Cosmological models with dynamical Λ in scalar-tensor theories

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In the context of a family of scalar-tensor theories with a dynamical Λ that is a binomial on the scalar field, the cosmological equations are considered. A general baryotropic state equation $p=(\gamma-1)\rho$, for a perfect fluid is used for the matter content of the Universe. Some Friedmann-Robertson-Walker exact solutions are found; they have a scale factor which shows exponential or power law dependence on time. For some models the singularity can be avoided. Cosmological parameters such as Ω_m, Ω_Λ, q_0, and t_0 are obtained and compared with observational data. [S0556-2821(99)00220-9]

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I. INTRODUCTION

Unification theories have a nonzero cosmological constant that is about 120 orders of magnitude larger than the observed value for Λ; this constitutes the cosmological constant problem [1,2]. In order to explain and solve such a problem, and to make compatible the actual observational data with the inflationary scenario and particle physics expectations, a time dependent cosmological constant was proposed [3]. This old idea has received a lot of attention (see, e.g., [4–43]). What people have in mind is to make the vacuum energy dynamical. In such a way, during the evolution of the Universe, the energy density of the vacuum decays into particles, thus leading to the decrease of the cosmological constant and obtaining as a result, although small, a creation of particles.

A broad summary of cosmological models with a time dependent cosmological ''constant'' is given by Overduin and Cooperstock [44], reexamining there the evolution of the scale factor when λ is given as function of t, $a(t)$, H, or q. A fairly general equation of state is considered and new numerical solutions are obtained, but as in most previous works, the time dependence of the cosmological term is introduced *ad hoc*.

An alternative is an effective time dependent cosmological ''constant'' in the context of scalar-tensor theories, which becomes a true constant for $t\gg 0$ [45]. Using Jordan-Brans-Dicke theory (JBD) in particular, the ''graceful exit'' problem of old inflationary cosmology might be improved. Determining the JBD parameter ω that according to solar system experiments is $\|\omega\|\approx 500$, which has been derived from timing experiments using the Viking space probe remains a problem [46]. A better estimation of this parameter should be obtained from measure of other cosmological parameters in order to constrain ω more strongly than by means of solar system experiments [47]. However, theories of the very early Universe such as string theory, are better described in the context of JBD, which shows that ω can take negative values [48].

Thus, scalar-tensor theories, and in particular JBD, are better theories in order to get, in a natural way, a time dependent cosmological constant. Clearly, recent observational results restrict this kind of theory, e.g., the type Ia supernova (SN Ia) results, which in 1998 show that $\Omega_\Lambda\sim 0.6$ [49] implying that our Universe is speeding up. Thus, a model which attempts to describe the cosmological constant behavior should take into account the observational evidence.

In a recent work [50] we investigated the effect of a time dependent cosmological constant, in a family of scalar-tensor theories. There, we get cosmological models in the coasting period, where the time dependence on the cosmological constant occurs in a natural way. In such models we assumed a simple relation $\lambda(\phi)=c\phi(t)^n$ (with c and n constants).

The existence of inflationary phase in scalar-tensor theories (STT) has been investigated by Pimentel and Stein-Schabes [51], finding inflationary phases for a polynomial cosmological constant in a general STT, which includes Brans-Dicke model with nonzero cosmological constant. On the other hand, Guendelman [52] has investigated the requirements of the potentials in order to have scale invariance. A form of the potential needed by the global invariance was found whose energy in the conformal Einstein frame has the characteristics for a suitable inflationary universe and Λ decaying scenario for the late universe.

Motivated by these ideas, we shall consider a general STT as in our previous work [50], but now we shall consider a binomial λ function on $\phi(t)$, in order to obtain exact solutions of the field equations, from which we obtain some kind of inflationary cosmological models and related cosmological parameters. In fact, we obtain in most of our solutions a power law growth for the cosmological scale factor $a(t)\sim t^\sigma$, where $\sigma\gtrsim 1$ implies inflationary models. As is known, this is a generic feature of a class of models that attempt dynamically to solve the cosmological constant problem. In our models σ is a free parameter (at least in most of our models), in order to be adjusted by physical conditions and to be in agreement with recent data for type Ia supernovae (SN Ia), which implies $\sigma\approx 1$, and which is consistent with the nucleosynthesis [53].

Most of our solutions predict an accelerated expansion; such solutions are in agreement with the SN Ia results, but Ω_m and Ω_Λ depend on free parameters of our model. In some specific cases we get solutions with exponential growth of the scale factor.

In Sec. II we obtain the field equations and introduce our main ansatz. Section III is devoted to obtaining an expression of the density parameter and the corresponding contributions of the matter and scalar field. In Sec. IV we consider the vacuum case and obtain a set of exact solutions as well as cosmological parameters. In Sec. V we consider the general case of a baryotropic equation of state and obtain exact solutions for this general case. As an example, we calculate specifically the case of a dust fluid and a stiff matter fluid. Also, we get and discuss the solutions of the radiation case in Sec. VI and false vacuum case in Sec. VII. Finally, in Sec. VIII we summarize our results.

II. FIELD EQUATIONS

We start with the action for the most general scalar-tensor theory of gravitation [54]

$$S=\frac{1}{16\pi G}\int d^4x\sqrt{-g}[\phi R-\phi^{-1}\omega g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+2\phi\lambda(\phi)] + S_{\rm NG}, \tag{2.1}$$

where $g=\det(g_{\mu\nu})$, G is Newton's constant, $S_{\rm NG}$ is the action for the nongravitational matter. We use the signature $(-,+,+,+)$. The arbitrary functions $\omega(\phi)$ and $\lambda(\phi)$ distinguish the different scalar-tensor theories of gravitation, $\lambda(\phi)$ is a potential function and plays the role of a cosmological constant, and $\omega(\phi)$ is the coupling function of the particular theory.

The explicit field equations are

$$G_{\mu\nu}=\frac{8\pi T_{\mu\nu}}{\phi}+\lambda(\phi)\Box g_{\mu\nu}+\omega\phi^{-2}\times\left(\phi_{,\mu}\phi_{,\nu}-\frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}\right)+\phi^{-1}(\phi_{;\mu\nu}-g_{\mu\nu}\Box\phi), \tag{2.2}$$

$$\Box\phi+\frac{1}{2}\phi_{,\lambda}\phi^{,\lambda}\frac{d}{d\phi}\ln\left(\frac{\omega(\phi)}{\phi}\right)+\frac{1}{2}\frac{\phi}{\omega(\phi)}\times\left[R+2\frac{d}{d\phi}[\phi\lambda(\phi)]\right]=0, \tag{2.3}$$

where $G_{\mu\nu}$ is the Einstein tensor. The last equation can be substituted by

$$\Box\phi+\frac{2\phi^2 d\lambda/d\phi-2\phi\lambda(\phi)}{3+2\omega(\phi)}=\frac{1}{3+2\omega(\phi)}\left(8\pi T-\frac{d\omega}{d\phi}\phi_{,\mu}\phi^{,\mu}\right), \tag{2.4}$$

where $T=T^\mu_\mu$ is the trace of the stress-energy tensor. In a previous work [50] it was demonstrated that the divergenceless condition of the stress-energy matter tensor is satisfied if the field equation (2.3) is satisfied too, although our field equations are given by Eqs. (2.2) and (2.4).

In what follows we shall assume $\omega(\phi)=$const, $\lambda=\lambda(\phi)$. The corresponding field equations with a perfect fluid for the matter content in the isotropic and homogeneous line element

$$ds^2=-dt^2+a^2(t)\left[\frac{dr^2}{1-kr^2}+r^2(d\theta^2+\sin^2\theta d\phi^2)\right] \tag{2.5}$$

will be considered. Thus the field equations are

$$3\left(\frac{\dot a}{a}\right)^2+\frac{3k}{a^2}-\lambda(\phi)-\frac{8\pi\rho}{\phi}-\frac{\omega}{2}\left(\frac{\dot\phi}{\phi}\right)^2+3\frac{\dot a}{a}\frac{\dot\phi}{\phi}=0, \tag{2.6}$$

$$-2\frac{\ddot a}{a}-\left(\frac{\dot a}{a}\right)^2-\frac{k}{a^2}+\lambda(\phi)-\frac{8\pi p}{\phi}-\frac{\omega}{2}\left(\frac{\dot\phi}{\phi}\right)^2-\frac{\ddot\phi}{\phi}-2\frac{\dot a}{a}\frac{\dot\phi}{\phi}=0, \tag{2.7}$$

$$\left[\frac{\ddot\phi}{\phi}+3\frac{\dot a}{a}\frac{\dot\phi}{\phi}\right](3+2\omega)-2\left(\lambda-\phi\frac{d\lambda}{d\phi}\right)-\frac{8\pi}{\phi}(\rho-3p)=0, \tag{2.8}$$

where we have assumed $\phi=\phi(t)$, and the derivatives respect t are denoted by a dot.

Assuming a baryotropic equation of state $p=(\gamma-1)\rho$ and transforming to the time τ defined as

$$\tau=\int\phi^{1/2}dt, \tag{2.9}$$

the set of equations (2.6)–(2.8) is rewritten in the following way:

$$3\left(\frac{a'}{a}\right)^2+\frac{3k}{a^2\phi}-\frac{\lambda(\phi)}{\phi}-\frac{8\pi\rho}{\phi^2}-\frac{\omega}{2}\left(\frac{\phi'}{\phi}\right)^2+3\frac{a'}{a}\frac{\phi'}{\phi}=0, \tag{2.10}$$

$$-2\frac{a''}{a}-\frac{\phi''}{\phi}-\left(\frac{a'}{a}\right)^2-\frac{1}{2}(1+\omega)\left(\frac{\phi'}{\phi}\right)^2-3\frac{a'}{a}\frac{\phi'}{\phi}-\frac{k}{a^2\phi}+\frac{\lambda(\phi)}{\phi}-\frac{8\pi\rho(\gamma-1)}{\phi^2}=0, \tag{2.11}$$

$$(3+2\omega)\left[\frac{\phi''}{\phi}+\frac{1}{2}\left(\frac{\phi'}{\phi}\right)^2+3\frac{a'}{a}\frac{\phi'}{\phi}\right]-2\left(\frac{\lambda(\phi)}{\phi}-\frac{d\lambda(\phi)}{d\phi}\right)-\frac{8\pi\rho(4-3\gamma)}{\phi^2}=0, \tag{2.12}$$

where the derivatives respect to τ are denoted by a prime. In what follows we shall consider two important assumptions:

$$a\phi^m=\alpha, \tag{2.13}$$

$$\lambda(\phi)=\lambda_1\phi^{n_1}+\lambda_2\phi^{n_2}, \tag{2.14}$$

where m, α, λ_1, λ_2, n_1, and n_2 are constants. The first assumption is a very well-known one (see, e.g., Ref. [55], and reference therein) and it has been used as a condition for the deceleration parameter to be constant for flat models in Brans-Dicke theory. Furthermore, with this condition our field equations simplify notoriously and allow us to obtain exact solutions. The second condition is the main assumption of the present work which is motivated by the cosmological no-hair theorem for scalar-tensor theories [51], in order to study inflationary solutions in a theory of gravitation with a naturally dynamical cosmological constant. In this work we always work in the Jordan frame, where G is variable, however, we could make a conformal transformation to the Einstein frame where G is constant and we have general relativity plus a minimally coupled scalar field, then our potential becomes an exponential one, i.e., $V_1+V_2\sim\exp(\epsilon n_1\phi_c)+\exp(\epsilon n_2\phi_c)$ (where ϵ is a constant and ϕ_c is a canonically defined scalar field). This is the type of potential according to Guendelman [52] that is necesary to have scale invariance in a theory of gravitation free of the cosmological constant problem, that is, one with an early expanding phase and a Λ decaying for late times. Details of the conformal transformation for STT can be seen in Ref. [51]. With these assumptions, from Eqs. (2.10)–(2.12) we get

$$\left(\frac{\phi'}{\phi}\right)^2\left[3m^2-3m-\frac{\omega}{2}\right]+\frac{3k}{\alpha^2}\phi^{2m-1}-\lambda_1\phi^{n_1-1}-\lambda_2\phi^{n_2-1}-\frac{8\pi\rho}{\phi^2}=0, \tag{2.15}$$

$$\frac{\phi''}{\phi}(2m-1)+\left(\frac{\phi'}{\phi}\right)^2\left[-3m^2+m-\frac{\omega}{2}-\frac{1}{2}\right]-\frac{k}{\alpha^2}\phi^{2m-1}+\lambda_1\phi^{n_1-1}+\lambda_2\phi^{n_2-1}-\frac{8\pi\rho(\gamma-1)}{\phi^2}=0, \tag{2.16}$$

$$\frac{\phi''}{\phi}(3+2\omega)+\left(\frac{\phi'}{\phi}\right)^2(3+2\omega)\left(\frac{1}{2}-3m\right)-2\lambda_1\phi^{n_1-1}-2\lambda_2\phi^{n_2-1}+2\lambda_1 n_1\phi^{n_1-1}+2\lambda_2 n_2\phi^{n_2-1}\frac{8\pi\rho(4-3\gamma)}{\phi^2}=0. \tag{2.17}$$

In the following sections we shall find exact solutions on different cases, as well as cosmological parameters which allow us to compare with actual observations of today value of Hubble parameter H_0, the actual value of deceleration parameter q_0, the density parameter Ω_m, as well as the value of the vacuum energy density parameter Ω_Λ.

III. THE DENSITY PARAMETER

Before we compute exact solutions of the set of field equations (2.15)–(2.17), we shall get a general equation for Ω_m and Ω_ϕ, according to our proposed model. Assuming $k=0$, Eq. (2.6) is written as

$$1=\frac{8\pi G}{3H^2}\left[\frac{\lambda(\phi)}{8\pi G}+\frac{\rho}{G\phi}+\frac{\omega}{16\pi G}\left(\frac{\dot\phi}{\phi}\right)^2-\frac{3H}{8\pi G}\frac{\dot\phi}{\phi}\right]. \tag{3.1}$$

Defining

$$\Omega_m=\frac{8\pi\rho_m}{3H^2}\frac{1}{\phi},$$

$$\Omega_\phi=\frac{1}{3H^2}\left[\lambda(\phi)+\frac{\omega}{2}\left(\frac{\dot\phi}{\phi}\right)^2-3H\frac{\dot\phi}{\phi}\right] \tag{3.2}$$

we get

$$1=\Omega_m+\Omega_\phi. \tag{3.3}$$

Taking into account the proposed relation (2.13), we get

$$\Omega_m=\frac{8\pi\rho_m}{3m^2}\frac{1}{\phi}\left(\frac{\phi}{\dot\phi}\right)^2,$$

$$\Omega_\phi=\frac{1}{3m^2}\left(\frac{\phi}{\dot\phi}\right)^2\lambda(\phi)+\frac{\omega}{6m^2}+\frac{1}{m}. \tag{3.4}$$

According to the SN Ia observations [49], the favored value of $\Omega_m\sim 0.4\pm 0.1$ is given as a constraint to a cosmological constant

$$\Omega_\Lambda=\frac{4}{3}\Omega_m+\frac{1}{3}\pm\frac{1}{6}, \tag{3.5}$$

which implies $\Omega_\Lambda\sim 0.85\pm 0.2$. This means that the SN Ia results are sensitive to the acceleration of the expansion, and constrain $4\Omega_m/3-\Omega_\Lambda$, which corresponds to the acceleration parameter at the median redshift of this objects, $z\sim 0.4$. Then the combination $\Omega_0=\Omega_m+\Omega_\Lambda$ is constrained by the microwave background radiation (CBR) anisotropy. So that, $\Omega_0\sim 1\pm 0.2$ obtained from COBE and other measurements (see, e.g., Ref. [56]), together with $\Omega_m\sim 0.4$, define a concordance region for $\Omega_\Lambda\sim 0.6$, becoming the best fit for the universe model [57,58].

In what follows, we shall compute both parameters of density in addition to the exact solutions of our field equations, in the different cases which we consider in this work.

IV. THE VACUUM CASE

Considering a vacuum case ($\rho=0$), we get from Eqs. (2.15)–(2.17) the corresponding set of equations

$$\left(\frac{\phi'}{\phi}\right)^2\left[3m^2-3m-\frac{\omega}{2}\right]+\frac{3k}{\alpha^2}\phi^{2m-1}-\lambda_1\phi^{n_1-1}-\lambda_2\phi^{n_2-1}=0, \tag{4.1}$$

$$\frac{\phi''}{\phi}(2m-1)+\left(\frac{\phi'}{\phi}\right)^2\left[-3m^2+m-\frac{\omega}{2}-\frac{1}{2}\right]-\frac{k}{\alpha^2}\phi^{2m-1}+\lambda_1\phi^{n_1-1}+\lambda_2\phi^{n_2-1}=0, \tag{4.2}$$

$$\frac{\phi''}{\phi}(3+2\omega)+\left(\frac{\phi'}{\phi}\right)^2(3+2\omega)\left(\frac{1}{2}-3m\right)-2\lambda_1\phi^{n_1-1}-2\lambda_2\phi^{n_2-1}+2\lambda_1 n_1\phi^{n_1-1}+2\lambda_2 n_2\phi^{n_2-1}=0. \quad (4.3)$$

Naturally for vacuum, the energy density is due to the contribution of the scalar field $\phi(t)$: $\Omega_\phi=1$. We found exact solutions of this set of equations in the following cases:

(1) $k=0$, $m=1/2$. This solution is not relevant for our purpose because $\lambda(\phi)$ becomes null. $m=2/3$.

$$\phi(t)=\phi_1 t^{1/2},$$
$$a(t)=a_1 t^{-1/3},$$
$$\lambda(\phi)=\lambda_1\phi(t)^{-4},$$
$$\omega=-7/3, \quad (4.4)$$

where $\phi_1=\sqrt{2c_1}$, $a_1=\alpha(2c_1)^{-1/3}$, $\lambda_1=c_1^2/2$, $\lambda_2=0$, and c_1 is an integration constant. This solution was written directly in terms of time t, according to Eq. (2.9). Here the expansion factor is decaying with the time, in conflict with observations, and ω has a negative values. A discussion of the meaning of negative values of ω is given in Eq. [48].

The Ricci scalar for this case is given by the following expression:

$$\Re=\frac{10}{3}t^{-2}, \quad (4.5)$$

where we can see that there is an initial singularity. The today deceleration parameter has a negative value, i.e., the present solution is an accelerated cosmological model

$$q_0=-4, \quad H_0=-\frac{1}{3}t_0^{-1}, \quad (4.6)$$

according to this solution, the actual Hubble parameter has negative values, then we conclude that this model has not physical meaning.

$m\neq 1/2,\ 2/3$. In this case we get two families of solutions: (a) $\omega\neq(3m^2+m-2)/(2-3m)$.

$$\phi(t)=\phi_1 t^{2\sigma},$$
$$a(t)=a_1 t^{-2m\sigma},$$
$$\lambda(\phi)=\lambda_1\phi(t)^{-\frac{1}{\sigma}}, \quad (4.7)$$

where

$$\mu\neq-2\nu,$$
$$\sigma=\mu/(\mu+2\nu),$$
$$a_1=\alpha\phi_1^{-m},$$
$$\phi_1=[c'^{1+\frac{\nu}{\mu}}(\mu+2\nu)/2]^{2\mu/(\mu+2\nu)},$$
$$\lambda_1=(3m^2-3m-\omega/2)\mu^2 c'^{2(1+\nu/\mu)}, \quad \lambda_2=0,$$
$$\mu\equiv 12m^2-14m+4,$$
$$\nu\equiv-12m^2+5m+4\omega-6m\omega+2, \quad (4.8)$$

and c' is an integration constant. According to the solution of the present case, λ is a monomial function on $\phi(t)$, although a time decaying one. Clearly the present solution requires $m\sigma<0$ in order to be expanding.

On the other hand, the Ricci scalar is given by

$$\Re=12m\sigma(4m\sigma+1)t^{-2}. \quad (4.9)$$

According to this expression, the corresponding solution for the curvature scalar has an initial singularity. For this model, the present deceleration and Hubble parameter are

$$q_0=-1-\frac{1}{2m\sigma}, \quad H_0=-2m\sigma(t_0)^{-1}, \quad t_0=\frac{1}{H_0}2m\|\sigma\|. \quad (4.10)$$

From these last equations we can see that the present model expands with acceleration if $m\sigma<-1/2$. On the other hand, assuming $H_0\sim 65\pm 5$ km s^{-1}Mpc^{-1} [59], we get $t_0\sim 15.05\pm 1.96\|m\sigma\|$ Gyr, then the estimated age from this model is small compared with actual accepted values of t_0.

(b) $\mu=-2\nu\Rightarrow\omega=(3m^2+m-2)/(2-3m)$. For this particular relation between ω and m, we get the following solution

$$\phi(t)=c'\exp[\phi_2 t],$$
$$a(t)=a_2\exp[-m\phi_2 t],$$
$$\lambda(\phi)=\lambda_1\phi(t), \quad (4.11)$$

where $\phi_2=c'^{1/2}\mu$, $a_2=\alpha c'^{-m}$, $\lambda_2=0$, $\lambda_1=4c'(3m^2-3m-\omega/2)\nu^2$ and c' is an integration constant. In this case we get an inflationary exponential solution provided that $m\phi_2<0\Rightarrow m<0$ or $1/2<m<2/3$. This model is nonsingular, as we can see from the Ricci scalar

$$\Re=24c'm^2(6m^2-7m+2). \quad (4.12)$$

The present deceleration and Hubble parameters are given by

$$q_0=-1, \quad H_0=c'm(12m^2-14m+4), \quad (4.13)$$

thus, this model expands with constant acceleration from a nonsingular state, and with constant Hubble parameter.

(2) $k\neq 0$. $m=1/2$:

$$\phi(t)=\phi_1 t^{-2},$$
$$a(t)=a_1 t,$$
$$\lambda(\phi)=\lambda_1\phi(t), \quad (4.14)$$

where

$$\phi_1=\alpha^2\frac{3+2\omega}{k},\quad a_1=\left(\frac{k}{3+2\omega}\right)^{1/2},$$

$$\lambda_1=\frac{2k}{\alpha^2},\qquad \lambda_2=0.$$

According to this solution, $a(t)$ grows linearly with the time at a constant rate. The cosmological "constant" λ decreases with the time. In order to have a real $a(t)$, we must have $k/(3+2\omega)>0$. This is a coasting cosmological solution which has initial singularities as it is shown from the corresponding Ricci scalar

$$\Re=12(3+2\omega)t^{-2}. \tag{4.15}$$

On the other hand, as we have said, the today deceleration parameter becomes null, and the Hubble parameter is given by

$$q_0=0,\quad H=t_0^{-1}, \tag{4.16}$$

then, $t_0=1/H_0\sim15.05$ Gyr, in relative agreement with actual observations.

V. EXACT SOLUTIONS FOR THE CASE WITH A BARYOTROPIC EQUATION OF STATE

In what follows we shall consider $\rho\neq0$, so that returning to the set of equations (2.15)–(2.17), we get from Eq. (2.15)

$$\frac{8\pi\rho}{\phi^2}=\left(\frac{\phi'}{\phi}\right)^2\left[3m^2-3m-\frac{\omega}{2}\right]+\frac{3k}{\alpha^2}\phi^{2m-1}-\lambda_1\phi^{n_1-1}-\lambda_2\phi^{n_2-1}. \tag{5.1}$$

Using this expression in Eqs. (2.16) and (2.17), the set of field equations is reduced to the following two field equations:

$$\frac{\phi''}{\phi}(2m-1)+\left(\frac{\phi'}{\phi}\right)^2\left[-2m-\omega-\frac{1}{2}-\gamma\left(3m^2-3m-\frac{\omega}{2}\right)\right]+\frac{(2-3\gamma)k}{\alpha^2}\phi^{2m-1}+\gamma(\lambda_1\phi^{n_1-1}+\lambda_2\phi^{n_2-1})=0, \tag{5.2}$$

$$\frac{\phi''}{\phi}(3+2\omega)+\left(\frac{\phi'}{\phi}\right)^2\left[-12m^2+3m-6\omega m+3\omega+\frac{3}{2}+3\gamma\left(3m^2-3m-\frac{\omega}{2}\right)\right]+3(3\gamma-4)\frac{k}{\alpha^2}\phi^{2m-1}+(2-3\gamma)(\lambda_1\phi^{n_1-1}+\lambda_2\phi^{n_2-1})+2(\lambda_1 n_1\phi^{n_1-1}+\lambda_2\phi^{n_2-1})=0. \tag{5.3}$$

From Eq. (5.2) with $\gamma\neq0$ (the false vacuum case will be considered in Sec. VII), we have

$$\lambda_1\phi^{n_1-1}+\lambda_2\phi^{n_2-1}=\frac{1-2m}{\gamma}\frac{\phi''}{\phi}-\frac{1}{\gamma}\left[-2m-\omega-\frac{1}{2}-\gamma\left(3m^2-3m-\frac{\omega}{2}\right)\right]\frac{\phi'^2}{\phi^2}-\frac{(2-3\gamma)k}{\gamma\alpha^2}\phi^{2m-1}. \tag{5.4}$$

Using this expression and its derivative in Eq. (5.3) we get the following equation:

$$\frac{2}{\gamma}(1-2m)\frac{\phi'''}{\phi}\frac{\phi}{\phi'}+\frac{\phi''}{\phi}\left[\frac{4}{\gamma}(m+\omega+1)+6m(2m-1)\right]+\left(\frac{\phi'}{\phi}\right)^2[-12m^2-3m-6m\omega]+\frac{k}{\alpha^2}\phi^{2m-1}\times\left[12m-\frac{4}{\gamma}(2m+1)\right]=0. \tag{5.5}$$

In order to solve this differential equation, we shall consider the value $m=1/2$, so that Eq. (5.5) is reduced to the

$$\frac{\phi''}{\phi}-\frac{3\gamma}{4}\left(\frac{\phi'}{\phi}\right)^2+\frac{3\gamma-4}{3+2\omega}\frac{k}{\alpha^2}=0. \tag{5.6}$$

From this differential equation we have the two possible cases: $k=0$ and $k\neq0$. The case $\gamma=4/3$ will be considered in Sec. VI.

(1) For $k=0$ we get the following solution which we write in terms of time t as

$$\begin{aligned}\phi(t)&=\phi_1 t^{\sigma},\\ a(t)&=a_1 t^{-\sigma/2},\\ \lambda(\phi)&=\lambda_1\phi(t)^{-2/\sigma},\\ \rho&=\rho_1 a(t)^{-3\gamma},\end{aligned} \tag{5.7}$$

where

$$\phi_1=c_1\left[-\frac{3\gamma-2}{3\gamma-4}c_1^{1/2}\right]^{4/(2-3\gamma)},$$

$$\sigma=\frac{4}{2-3\gamma},$$

$$a_1=\frac{\alpha}{c_1^{1/2}}\left[-\frac{3\gamma-2}{3\gamma-4}c_1^{1/2}\right]^{2/(3\gamma-2)},$$

$$\rho_1=\frac{1}{\pi}\frac{1-2\omega}{\gamma(4-3\gamma)^2}c_1^{2-3\gamma/2},\quad \alpha^{3\gamma}$$

$$\lambda_1=\left[\omega\left(\frac{1}{\gamma}-\frac{1}{2}\right)-\frac{1}{2}\left(\frac{1}{\gamma}+\frac{3}{2}\right)\right]\left(-\frac{4}{4-3\gamma}\right)^2c_1^{2-3\gamma/2},$$

$$\lambda_2=0,$$

and c_1 is an integration constant. From this solution the expansion condition is $\sigma<0 \Rightarrow \gamma>2/3$. In this case, the Ricci scalar and deceleration and Hubble parameters are given by the following expressions:

$$\mathfrak{R}=36\frac{(2-\gamma)}{(3\gamma-2)^2}\frac{1}{t^2}, \tag{5.8}$$

$$q_0=\frac{3}{2}\gamma-2, \quad H_0=\frac{2}{3\gamma-2}\frac{1}{t_0}, \quad \Rightarrow t_0=\frac{2}{3\gamma-2}\frac{1}{H_0}. \tag{5.9}$$

Causality requires $0\leqslant\gamma\leqslant 2$, so that this is an accelerated model for $\gamma<4/3$, just at $\gamma=4/3$, $q_0=0$. On the other hand, $t_0\sim 1/H_0\sim 15.05$ Gyr for $\gamma\sim 4/3$. Then this is a kind of solution where the cosmic expansion is driven by the big-bang impulse.

The energy density parameter and the contribution of the scalar field $\phi(t)$ are given as follows:

$$\Omega_m=\frac{2}{3\gamma}-\frac{4\omega}{3\gamma},$$

$$\Omega_\phi=\frac{4\omega}{3\gamma}-\frac{2}{3\gamma}+1. \tag{5.10}$$

In order to have positive values of Ω_m, $\omega<1/2$ is required, including negative values of ω. On the other hand, $\Omega_m \sim 0.4\pm 0.1$, and $0\leqslant\gamma\leqslant 2$, from our equations for Ω_m and Ω_ϕ, we get a restriction for ω: $-0.1\leqslant\omega\leqslant 1/2$.

(2) Considering now the case $k\neq 0$ from Eq. (5.6), we get the following solution in terms of the parameter τ. For $k/(3+2\omega)>0$

$$\phi(\tau)=c_1[\cosh(\beta\tau)]^\sigma, \quad k\neq 0, \quad \gamma\neq 0,4/3, \tag{5.11}$$

where now

$$\sigma=\frac{4}{4-3\gamma}, \quad \beta=\frac{(3\gamma-4)}{2\alpha}\sqrt{\frac{k}{3+2\omega}},$$

and c_1 is an integration constant. According to Eq. (2.13) we get

$$a(\tau)=a_1[\cosh(\beta\tau)]^{-\sigma/2}, \quad k\neq 0, \quad \gamma\neq 0,4/3. \tag{5.12}$$

The Ricci scalar on this case is given by

$$\mathfrak{R}=r_1(r_2+r_3)\cosh^\sigma[\beta\tau]-r_1r_2\cosh^{\sigma-2}[\beta\tau], \tag{5.13}$$

where $a_1=\alpha c_1^{-1/2}$, $r_1=(3c_1/\alpha^2)[k/(3+2\omega)]$, $r_2=3(2+\gamma)$, and $r_3=3\gamma+2+4\omega$. In order to know the singularities of this solution, we calculate the nonzero curvature invariant [60], which for this case, are given by

$$R_1=\frac{3}{4}\left[\frac{\ddot{a}}{a}-\left(\frac{\dot{a}}{a}\right)^2-\frac{k}{a^2}\right]^2,$$

$$R_2=-\frac{1}{\sqrt{3}}R_1^{3/2},$$

$$R_3=\frac{7}{12}R_1^2, \tag{5.14}$$

then, it is enough to calculate R_1:

$$R_1=s_1(s_2-s_3)^2\cosh^{2\sigma-4}[\beta\tau]+s_1(s_3+s_4)^2\cosh^{2\sigma}[\beta\tau] +2s_1(s_2-s_3)(s_3+s_4)\cosh^{2\sigma-2}[\beta\tau], \tag{5.15}$$

where

$$s_1=\frac{3}{4}\frac{c_1^2}{\alpha^4(3+2\omega)^2},$$

$$s_2=k\left(2-\frac{3}{2}\gamma\right),$$

$$s_3=k,$$

$$s_4=k(3+2\omega).$$

From Eq. (5.13), $\mathfrak{R}\to\infty$ provided that $\tau\to\pm\infty$, and at least one exponent is positive, i.e., $\sigma>0$ or $\sigma>2$. On the other hand, from the curvature invariant, Eq. (5.15), $R_1\to\infty$ requires that $\tau\to\pm\infty$, and $\sigma>2$ $(\gamma>2/3)$, $\sigma>0$ $(\gamma<4/3)$ or $\sigma>1$ $(\gamma>0)$. So that the solutions of this case, are singular for $0<\gamma<4/3$, and $\tau\to\pm\infty$.

For $\dfrac{k}{3+2\omega}<0$

$$\phi(\tau)=c_1[\cos(\beta\tau)]^\sigma, \quad k\neq 0, \quad \gamma\neq 0,4/3, \tag{5.16}$$

$$a(\tau)=a_1[\cos(\beta\tau)]^{-\frac{\sigma}{2}}, \quad k\neq 0, \quad \gamma\neq 0,4/3, \tag{5.17}$$

where σ and a_1 are defined as in the paragraph under Eqs. (5.11) and (5.13). In this case,

$$\beta=\frac{3\gamma-4}{2\alpha}\sqrt{\left\|\frac{k}{3+2\omega}\right\|},$$

$$\mathfrak{R}=r_1(r_3-r_2)\cos^\sigma[\beta\tau]+r_1r_2\cos^{\sigma-2}[\beta\tau], \tag{5.18}$$

where now

$$r_1=\frac{3c_1}{\alpha^2}\left\|\frac{k}{3+2\omega}\right\|,$$

$$r_2=3(2-\gamma),$$

$$r_3=(4-3\gamma)+2\|3+2\omega\|.$$

According to Eq. (5.14), as in the previous case, we need to calculate R_1 only, which for this case is given by

$$R_1=s_1(s_2+s_3)^2\cos^{2\sigma-4}[\beta\tau]+s_1(s_4-s_3)^2\cos^{2\sigma}[\beta\tau] +2s_1(s_2+s_3)(s_4-s_3)\cos^{2\sigma-2}[\beta\tau], \tag{5.19}$$

where

$$s_1=\frac{3}{4}\,\frac{c_1^2}{\alpha^4|3+2\omega|^2},$$

$$s_2=\|k\|\left(2-\frac{3}{2}\gamma\right),$$

$$s_3=\|k\|,$$

$$s_4=k\|3+2\omega\|.$$

From Eqs. (5.18) and (5.19), $\Re\to\infty$ as well as $R_1\to\infty$, if $[\beta\tau]\to\pm(2n+1)\pi/2$ and at least one exponent on the respective expressions of $\Re$ and R_1 is negative: $\sigma<2$ $(\gamma<2/3)$, $\sigma<1$ $(\gamma<0)$, or $\sigma<0$ $(\gamma>4/3)$. Then we have the two ranges of γ for which the solutions of the present case, are singular: $0<\gamma<2/3$ and $4/3<\gamma\leqslant 2$, since furthermore causality requires γ to be in the interval $0\leqslant\gamma\leqslant 2$.

On the other hand, from equations (5.1) and (5.4) we get respectively, for both possibilities $k/(3+2\omega)<0$ or $k/(3+2\omega)>0$

$$\rho=\rho_1 a(\tau)^{-3\gamma}, \tag{5.20}$$

$$\lambda(\phi)=\lambda_1\phi(\tau)^{\frac{3}{2}\gamma-1}+\lambda_2\phi(\tau), \tag{5.21}$$

where

$$\rho_1=\frac{kc_1^{2-3\gamma/2}}{4\pi\alpha^2\gamma}\alpha^{3\gamma},\quad \lambda_1=\frac{kc_1^{2-3\gamma/2}}{\alpha^2}\left(1-\frac{2}{\gamma}\right),\quad \lambda_2=\frac{2k}{\alpha^2}.$$

As we can see, $\lambda(\phi)$ remains as a binomial function of ϕ if $\gamma\neq 4/3$. In order to analyze the behavior of the obtained solution in terms of the cosmological time t, we shall give some examples.

A. Dust fluid

One interesting application of a baryotropic equation of state corresponds to a dust fluid $(\gamma=1)$, on that case the solution reads as follows.

(1) $k=0$:

$$\phi(t)=\phi_1 t^{-4},$$

$$a(t)=a_1 t^2,$$

$$\lambda(\phi)=\lambda_1\phi(t)^{1/2},$$

$$\rho=\rho_1 a(t)^{-3}, \tag{5.22}$$

where $\phi_1=c_1^{-1}$, $a_1=\alpha c_1^{1/2}$, $\lambda_1=4(2\omega-5)c_1^{1/2}$, $\lambda_2=0$, $\rho_1=[(1-2\omega)/\pi]c_1^{1/2}\alpha^3$, and c_1 is an integration constant. This is an extended inflationary solution, with a time decaying cosmological constant and initial singularity, as it is shown by the corresponding Ricci scalar

$$\Re=36t^{-2}. \tag{5.23}$$

The expansion takes place with a constant acceleration

$$q_0=-\frac{1}{2},\quad H_0=\frac{2}{t_0},\quad\Rightarrow\quad t_0\sim\frac{2}{H_0}. \tag{5.24}$$

With $H_0\sim 65\pm 5$ km s^{-1}Mpc^{-1}, we obtain $t_0\sim 30.1$ Gy, which is too big value compared with the globular cluster age.

The density parameters for a dust fluid are given by

$$\Omega_m=\frac{2}{3}-\frac{4\omega}{3},$$

$$\Omega_\phi=\frac{4\omega}{3}+\frac{1}{3}. \tag{5.25}$$

In this case, as in the general case, in order to have a positive values of Ω_m, it is required that $\omega<1/2$, including negative values. On the other hand, according to observational results, $\Omega_m\sim 0.4\pm 0.1$ and $\Omega_\Lambda\sim 0.6$, then ω is restricted to be $\omega\sim 1/5$.

(2) $k\neq 0$, for $k/(3+2\omega)>0$. The solutions in terms of the time t is given by

$$\phi(t)=c_1[1-\phi_1 t^2]^{-2},$$

$$a(t)=a_1[1-\phi_1 t^2],$$

$$\lambda(\phi)=\lambda_1\phi(t)+\lambda_2\phi(t)^{1/2},$$

$$\rho=\rho_1 a(t)^{-3}, \tag{5.26}$$

where

$$a_1=\alpha/c_1^{1/2},\quad \phi_1=\frac{c_1}{4\alpha^2}\,\frac{k}{3+2\omega},\quad \lambda_1=\frac{2k}{\alpha^2},$$

$$\lambda_2=-\frac{kc_1^{1/2}}{\alpha^2},\quad \rho_1=\frac{kc_1^{1/2}}{4\pi\alpha^2}\alpha^3,$$

and c_1 is an integration constant. Clearly $a(t)$ and $\phi(t)$ must be positive, in order to be physically significant; this requirement restricts the range of values which t can take: $t<\phi_1^{-1/2}$, and $c_1>0$ as we can see from the definition of ϕ_1. In this case the cosmological term, in spite of being a binomial function on ϕ, decays with the time.

The corresponding Ricci scalar and curvature invariant show that this solution is singular:

$$\Re=\frac{6}{[1-\phi_1 t^2]^2}\left[6\phi_1^2 t^2-2\phi_1+\frac{k}{a_1^2}\right], \tag{5.27}$$

$$R_1=\frac{3}{4}\,\frac{1}{[1-\phi_1 t^2]^4}\left[-2\phi_1^2 t^2-2\phi_1-\frac{k}{a_1^2}\right]^2. \tag{5.28}$$

According to our analysis of the general solution, for $\sigma=4/(4-3\gamma)$, with $\gamma=1$ we get $\sigma=4$. Taking into account the general equations of the Ricci scalar and curvature invariant, Eqs. (5.13) and (5.15), $\Re\to\infty$ and $R_1\to\infty$ for $\sigma=4$ and $\tau\to\pm\infty$, which corresponds to a finite value of t, according to the time dependent solution (5.26). Furthermore $\gamma=1<4/3$, is consistent with our singularity requirement in our discussion for the general case with $k/(3+2\omega)>0$.

The today values of the deceleration and Hubble parameters are given by the following expressions:

$$q_0=\frac{1}{2}\left(\frac{1}{\phi_1,t_0^2}-1\right),\quad H_0=\frac{2\phi_1 t_0}{\phi_1 t_0^2-1},$$

$$t_0=\frac{1}{H_0}\pm\sqrt{\frac{1}{H_0^2}+\frac{1}{\phi_1}}. \tag{5.29}$$

Because $\phi_1 t_0^2<1$, then $q_0>0$. This model expands from $t=-\phi_1^{-1/2}$ until $t=0$, then it contracts until $t=\phi_1^{-1/2}$, in both cases with positive deceleration parameter. The numerical value for t_0 depends on the values of the free constants.

For $k/(3+2\omega)<0$. The corresponding solution in terms of the time t is given as

$$\phi(t)=c_1[1+\phi_1 t^2]^{-2},$$

$$a(t)=a_1[1+\phi_1 t^2],$$

$$\lambda(\phi)=\lambda_1\phi(t)+\lambda_2\phi(t)^{1/2},$$

$$\rho=\rho_1 a(t)^{-3}, \tag{5.30}$$

where

$$\phi_1=\frac{c_1}{4\alpha^2}\left\|\frac{k}{3+2\omega}\right\|,\quad a_1=\alpha/c_1^{1/2},\quad \lambda_1=\frac{2k}{\alpha^2},$$

$$\lambda_2=-\frac{kc_1^{1/2}}{\alpha^2},\quad \rho_1=\frac{kc_1^{1/2}}{4\pi\alpha^2}\alpha^3,$$

and c_1 is an integration constant. In this case we have not restrictions on the values which t can take. If $-1/\phi_1<(t_0+c)^2$, the expansion takes place with nonconstant acceleration and without singularity, as it is shown from the corresponding Ricci scalar and curvature invariant

$$\Re=\frac{6}{[1+\phi_1 t^2]^2}\left[6\phi_1^2 t^2+2\phi_1+\frac{k}{a_1^2}\right], \tag{5.31}$$

$$R_1=\frac{3}{4}\frac{1}{[1+\phi_1 t^2]^4}\left[-2\phi_1^2 t^2+2\phi_1-\frac{k}{a_1^2}\right]^2. \tag{5.32}$$

According to our general analysis of this case, from Eqs. (5.18) and (5.19), $\Re$ and R_1 do not diverge for $\sigma=4$ and $[\beta\tau]\to\pm(2n+1)\pi/2$, in agreement with the time dependent solution (5.30) which has not singularities. $\gamma=1<4/3$ in this model, which is consistent with our condition $\gamma>4/3$ as requirement for existence of singularities.

For this solution, the corresponding present values of the density and Hubble parameters are given as

$$q_0=-\frac{1}{2}-\frac{1}{2\phi_1}t_0^{-2},\quad H_0=\frac{2\phi_1 t_0}{\phi_1 t_0^2+1},$$

$$t_0=\frac{1}{H_0}\pm\sqrt{\frac{1}{H_0^2}-\frac{1}{\phi_1}}. \tag{5.33}$$

As in the previous case, the values of t_0 depends on the free constants of our model, but in this case, for $H_0^2\sim\phi_1\Rightarrow t_0\sim 1/H_0$.

B. Stiff matter fluid

Another interesting application of a baryotropic equation of state is a stiff matter fluid for which $\gamma=2$. In such a case, the solutions (5.7), (5.11), (5.12), (5.16), and (5.17) are the following ones.

(1) $k=0$. The solution in terms of the physical time t is given by

$$\phi(t)=\phi_1(c_-t)^{-1},$$

$$a(t)=a_1(c-t)^{1/2},$$

$$\lambda(\phi)=\lambda_1\phi(t)^2,$$

$$\rho=\rho_1 a(t)^{-6}, \tag{5.34}$$

where now

$$\phi_1=\frac{c_1^{1/2}}{2},\quad a_1=\frac{\alpha\sqrt{2}}{c_1^{1/4}},\quad \lambda_1=-\frac{4}{c_1},$$

$$\lambda_2=0,\quad \rho_1=\frac{1}{8\pi}\frac{1-2\omega}{c_1}\alpha^6.$$

This solution has a physical meaning for $t<c$, where c is an integration constant and the scale factor increases very slowly with the time, and with constant deceleration. The Ricci scalar and Hubble parameter, are given by

$$\Re=0, \tag{5.35}$$

$$q_0=1,\quad H_0=\frac{1}{2}\frac{1}{t_0-c}, \tag{5.36}$$

Naturally this model is not valid today because it shrinks for the allowed range of t. The corresponding density parameters are

$$\Omega_m=\frac{1}{3}-\frac{2\omega}{3},$$

$$\Omega_\phi=\frac{2}{3}+\frac{2\omega}{3}. \tag{5.37}$$

In order to have a positive values of Ω_m, then $\omega<1/2$, as we have claimed in the discussion of the general solution. The observational results $\Omega_m=0.4$ and $\Omega_\Lambda=0.6$, determine $\omega=-0.1$.

(2) $k\neq 0$. for $k/(3+2\omega)>0$. The solution of this case, in terms of the time t is given as

$$\phi(t)=c_1[1+\phi_1 t^2]^{-1},$$
$$a(t)=a_1[1+\phi_1 t^2]^{1/2},$$
$$\lambda(\phi)=\lambda_1\phi(t),$$
$$\rho=\rho_1 a(t)^{-6}, \tag{5.38}$$

where $\phi_1=(c_1/\alpha^2)[k/(3+2\omega)]$, $(\phi_1>0$ for $c_1>0)$, $a_1=\alpha c_1^{-1/2}$, $\lambda_1=2k/\alpha^2$, $\lambda_2=0$, $\rho_1=(k/8\pi c_1)\alpha^4$, and c_1 is an integration constant. Here $a(t)$ grows with the time, from a minimum radius a_1 there is a nonsingular state. The expansion takes place with nonconstant acceleration, as we can see from the corresponding Ricci scalar and curvature invariant, which are given by

$$\Re=\frac{6}{[1+\phi_1 t^2]}\left(\phi_1+\frac{kc_1}{\alpha^2}\right), \tag{5.39}$$

$$R_1=\frac{3}{4}\frac{1}{[1+\phi_1 t^2]^4}\left[-\phi_1 t^2\left(\phi_1+\frac{c_1 k}{\alpha^2}\right)+\phi_1-\frac{kc_1}{\alpha^2}\right]^2. \tag{5.40}$$

In this case $\gamma=2$ implies $\sigma=-2$. From Eqs. (5.13) and (5.15) with this value of σ, neither $\Re$ nor R_1 diverges for $\tau\to\pm\infty$ which is consistent with our requirement $\gamma>4/3$ for the avoidance of singularities. This analysis is in agreement with the inspection of Eqs. (5.39) and (5.40).

The deceleration and Hubble parameters are

$$q_0=-\frac{1}{\phi_1}t_0^{-2},$$

$$H_0=\frac{c_1}{\alpha^2}\frac{k}{3+2\omega}\frac{t_0}{1+\dfrac{c_1}{\alpha^2}\dfrac{k}{3+2\omega}t_0^2},$$

$$t_0=\frac{1}{2H_0}\pm\sqrt{\frac{1}{4H_0^2}-\frac{\alpha^2}{c_1}\frac{3+2\omega}{k}}. \tag{5.41}$$

For $k/(3+2\omega)<0$:

$$\phi(t)=c_1[1-\phi_1 t^2]^{-1},$$
$$a(t)=a_1[1-\phi_1 t^2]^{1/2},$$
$$\lambda(\phi)=\lambda_1\phi(t),$$
$$\rho=\rho_1 a(t)^{-6}, \tag{5.42}$$

where $\phi_1=c/\alpha^2\|k/(3+2\omega)\|$. In order to have a physically solution, it is required that $\phi_1<0\Rightarrow c<0$. The corresponding Ricci scalar and curvature invariant in the present case, are given by

$$\Re=\frac{6}{[1-\phi_1 t^2]}\left(\frac{kc_1}{\alpha^2}-\phi_1\right), \tag{5.43}$$

$$R_1=\frac{3}{4}\frac{1}{[1-\phi_1 t^2]^4}\left[-\phi_1 t^2\left(\phi_1-\frac{c_1 k}{\alpha^2}\right)-\phi_1-\frac{kc_1}{\alpha^2}\right]^2. \tag{5.44}$$

In this case $\gamma=2$, then $\sigma=-2$; such that in the general solution [Eqs. (5.16)–(5.19)], $\Re\to\infty$ and $R_1\to\infty$, i.e., the solution for this case is singular provided that $[\beta\tau]\Rightarrow\pm(2n+1)\pi/2$, which corresponds to $t=\phi_1^{-1/2}$. According to our singularity discussion under Eq. (5.19), for $\gamma=2$ the corresponding solution should being singular, as we can verify from inspection of Eqs. (5.43) and (5.44).

The present deceleration and Hubble parameters are

$$q_0=\frac{1}{\phi_1}t_0^{-2},$$

$$H_0=-\frac{c_1}{\alpha^2}\left\|\frac{k}{3+2\omega}\right\|\frac{t_0}{1-c_1/\alpha^2\|k/(3+2\omega)\|t_0^2},$$

$$t_0=\frac{1}{2H_0}\pm\sqrt{\frac{1}{4H_0^2}+\frac{\alpha^2}{c_1}\frac{3+2\omega}{k}}. \tag{5.45}$$

As we can see, $\phi_1<0\Rightarrow q_0<0$, then this model is accelerated. The numerical values of H_0 and t_0 depend on the values of the free constants.

VI. THE RADIATION CASE

We shall consider the radiation case for which $\gamma=4/3$, so that returning to Eqs. (2.15)–(2.17) and following a similar procedure as in Sec. V, we get Eq. (5.5) with $\gamma=4/3$. In order to solve this differential equation, we shall assume $m=1/2$, then we have for this case

$$\frac{\phi''}{\phi}-\left(\frac{\phi'}{\phi}\right)^2=0, \tag{6.1}$$

for which $\omega\neq-3/2$. This differential equation has the following solution:

$$\phi(\tau)=c_1 e^{c\tau}, \tag{6.2}$$

where c and c_1 are integration constants. According to Equations (2.13), (5.1), and (5.4) with $\gamma=4/3$ and $m=1/2$ we get, respectively,

$$a(\tau)=a_1 e^{-c\tau/2},$$
$$\lambda(\phi)=\lambda_1\phi(\tau),\quad \lambda_2=0,$$
$$\rho=\rho_1 a(\tau)^{-4}, \tag{6.3}$$

where $a_1=\alpha c_1^{-1/2}$, $\lambda_1=[(c^2/8)(3+2\omega)+3/2(k/\alpha^2)]$ and $\rho_1=3/16\pi[k/\alpha^2-\omega c^2/2-3c^2/4]\alpha^4$. In terms of the time t, this solution is given as follows:

$$\phi(t)=\phi_1 t^{-2},$$

$$a(t)=a_1 t,$$

$$\lambda(\phi)=\lambda_1\phi(t),$$

$$\rho=\rho_1 a(t)^{-4}, \tag{6.4}$$

where $\phi_1=4/c^2$, $a_2=\alpha c/2$, and $\lambda_2=0$. This is a singular solution according to the Ricci scalar which for this case is given by

$$\mathfrak{R}=6\left(1+\frac{k}{a_1^2}\right)t^{-2}. \tag{6.5}$$

The deceleration parameter becomes null, instead the today Hubble parameter is given as

$$q_0=0, \quad H_0=\frac{1}{t_0}, \tag{6.6}$$

and therefore $t_0=1/H_0\sim 15.05$ Gyr.

The case $k=0$ is not excluded from the solution (6.4). For $k=0$ we get the density parameter from the matter and the scalar field as

$$\Omega_m=-\omega-\frac{3}{2},$$

$$\Omega_\phi=\omega+\frac{5}{2}. \tag{6.7}$$

It is required that $\omega<-3/2$, in order to have $\Omega_m>0$, and the observational accepted values of Ω_m and Ω_ϕ, determine $\omega=-1.9$.

VII. THE FALSE VACUUM CASE

We analyze now the case $\gamma=0$. If we follow a similar procedure as in Sec. V, then we get from Eqs. (5.2) and (5.3) with $\gamma=0$, the following set of equations:

$$\frac{\phi''}{\phi}(2m-1)+\left(\frac{\phi'}{\phi}\right)^2\left[-2m-\omega-\frac{1}{2}\right]+\frac{2k}{\alpha^2}\phi^{2m-1}=0, \tag{7.1}$$

$$\frac{\phi''}{\phi}(3+2\omega)+\left(\frac{\phi'}{\phi}\right)^2\left[-12m^2+3m-6\omega m+3\omega+\frac{3}{2}\right]+\frac{12k}{\alpha^2}\phi^{2m-1}+2\frac{\lambda(\phi)}{\phi}+2\frac{d\lambda(\phi)}{d\phi}=0. \tag{7.2}$$

We consider first the case where $k=0$. The set of equations (7.1), (7.2) with $k=0$ has the two possible set of solution depending on the relation between m and ω.

(1) $\omega=-1-m$:

$$\phi(t)=c_1\exp[\phi_1 t],$$

$$a(t)=a_1\exp[-m\phi_1 t],$$

$$\lambda(\phi)=\frac{\lambda_1}{\phi(t)}+\lambda_2,$$

$$\rho=-\frac{c_1}{8\pi}, \tag{7.3}$$

where

$$\phi_1=c_3^{1/2}(2m-1), \quad a_1=\alpha c_1^{-m},$$

$$\lambda_1=c_1, \quad \lambda_2=\frac{c_1}{2}(2m-1)^3(3m-1).$$

This is an inflationary solution if $m\phi_1<0$. This condition means $0<m<1/2$, which implies a condition on the range of the values of ω: $1/2<\omega<1$. In order to have physical solutions, $c_1>0$, which means $\rho<0$. Of course, these solutions have not singularities, as we can see from the Ricci scalar

$$\mathfrak{R}=12c_1m^2(2m-1)^2. \tag{7.4}$$

The deceleration and Hubble parameters are given by

$$q_0=-1, \quad H_0=m(1-2m)c_1^{1/2}, \tag{7.5}$$

thus the model is accelerated. On the other hand, the density parameter due to the matter and scalar field, are

$$\Omega_m=-\frac{1}{3m^2}c_2\phi_1^{-2}\exp[-\phi_1 t],$$

$$\Omega_\phi=\frac{1}{3m^2}c_2\phi_1^{-2}\exp[-\phi_1 t]+1. \tag{7.6}$$

Here, it is required $c_2/c_3<0$, in order to have a positive values of Ω_m. As we have seen, $0<m<1/2\Rightarrow m>0$ and $\phi_1<0$, which means that Ω_m and Ω_ϕ increase exponentially with the time, keeping $\Omega_m+\Omega_\phi=1$.

(2) $\omega\neq-1-m$. With this condition and $k=0$, the corresponding exact solutions to the set of Eqs. (7.1)–(7.2) are

$$\phi(t)=\phi_1(c-t)^\sigma,$$

$$a(t)=a_1(c-t)^{-m\sigma},$$

$$\lambda(\phi)=\lambda_1\phi(t)^{n_1}+\frac{\lambda_2}{\phi(t)},$$

$$\rho=-\frac{c_1}{8\pi}. \tag{7.7}$$

where

$$\sigma=\frac{1-2m}{m+\omega+1},$$

$$\phi_1=c_1[c_1^{1/2}(m+\omega+1)]^{\sigma},$$

$$a_1=\alpha\phi_1^{-m},$$

$$n_1=\frac{2\omega+3}{2m-1}+1=-\frac{2}{\sigma},$$

$$\lambda_1=-\frac{c_1^{1-n_1}}{4}\frac{(2m-1)^2}{2m+\omega+1/2}[(3+2\omega)(2m+\omega+1/2)+(2m-1)(-12m^2+3m-6m\omega+3\omega+3/2)],$$

$$\lambda_2=c_2.$$

This solution corresponds to an extended inflationary model if $m\sigma<0$. In order to have a physical solution, it is required $t<c$. The cosmological constant is a decaying function of time: $\lambda\sim t^{-2}+t^{-\sigma}$, with $\sigma>0$. But according to the above mentioned condition $m\sigma<0$, it is clear that m should be negative.

This set of solutions is singular just at $t=c$, as we can see from the Ricci scalar

$$\mathfrak{R}=6m(2m-1)\frac{4m^2-3m-\omega-1}{(m+\omega+1)^2}\frac{1}{(c-t)^2}. \tag{7.8}$$

The present deceleration and Hubble parameters for this model, are given as

$$q_0=1-\frac{m+\omega+1}{m(2m-1)},\quad H_0=-\frac{m(1-2m)}{m+\omega+1}(c-t_0)^{-1},$$

$$t_0=c-\frac{m(1-2m)}{m+\omega+1}\frac{1}{H_0}. \tag{7.9}$$

According to these results, the expansion of this model takes place in accelerated way if $q_0=1+1/m\sigma<0$, but we have seen that $m\sigma<0$, then $\|m\sigma\|<1$ is the required condition for accelerated expansion. The value of t_0 clearly depend on the value of m and ω, but we know that, because the restriction on t, this model is not relevant at present times.

The corresponding density parameters due to the matter and scalar field for the present case, are given by

$$\Omega_m=\frac{1}{3m^2}\frac{1}{\sigma^2}\phi_1^{-2/\sigma}[8\pi\rho_1-c_2\phi(t)^{(4m+2\omega+1)/(1-2m)}],$$

$$\Omega_\phi=\frac{1}{3m^2}\frac{1}{\sigma^2}\phi_1^{-2/\sigma}[\lambda_1+\lambda_2\phi(t)^{(4m+2\omega+1)/(1-2m)}]+\frac{\omega}{6m^2}+\frac{1}{m}. \tag{7.10}$$

Both values depend on the free constants, but for all time, it is satisfied that $\Omega_m+\Omega_\phi=1$.

Now we consider the case where $k\neq 0$. In order to solve the set of equations (7.1), (7.2) with $k\neq 0$, we shall assume $m=1/2$, then we have, respectively,

$$\left(\frac{\phi'}{\phi}\right)^2(3+2\omega)-\frac{4k}{\alpha^2}=0, \tag{7.11}$$

$$\frac{\phi''}{\phi}(3+2\omega)-\frac{12k}{\alpha^2}+2\frac{\lambda(\phi)}{\phi}+2\frac{d\lambda(\phi)}{d\phi}=0. \tag{7.12}$$

From Eq. (7.11) we get

$$\frac{\phi'}{\phi}=\left[\frac{4k}{\alpha^2(3+2\omega)}\right]^{1/2}. \tag{7.13}$$

The solution to this equation is given by

$$\phi(\tau)=c_1\exp^{(2/\alpha)\sqrt{\frac{k}{3+2\omega}}\tau}. \tag{7.14}$$

From Eq. (7.13) we get

$$\frac{\phi''}{\phi}=\frac{4k}{\alpha^2(3+2\omega)}. \tag{7.15}$$

We use this last equation in Eq. (7.12) from which we get a differential equation for $\lambda(\phi)$:

$$\frac{d\lambda(\phi)}{d\phi}+\frac{\lambda(\phi)}{\phi}-\frac{4k}{\alpha^2}=0, \tag{7.16}$$

the solution of this equation reads as follows:

$$\lambda(\phi)=\frac{2k}{\alpha^2}\phi(\tau)+\frac{c1}{\phi(\tau)}. \tag{7.17}$$

From Eqs. (2.13) and (5.1) with $m=1/2$, we get, respectively,

$$a(\tau)=a_1\exp^{-\frac{1}{\alpha}\sqrt{\frac{k}{3+2\omega}}\tau}, \tag{7.18}$$

$$\rho=-\frac{c_1}{8\pi}, \tag{7.19}$$

where $a_1\equiv\alpha c_1^{-1/2}$. In terms of t, this solution becomes

$$\phi(t)=\phi_1 t^{-2},$$

$$a(t)=a_1 t,$$

$$\lambda(\phi)=\lambda_1\phi(t)+\lambda_2\phi(t)^{-1},$$

$$\rho=-\frac{c_1}{8\pi}, \tag{7.20}$$

where $\phi_1=(\alpha^2/k)(3+2\omega)$, $a_1=\sqrt{k/(3+2\omega)}$, $\lambda_1=2k/\alpha^2$ and $\lambda_2=c_1$. This solution has an initial singularity, as we can see from the Ricci scalar

$$\mathfrak{R}=12(2+\omega)t^{-2}. \tag{7.21}$$

$\lambda(\phi)$ increases with the time, in contradiction with the actual observations. However, for a particular combination of the today values of the free constants in this model we can get a small value of λ_0

$$\left[\frac{2}{\alpha^2}\left\|\frac{k}{c}\right\|\phi_1^2\right]_0^{1/4}\sim t_0. \tag{7.22}$$

The today values of the corresponding deceleration and Hubble parameters are given by

$$q_0=0, \quad H_0=\frac{1}{t_0}, \tag{7.23}$$

such that $t_0=1/H_0\sim 15.05$ Gyr.

VIII. FINAL REMARKS

We have considered a Brans-Dicke scalar-tensor theory, obtaining Friedmann-Robertson-Walker cosmological models with time dependent cosmological constant. The time dependence occurs in a natural way. Two ansatz were proposed: $a\phi^m=\alpha$, with α constant, and $\lambda(t)=\lambda_1\phi(t)^{n_1}+\lambda_2\phi(t)^{n_2}$, in order to get exact inflationary solutions of the field equations, with a general state equation $p=(\gamma-1)\rho$. Our set of exact solutions depend on the values of γ, k, m and ω.

We classify the exact solutions of each case which we deal, according to the values of the free constants of our model. For vacuum with $k=0$ and $m=1/2$, we get a nonrelevant solution (according to our goal), with $\lambda=0$. For $k=0$, $m=2/3$, we get a singular solution for which the scale factor decreases with the time as $a(t)\sim t^{-1/3}$, in accelerated way, but its predicted age is a negative value, then we conclude that this model has not physical meaning today.

Furthermore for vacuum, we get for a flat case, an extended inflationary solution with initial singularity, and an exponential inflationary solution without singularity. For a not flat case we get a coasting singular solution. In all this models, the values of t_0 and Ω_ϕ (usually called Ω_Λ) are similar to the actual accepted values.

We obtain exact solutions for a general equation of state $p=(\gamma-1)\rho$. In the flat case ($k=0$) we get an extended inflationary solution with initial singularity. The expansion of this model occurs in an accelerated way, independently of the equation of state. The values of t_0, Ω_m, and Ω_ϕ depend on the value of the undetermined constants γ and ω.

The solution of the nonflat case cannot be expressed in terms of the cosmological time, but in terms of the parameter τ. In such models the initial singularities can be avoided for some values of γ.

As examples of our general solutions, we calculate the models for a dust and stiff matter fluid. For a flat dust model we get a slow extended inflationary solution with acceleration and time decaying cosmological constant. This solution has an initial singularity and its estimated age is $t_0\sim 2/H_0$, which is too big according to actual known values. A bigger growth rate is required in order to have smaller values of t_0 and to be consistent with actual observations.

On the other hand, for a non-flat dust model we get a slowly expanding solution $a(t)\sim t^2$, with nonconstant acceleration or deceleration depending on the relation between our free constants $k/(3+2\omega)>0$ or $k/(3+2\omega)<0$, respectively. The models are singular or nonsingular depending on the free constants α, k, and ω.

As another application of our solutions with a general state equation, we consider a stiff matter fluid for which $\gamma=2$. The solution of the corresponding flat case is a slowly expanding model $a(t)\sim t^{1/2}$, with acceleration, without singularities and with a time decaying cosmological ''constant'': $\lambda\sim t^{1/2}$. The validity of this model is restricted to some values of t. The actual age of the Universe predicted by this model, $t_0\sim 1/2H_0\sim 7.5$ Gyr, with $H_0\sim 65\pm 5$ km s^{-1} Mpc^{-1}; clearly this model is not applicable today.

In the nonflat case for a stiff matter fluid, we get cosmological models for which the scale factor grows with the time from a minimum radius, and the cosmological ''constant'' decreases with the time, but as in the dust model, we have a set of free constants, whose values should determine the characteristics of this model. Thus, for $k/(3+2\omega)>0$ we have a nonconstant accelerated model without singularities, while for $k/(3+2\omega)<0$, our solution is a nonconstant accelerated model with initial singularities whose validity is restricted to some values of t. In both cases the actual predicted age of the Universe depends on the numerical value of the free constants.

We solve separately the case of a radiation fluid ($\gamma=4/3$) and a false vacuum fluid ($\gamma=0$). For the radiation fluid we get a coasting model $a(t)\sim t$, with a decaying cosmological ''constant'' and with initial singularity, independently of the curvature. The value of $t_0\sim 15.05$ Gyr, obtained from this model is in fair agreement with actual observations.

For a false vacuum fluid, we obtain a set of solutions which we classify depending on the range of values which our free constants may take. For the flat case, we get two family of solutions, one of them, corresponds to an exponential inflationary model with acceleration and without singularities. Another set of solutions is a kind of power law inflationary model $a(t)\sim(c-t)^\epsilon$, where ϵ depend on the ''free'' constants m and ω, restricted by physical requirements. For this last solution the cosmological constant is a binomial function of t, which decreases under specific conditions on the free constants.

Additionally, we get a solution for the not flat case of a false vacuum fluid. In such a case we get a coasting model $a(t)\sim t$, which has initial singularity and with cosmological ''constant'' which is a binomial function of t: $\lambda\sim\lambda_1 t^{-2}+\lambda_2 t^2$. Such a cosmological ''constant'' would increases with the time in contradiction with the actual accepted value of λ. For a particular combination of the today values of the free constants in this model, it is possible to obtain small values of λ_0. The actual age predicted by this model is $t_0\sim 15.05$ Gyr, which is again in fair agreement with the actual accepted value.

Then, as we can see from the description of our exact solutions, some of them have not physical meaning today, some others are restricted to be valid during a specific period. Most of them are valid from an initial singularity until today, predicting an inflationary epoch, a cosmological ''constant'' which decreases with the time and the today observed acceleration, as well as an actual age of the universe which is in reasonable agreement with the actual observations.

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