} \times \exp[-\phi/4\alpha]$ . The most general form of  $\Lambda$  is

$$\Lambda = \int d^3x \sqrt{f} [L(\phi) \mathcal{R}^{(3)} + M(\phi) \nabla_i \phi \nabla^i \phi] \tag{38}$$

where a tilde over any quantity represents that it is calculated using the  $f_{ij}$  metric.  $\tilde{\mathcal{R}}^{(3)}$  is the Ricci scalar curvature of  $f_{ij}$ , while  $L$  and  $M$  are some functions of the scalar field. The above expression is the most general form to make  $\Lambda$  3-diffeomorphic invariant and contains terms with spatial gradients of order two. Note that terms like  $\tilde{\nabla}^2 \phi$  can be transformed to  $\tilde{\nabla}_i \phi \tilde{\nabla}^i \phi$  by integration by part.

Now the equations (35) and  $IV[\Lambda]=0$  can be solved for  $L$  and  $M$ . The solution can be transformed back to the original metric  $h_{ij}$  using the inverse of the above conformal transformation. The result is

$$\Lambda = C \int d^3x \sqrt{h} e^{\phi/4\alpha} [\mathcal{R}^{(3)} - \frac{1}{6} \nabla_i \phi \nabla^i \phi], \quad (39)$$

where  $C$  is a constant.

For writing down  $\mathcal{S}^{(2)}$  and  $\Omega^{(2)}$  it is necessary to know  $\mathcal{S}_c^{(2)}$ . From [2], we have

$$\mathcal{S}_c^{(2)} = \frac{3}{10} \int d^3x \sqrt{h} e^{\alpha\phi} [\mathcal{R}^{(3)} - \nabla_i \phi \nabla^i \phi]. \quad (40)$$

Thus we have

$$\mathcal{S} = \mathcal{S}^{(0)} + \mathcal{S}^{(2)} + \dots = \int d^3x \sqrt{h} \left\{ e^{\alpha\phi} \left[ A + \frac{3}{10(1+B^2)} (\mathcal{R}^{(3)} - \nabla_i \phi \nabla^i \phi) \right] + BC e^{\phi/4\alpha} (\mathcal{R}^{(3)} - \frac{1}{6} \nabla_i \phi \nabla^i \phi) \right\} \dots, \quad (41)$$

$$\Omega = \Omega^{(0)} + \Omega^{(2)} + \dots = \int d^3x \sqrt{h} \left\{ e^{\alpha\phi} \left[ AB - \frac{3B}{10(1+B^2)} (\mathcal{R}^{(3)} - \nabla_i \phi \nabla^i \phi) \right] + C e^{\phi/4\alpha} (\mathcal{R}^{(3)} - \frac{1}{6} \nabla_i \phi \nabla^i \phi) \right\} \dots. \quad (42)$$

It is worth noting that the first two terms in  $\mathcal{S}$  which are scaled by  $e^{\alpha\phi}$  are of the same form as the classical solution up to second order. In fact quantum effects are introduced via the third term and the renormalization of the factor  $3/10$  in the second term to  $3/10(1+B^2)$ . An important property of the solution is the factor  $e^{4\phi/\alpha}$  of the third term which differs from the factor of the first two terms. The presence of the third term leads to new couplings between the matter field and the metric in Bohmian equations of motion, leading to highly quantic solutions.

#### IV. CONCLUDING REMARKS

As we saw, the Bohmian equations of motion for quantum gravity, i.e., the quantum Einstein-Hamilton-Jacobi equation and continuity equation, can be solved in principle as an expansion with respect to spatial gradients. We derived the solution up to the second order. As a result, since our solution contains spatial gradients, it is useful for discussing inhomogeneous space-times such as black holes, which are

partly discussed in the framework of Bohmian quantum gravity [10]. In a forthcoming paper we shall apply the result to black holes.

A point must be noted here. As we asserted previously, according to Bohm's theory, in the classical limit quantum potential and its gradient are small compared to classical potential and its gradient. This can be achieved both in the case where the norm of the wave function varies slowly and in the case where it varied highly. This is because of the fact that quantum potential is proportional to the fraction of second derivatives of the norm of the wave function and the norm itself (see, e.g., [7]). As a result, classical limit and long wavelength limit (i.e., considering only a few terms in the expansion with respect to spatial gradients) are not the same. So in Bohmian quantum gravity the comparison of the characteristic length of fluctuations with the theory's characteristic length (e.g., Hubble's radius) does not lead us to anything about the fact that the limit is either classic or quantum.

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