Chiral baryon in the coherent pair approximation

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We reexamine the work of Goeke *et al.*, who considered a chiral model for the nucleon based on the linear sigma model with scalar-isoscalar scalar-isovector mesons coupled to quarks, and solved it using the coherent-pair approximation. In this way the quantum pion field can be treated in a nonperturbative fashion. In this work we review this model and the coherent pair approximation, correcting several errors in the earlier work. We minimize the expectation value of the chiral Hamiltonian in the ansatz coherent-pair ground-state configuration, and solve the resulting equations for nucleon quantum numbers. We calculate the canonical set of nucleon observables and compare it with the Hedgehog model and experiment. Using the corrected equations yields slightly different values for nucleon observables, but does not correct the large virial deviation in the π-nucleon coupling. Our results therefore do not significantly alter the conclusions of Goeke *et al.*
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I. INTRODUCTION

It is widely believed that quantum chromodynamics (QCD) is the fundamental theory underlying strong interaction. Regrettably, reliable first-principles calculations of hadronic structure and reactions based on QCD are still some time off. Nevertheless, simpler QCD-motivated phenomenological models have been proposed which preserve the important property of chiral symmetry but sacrifice confinement. Familiar examples include Skyrme-Witten models [1] and hybrid chiral-soliton models such as those of Goeke, Harvey, Grümmer, and Urbano [2] (GHGU), Kahana and Ripka [3], Birse and Banerjee [4], Birse [5], Fiolhais, Goeke, Grümmer, and Urbano [6], and others (see Ref. [7]).

Such approaches, in particular Ref. [4], argue that spontaneous symmetry breaking of the QCD Lagrangian gives rise to an effective chiral Lagrangian of the Gell-Mann–Lévy sigma-model form involving explicit quark, scalar-isoscalar meson (sigma, σ), and pseudoscalar-isovector meson (pion, $\vec{\pi}$) degrees of freedom. There is no longer confinement in the model, and nucleons appear as bound states of a three-quark system. The bound states of the model have been solved in mean field using the ''hedgehog'' ansatz, which assumes a configuration-space-isospin correlation for the pion field, $\vec{\pi}=\hat{r}\pi$, and for the quarks. One drawback to this ansatz is that it breaks both rotational and isospin invariance (although the ''grand spin'' $\vec{K}=\vec{I}+\vec{J}$ remains conserved) requiring some projection onto physical states at the end. Considerable attention has been given to the problem of projection in the calculation of observables [8].

In spite of this drawback, the model is quite successful at predicting baryon properties. Constraining the pion mass and decay constant with experimental values, the model contains but two additional free parameters (which can be written in terms of the effective quark and sigma masses), yet the model makes quite respectable predictions for a host of hadronic properties [mass, magnetic moments, sigma commutator, pion-nucleon coupling constant ($g_{\pi NN}$), and axial vector coupling constant (g_A), as well as weak and electromagnetic form factors].

Another criticism of this approach, however, is the approximate treatment of the pion, whose very light mass argues for a quantum, as opposed to mean-field, treatment. However, treating light strongly coupled fields is in general difficult as perturbative methods are unreliable. In part to address such issues, the coherent-pair approximation was developed by Bolsterli [9] and used by GHGU to treat pions as true quantum fields nonperturbatively. Besides the nonperturbative inclusion of quantum pionic degrees of freedom, another advantage is that one need not invoke a hedgehog ansatz for the pion field. Specifically, the permutation symmetry of the quarks induces the space-isospin correlations in the pion field. The results reported by GHGU, however, were somewhat dissappointing when compared to the generalized hedgehog ansatz approach [6] with the principal problem being an apparent lack of self-consistency in the pionic sector. We find this approach promising, and in this work revisit the hybrid chiral model of Goeke *et al.*, correcting several errors in the hope that better self-consistency and a more favorable phenomenology will result. Following the groundbreaking work of GHGU, we minimize the chiral Hamiltonian with respect to the coherent-pair Fock-space ansatz ground-state configuration, then calculate nucleon properties, and compare with other chiral models.

For completeness and ease of comparison the organization of this paper closely follows that of GHGU. We present the starting Lagrangian and the major intermediate results, refer the reader to the original paper for details, and note where our results differ from those of GHGU. Where appropriate, we will give the corresponding GHGU equation references. We present the background theory in Sec. II, a summary of the coherent-pair approximation results in Sec. III, the variational equations in Sec. IV, the derived nucleon properties in Sec. V, the numerical results in Sec. VI, and our summary and conclusions in Sec. VII. We find that the corrected equations do not significantly alter the results originally reported by GHGU, and the problems with self-consistency in the pionic sector persist. When compared with the hedgehog model, the coherent-pair approximation shows systematically smaller mesonic contributions to the nucleon observables and the energy densities, which is probably related to the poor pionic self-consistency.

II. CHIRAL QUARK-MESON MODEL

We begin with the chiral model Lagrangian of Gell-Mann and Lévy [10] but with explicit quark degrees of freedom. (Discussion of how such a form may be argued from QCD are given in Refs. [1] and [5].) After chiral symmetry breaking, inducing a pion mass, the Lagrangian can be written [GHGU, Eq. (2.1)] as

$$\mathcal{L}(x)=i\bar{\hat{\psi}}\partial\!\!\!/\hat{\psi}+\tfrac{1}{2}(\partial_\mu\hat{\sigma}\partial^\mu\hat{\sigma}+\partial_\mu\hat{\vec{\pi}}\cdot\partial^\mu\hat{\vec{\pi}}) +g\bar{\hat{\psi}}(\hat{\sigma}+i\gamma_5\vec{\tau}\cdot\vec{\pi})\hat{\psi}-\mathcal{U}(\hat{\sigma},\hat{\vec{\pi}}), \tag{1}$$

with

$$\mathcal{U}(\hat{\sigma},\hat{\pi})=\frac{\lambda^2}{4}(\hat{\sigma}^2+\hat{\vec{\pi}}^2-\nu^2)^2-f_\pi m_\pi^2\hat{\sigma}, \tag{2}$$

where the carat denotes a quantum field, f_π is the pion decay constant, m_π is the pion mass, and ν, g, and λ are constants to be determined. In the standard scenario spontaneous symmetry breaking generates masses for the quark and sigma fields and the linear sigma term, which breaks the chiral symmetry and generates the small pion mass which would be zero otherwise as the Goldstone boson of the theory. The vacuum then has a unique nonvanishing scalar field expectation value:

$$\frac{\partial U}{\partial\hat{\vec{\pi}}}=0\Rightarrow\vec{\pi}_0=0,\quad \frac{\partial U}{\partial\hat{\sigma}}=0\Rightarrow\sigma_0=f_\pi. \tag{3}$$

Then the three undetermined constants in the original Lagrangian can be written in terms of the three effective masses: $m_q=-g\sigma_0$, $m_\sigma^2=\lambda^2(3\sigma_0^2-\nu^2)$, and $m_\pi^2=\lambda^2(\sigma_0^2-\nu^2)$.

We take the experimental values $f_\pi=93\,\mathrm{MeV}$ and $m_\pi=139.6\,\mathrm{MeV}$, leaving g and m_σ as the only free parameters which must be determined. The additional parameters introduced by the coherent-pair approximation are constrained by minimization.

Introducing the conjugate momenta, one formally converts the Lagrangian density to a Hamiltonian density [GHGU, Eq. (2.10)] into

$$\hat{H}(r)=\tfrac{1}{2}\{\hat{P}_\sigma(r)^2+[\nabla\hat{\sigma}(r)]^2+\hat{\vec{P}}_\pi(r)^2+[\nabla\hat{\vec{\pi}}(r)]^2\} +\mathcal{U}(\hat{\sigma},\hat{\vec{\pi}})+\hat{\psi}^\dagger(r)(-i\alpha\vec{\nabla})\hat{\psi}(r) -g\hat{\psi}^\dagger(r)[\beta\hat{\sigma}(r)+i\beta\gamma_5\vec{\tau}\cdot\hat{\vec{\pi}}]\hat{\psi}(r), \tag{4}$$

where $\vec{\alpha}$ and β are the usual Dirac matrices. In the above expression $\hat{\psi}$, $\hat{\sigma}$, and $\hat{\vec{\pi}}$ are quantized field operators, with the appropriate static angular momentum expansions [GHGU, Eqs. (2.11), (2.17), and (2.13)],

$$\hat{\sigma}(r)=\int\frac{d^3k}{[(2\pi)^3 2\omega_\sigma(k)]^{1/2}}[\hat{c}^\dagger(k)e^{-ik\cdot r}+\hat{c}(k)e^{+ik\cdot r}], \tag{5}$$

$$\hat{\pi}(r)=\left[\frac{2}{\pi}\right]^{1/2}\int_0^\infty dk\,k^2\left[\frac{1}{2\omega_\pi(k)}\right]^{1/2}\sum_{lmw}j_l(kr)Y_{lm}^*(\Omega_r) \times[\hat{a}_{lm}^{1w\dagger}(k)+(-)^{m+w}\hat{a}_{l-m}^{1-w}(k)], \tag{6}$$

$$\hat{\psi}(r)=\sum_{njmw}(\langle r|njmw\rangle\hat{d}_{njm}^{(1/2)w}+\langle r|\overline{nj}mw\rangle\hat{d}_{njm}^{1/2\,w\dagger}), \tag{7}$$

where $|njmw\rangle$ and $|\overline{nj}mw\rangle$ form a complete set of quark and antiquark spinors with angular momentum quantum numbers and spin-isospin quantum numbers j, m, and w, respectively. The notation is slightly altered from that of GHGU in that isospin labels appear as superscripts and spin labels appear as subscripts. The corresponding conjugate momentum fields have the expansions [GHGU, Eqs. (2.14) and (2.18)]

$$\hat{P}_\sigma(r)=i\int_0^\infty d^3k\left[\frac{\omega_\sigma(k)}{2(2\pi)^3}\right]^{1/2}[\hat{c}^\dagger(k)e^{-i\vec{k}\cdot\vec{r}}-\hat{c}(k)e^{+i\vec{k}\cdot\vec{r}}], \tag{8}$$

$$\hat{P}_\pi(r)=i\left[\frac{2}{\pi}\right]^{1/2}\int_0^\infty dk\,k^2\left[\frac{\omega_\pi(k)}{2}\right]^{1/2}\sum_{lmw}j_l(kr)Y_{lm}^*[\hat{a}_{lm}^{1w\dagger}(k) -(-)^{m+w}\hat{a}_{l-m}^{1-w}(k)]. \tag{9}$$

Here $\hat{c}(k)$ destroys a σ quantum with momentum $\vec{k}$ and frequency $\omega_\sigma(k)=(k^2+m_\sigma^2)^{1/2}$, and $\hat{a}_{tlm}^{1/w}(k)$ destroys a pion with momentum $\vec{k}$ and corresponding $\omega_\pi=(k^2+m_\pi^2)^{1/2}$ in isospin-angular momentum state $\{lm;tw\}$. For convenience one constructs the configuration space pion field functions needed for the subsequent variational treatment by defining the alternative basis operators

$$\hat{b}_{lm}^{1w}=\int dk\,k^2\xi_l(k)\hat{a}_{lm}^{1w}(k), \tag{10}$$

where $\xi_l(k)$ is the variational function. Taking this over to configuration space defines the pion field function [GHGU, Eq. (3.11)]

$$\phi_l(r)=\frac{1}{2\pi}\int_0^\infty dk\,k^2\frac{\xi_l(k)}{[\omega_\pi(k)]^{1/2}}j_l(r). \tag{11}$$

In the following only the $l=1$ value is used, and the angular momentum label will be dropped.

III. GROUND-STATE CONFIGURATION ANSATZ

The ansatz Fock state for the nucleon is taken to be [GHGU, Eq. (4.23)]

$$|NT_3J_z\rangle=[\alpha(|n\rangle\otimes|P_0^0\rangle)_{T_3J_z}+\beta(|n\rangle\otimes|P_1^1\rangle)_{T_3J_z} + \gamma(|\delta\rangle\otimes|P_1^1\rangle)_{T_3J_z}]|\Sigma\rangle, \tag{12}$$

where $|\Sigma\rangle$ is the coherent sigma field state with the property [GHGU, Eq. (3.7)]. $\langle\Sigma|\hat{\sigma}(r)|\Sigma\rangle=\sigma(r)$, and $|P_0^0\rangle$ ($|P_{1m}^{1w}\rangle$) are pion coherent-pair states to be determined. The normalization of the nucleon state requires $\alpha^2+\beta^2+\gamma^2=1$. The permutation symmetric form of the SU(2)×SU(2)×SU(2) quark wave functions imply that the source terms in the pion field equations will induce an angular momentum-isospin correlation for the pion field. Thus, since the pion is an isovector, the only allowed angular momentum of the pion will be $l=1$, so in the treatment to follow only the $l=1$ term of the pion field expressions Eqs. (6) and (9) is retained. We remark at this point that the different treatments of the sigma (classical) and pion (quantum) fields introduce chiral violations of an uncontrolled nature which can only be tested by a full quantum treatment.

One constructs a pionic coherent-pair state with quantum numbers of the vacuum as follows [9]. Consider the scalar-isoscalar coherent state

$$|P_0^0\rangle=\sum_n\frac{f_n}{(2n)!}[\hat{b}_1^{1\dagger}:\hat{b}_1^{1\dagger}]^n|0\rangle, \tag{13}$$

where the double-dot notation refers to spin-isospin (i.e., $\{m,w\}$) scalar contractions. A coherent state with spin-isospin=$\{11\}$, $|P_{1m}^{1w}\rangle$, is constructed by operation of $(-)^{m+w}\hat{b}_{1-m}^{1-w}$ upon $|P_0^0\rangle$. Sequential such operations appropriately contracted are assumed to close, yielding a recurrence relation for f_n, namely,

$$(-)^{m+w}\hat{b}_{1-m}^{1-w}|P^{00}\rangle=a|P_{1m}^{1w}\rangle. \tag{14}$$

A contracted second application gives

$$\sum_{m,w}\hat{b}_{1m}^{1w}|P_{1m}^{1w}\rangle=\tilde{L}b|P_0^0\rangle, \tag{15}$$

where the spin-isospin multiplicity is $\tilde{L}=(2L+1)(2T+1)=9$, and where closure is forced by associating the vacuum quantum number multipion state uniquely with $|P_0^0\rangle$. The coherence is determined by the parameter x, defined by

$$(\hat{b}_1^{1\dagger}:\hat{b}_1^{1\dagger})|P\rangle=x|P\rangle. \tag{16}$$

Here $|P\rangle$ can be either $|P_0^0\rangle$ or $|P_1^1\rangle$ and $x=\tilde{L}ab=9ab$ serves as a (as yet free) coherence parameter and the symbol $b_1^{1\dagger}:b_1^{1\dagger}$ indicates the coupling to a scalar-isoscalar. Inserting these into Eqs. (13) and (16), we obtain the recursion relation [GHGU, Eq. (4.11)]

$$f_{n+1}=\frac{x(2n+1)}{(\tilde{L}+2n)}f_n, \tag{17}$$

which can be solved to give [GHGU, Eq. (4.12)]

$$f_n=\frac{x^n(2n-1)!!(\tilde{L}-2)!!}{(\tilde{L}-2+2n)!!}f_0, \tag{18}$$

where f_0 is given by the normalization of $|P_0^0\rangle$. The a and b parameters can be expressed as functions of x from the normalization of $|P_1^1\rangle$. We obtained values for the function f_0, $a(x)$ and $c(x)$ (b is determined from a) different from those of GHGU. We find

$$a(x)=\frac{1}{3}\left[\frac{(105+45x^2+x^4)\sinh x-(105+10x^2)x\cosh x}{-(15+6x^2)\sinh x+(15+x^2)x\cosh x}\right]^{1/2}, \tag{19}$$

$$c(x)=\frac{1}{3}\left[1+\frac{-(945+420x^2+15x^4)\sinh x+(945+105x^2+x^4)x\cosh x}{(105+45x^2+x^4)\sinh x-(105+10x^2)x\cosh x}\right]^{1/2}, \tag{20}$$

and

$$f_0^{-2}=(\tilde{L}-2)!!2^{\bar{x}-1}\partial_y^{\bar{x}-1}\cosh(x), \tag{21}$$

with $y=x^2$ and

$$\bar{x}=\frac{(2\tilde{L}+1)}{2},\quad \partial_y=\frac{\partial}{\partial y}=\left(\frac{1}{2x}\right)\frac{\partial}{\partial x}, \tag{22}$$

which should be contrasted with GHGU [Eqs. (4.13), (4.21), and (4.22)]. In addition we found that we needed another factor for the $|P_1^1\rangle$ matrix element of the four-pion term (implied summation over repeated indices),

$$\langle P_1^1|\hat{b}_\alpha^\dagger\hat{b}_\beta^\dagger\hat{b}_\beta\hat{b}_\alpha|P_1^1\rangle = 81d(x)^2, \tag{23}$$

where greek subscripts include both spin and isospin, and $d(x)$ is given by

$$d(x)=\frac{1}{9}\left[\frac{(7560+3465x^2+165x^4+x^6)\sinh x-(7560+945x^2+18x^4)x\cosh x}{(105+45x^2+x^4)\sinh x-(105+10x^2)x\cosh x}\right]^{1/2}. \tag{24}$$

IV. FIELD EQUATIONS

The total energy of the baryon is given by

$$E_B=\left\langle BT_3J_z\left|\int_0^\infty d^3r{:}\hat{H}(r){:}\right|BT_3J_z\right\rangle, \tag{25}$$

where $B=N$ or Δ. The field equations are obtained by minimizing the total energy of the baryon with respect to variations of the fields, $\{u(r),w(r),\sigma(r),\phi(r)\}$, as well as the Fock-space parameters $\{\alpha,\beta,\gamma\}$ subject to the normalization conditions. The total energy of the system is written as

$$E_B=4\pi\int_0^\infty dr\, r^2\mathcal{E}_B(r). \tag{26}$$

We find the following result for the energy density which differs from GHGU, Eq. (5.3). The differences can be traced to different results for coherent-pair matrix elements. Writing the quark Dirac spinor as

$$\psi_{(1/2)m}^{(1/2)w}(\vec{r})=\begin{pmatrix}u(r)\\ v(r)\vec{\sigma}\cdot\hat{r}\end{pmatrix}\chi_{(1/2)\,m}\xi^{(1/2)\,w}, \tag{27}$$

the energy density is given by

$$\begin{aligned}\mathcal{E}_B(r)=&\frac{1}{2}\left(\frac{d\sigma}{dr}\right)^2+\frac{\lambda^2}{4}[\sigma^2(r)-\nu^2]^2-m_\pi^2f_\pi\sigma(r)\\&+3\left[u(r)\left(\frac{dv}{dr}+\frac{2}{r}v(r)\right)-v(r)\frac{du}{dr}\right.\\&\left.+g\sigma(r)[u^2(r)-v^2(r)]\right]\\&+(N_\pi+x)\left[\left(\frac{d\phi}{dr}\right)^2+\frac{2}{r^2}\phi^2(r)\right]\\&+(N_\pi-x)\phi_p^2(r)-\alpha\delta(a+b)u(r)v(r)\phi(r)\\&+\lambda^2\{x^2+2xN_\pi+81[\alpha^2a^2c^2+(\beta^2+\gamma^2)d^2]\}\phi^4(r)\\&+\lambda^2(N_\pi+x)[\sigma^2(r)-\nu^2]\phi^2(r),\end{aligned} \tag{28}$$

where N_π is the average pion number [GHGU, Eq. (5.12)]

$$N_\pi=9[\alpha^2a^2+(\beta^2+\gamma^2)c^2], \tag{29}$$

and where δ takes the following values for nucleon or delta quantum numbers:

$$\delta_N=(5\beta+4\sqrt{2}\gamma)/\sqrt{3}, \tag{30}$$

$$\delta_\Delta=(2\sqrt{2}\beta+5\gamma)/\sqrt{3}. \tag{31}$$

The function $\phi_p(r)$ is obtained from $\phi(r)$ by the double folding

$$\phi_p(r)=\int_0^\infty\omega(r,r')\phi(r')r'^2dr', \tag{32}$$

with

$$\omega(r,r')=\frac{2}{\pi}\int_0^\infty dk\,k^2\omega(k)j_1(kr)j_1(kr'). \tag{33}$$

For fixed α, β, and γ, the stationary functional variations are expressed by

$$\delta\left[\int_0^\infty dr\,r^2\{\mathcal{E}_B(r)-3\varepsilon[u^2(r)+v^2(r)]-2\kappa\phi\phi_p(r)\}\right]=0, \tag{34}$$

where the Lagrangian parameter κ enforces the pion normalization condition

$$8\pi\int_0^\infty\phi(r)\phi_p(r)r^2dr=1, \tag{35}$$

and the Lagrangian parameter ε fixes the quark normalization

$$4\pi\int_0^\infty dr\,r^2[u^2(r)+v^2(r)]=1. \tag{36}$$

Minimizing the Hamiltonian yields the four nonlinear coupled differential equations

$$\frac{du}{dr}=-(g\sigma+\varepsilon)v(r)-\frac{2}{3}\alpha\delta(a+b)g\phi(r)u(r),$$

$$\frac{dv}{dr}=-\frac{2}{r}v(r)-[g\sigma(r)-\varepsilon]u(r)+\frac{2}{3}\alpha\delta(a+b)g\phi(r)v(r),$$

$$\begin{aligned}\frac{d^2\sigma}{dr^2}=&-\frac{2}{r}\frac{d\sigma}{dr}-m_\pi^2f_\pi+3g[u^2(r)-v^2(r)]\\&+2\lambda^2(N_\pi+x)\phi^2(r)\sigma(r)+\lambda^2[\sigma^2(r)-\nu^2]\sigma(r),\end{aligned}$$

$$\frac{d^2\phi}{dr^2}=-\frac{2}{r}\frac{d\phi}{dr}+\frac{2}{r^2}\phi(r)+\frac{1}{2}\left(1-\frac{x}{N_\pi}\right)m_\pi^2\phi(r)+\frac{\lambda^2}{2}\left(1+\frac{x}{N_\pi}\right)[\sigma^2(r)-\nu^2]\phi(r)+\frac{\lambda^2}{N_\pi}\{x^2+2N_\pi x+81[\alpha^2a^2c^2+(\beta^2+\gamma^2)d^2]\}\phi^3(r)-\frac{\alpha}{N_\pi}(a+b)g\,\delta u(r)v(r)-\frac{\kappa}{N_\pi}\phi_p(r), \tag{37}$$

with eigenvalues ε and κ. These consist of two quark equations for u and v where $\sigma(r)$ and $\phi(r)$ appear as potentials, and two Klein-Gordon equations with $u(r)v(r)$ and $[u^2(r)-v^2(r)]$ as source terms. The boundary conditions are, for $r\rightarrow 0$,

$$v=\frac{d\sigma}{dr}=\phi=\frac{du}{dr}=0, \tag{38}$$

and, for $r\rightarrow\infty$,

$$[r(gf_\pi-\varepsilon)^{1/2}+(gf_\pi+\varepsilon)^{(-1/2)}]u(r)-r(gf_\pi+\varepsilon)^{1/2}v(r)=0,$$
$$(2+2m_\pi r+m_\pi^2r^2)\phi(r)+r(1+m_\pi r)\phi'(r)=0, \tag{39}$$
$$r\sigma'(r)+[\sigma(r)-f_\pi](1+m_\sigma r)=0,$$

which has one sign in the first equation different from GHGU, Eq. (5.18). The field equations are solved for fixed coherence parameter x and fixed Fock-space parameters, $\{\alpha,\beta,\gamma\}$. Then the expectation value of the energy is minimized with respect to $\{\alpha,\beta,\gamma\}$ by diagonalization of the ''energy matrix,''

$$\begin{bmatrix} H_{\alpha\alpha} & H_{\alpha\beta} & H_{\alpha\gamma}\\ H_{\alpha\beta} & H_{\beta\beta} & H_{\beta\gamma}\\ H_{\alpha\gamma} & H_{\beta\gamma} & H_{\gamma\gamma}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}=E\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}. \tag{40}$$

Each H entry of the matrix is related to a corresponding density, $E(r)$, as follows:

$$H_{\alpha\beta}=4\pi\int_0^\infty r^2E_{\alpha\beta}(r)dr, \tag{41}$$

and analogously for the other entries. The $E_{\alpha\beta}(r)$ functions for a nucleon are

$$E_{\alpha\alpha}(r)=E_0(r)+18a^2\phi_p^2(r)+9a^2\lambda^2(2x+9c^2)\phi^4(r)+9a^2\lambda^2[\sigma^2(r)-\nu^2]\phi^2(r), \tag{42}$$

$$E_{\beta\beta}(r)=E_0(r)+18c^2\phi_p^2(r)+9\lambda^2(2xc^2+9d^2)\phi^4(r)+9c^2\lambda^2[\sigma^2(r)-\nu^2]\phi^2(r), \tag{43}$$

$$E_{\alpha\beta}(r)=-2g(a+b)\phi(r)u(r)v(r)\frac{5}{\sqrt{3}}, \tag{44}$$

$$E_{\alpha\gamma}(r)=-2g(a+b)\phi(r)u(r)v(r)4\sqrt{\frac{2}{3}}, \tag{45}$$

$$E_{\gamma\gamma}(r)=E_{\beta\beta}(r), \tag{46}$$

$$E_{\beta\gamma}(r)=0. \tag{47}$$

The $E_{\alpha\beta}(r)$ functions for a delta are the same, except for

$$E_{\alpha\beta}(r)=-2g(a+b)\phi(r)u(r)v(r)\frac{2\sqrt{2}}{\sqrt{3}},$$

$$E_{\alpha\gamma}(r)=-2g(a+b)\phi(r)u(r)v(r)\frac{5}{\sqrt{3}},$$

which expresses the difference in the δ terms [Eqs. (32)–(33)] appropriate to nucleon and delta respectively.

In the above expressions, $E_0(r)$ is given by

$$E_0(r)=\frac{1}{2}\left[\frac{d\sigma}{dr}\right]^2+\frac{\lambda^2}{4}[\sigma^2(r)-\nu^2]^2-m_\pi^2f_\pi\sigma(r)+U_0+\lambda^2x^2\phi^4(r)+3g\sigma(r)[u^2(r)-v^2(r)]+3\left[u(r)\left(\frac{dv}{dr}+\frac{2}{r}v(r)\right)-v(r)\frac{du}{dr}\right]-xm_\pi^2\phi^2(r)+\lambda^2x[\sigma^2(r)-\nu^2]\phi^2(r), \tag{48}$$

where $U_0=m_\pi^2[f_\pi^2-m_\pi^2/(4\lambda^2)]$. This can be rewritten as

$$E_0(r)=\frac{1}{2}[\sigma'(r)]^2+\lambda^2x^2\phi(r)^4+3g\sigma(r)[u(r)^2-v(r)^2]+\frac{\lambda^2}{4}[\sigma(r)^2-f_\pi^2]^2+\frac{m_\pi^2}{2}[\sigma(r)^2-f_\pi^2]-m_\pi^2f_\pi[\sigma(r)-f_\pi]+\lambda^2x[\sigma(r)^2-f_\pi^2]\phi(r)^2-2xm_\pi^2\phi(r)^2, \tag{49}$$

which shows an explicit rapid decay as $r\rightarrow\infty$.

We solve this set of equations in the same iterative manner as GHGU. The iteration procedure is implemented as follows. For fixed values of x and α, β, and γ, the above differential equations with the corresponding boundary conditions are solved by using the same numerical package (COLSYS [11]) as used in the original GHGU paper. Then the energy matrix is diagonalized and the minimum eigenvector chosen. These solutions are then mixed with the previous solution and repeated until self-consistency is achieved. The procedure is started with an initial guess. We used the so-called ''chiral circle'' meson field forms [4], but we found that the actual starting point does not matter, provided the iterations converge.

V. NUCLEON PROPERTIES

In this section we review the several nucleon observables which will be calculated from the solutions arising from the procedure described in Sec. IV, noting differences from GHGU where they occur. From the electromagnetic current operator

$$\hat{J}^{\mu}_{\rm em}=\bar{\hat{\psi}}(\tfrac{1}{6}+\tfrac{1}{2}\tau_3)\hat{\psi}+\epsilon_{3\alpha\beta}\hat{\phi}_\alpha\partial^\mu\hat{\phi}_\beta, \tag{50}$$

one derives the charge and magnetic moment densities for the neutron and proton;

$$\frac{\rho_{\rm p}(r)}{4\pi e}=\alpha^2(u^2+v^2)+\beta^2\left[\frac{1}{3}(u^2+v^2)+\frac{4}{3}\phi\phi_p\right]+\gamma^2\left[\frac{4}{3}(u^2+v^2)-\frac{2}{3}\phi\phi_p\right], \tag{51}$$

$$\frac{\rho_{\rm n}(r)}{4\pi e}=\beta^2\left[\frac{2}{3}(u^2+v^2)-\frac{4}{3}\phi\phi_p\right]+\gamma^2\left[-\frac{1}{3}(u^2+v^2)+\frac{2}{3}\phi\phi_p\right], \tag{52}$$

$$\frac{\mu_{\rm p}(r)}{4\pi e}=\frac{ruv}{81}(54\alpha^2+2\beta^2+\gamma^2+32\sqrt{2}\beta\gamma)+\frac{x}{729a^2}(9a^2+x)(4\beta^2+\gamma^2)\phi^2, \tag{53}$$

$$\frac{\mu_{\rm n}(r)}{4\pi e}=\frac{ruv}{81}\left(-36\alpha^2-8\beta^2+\frac{1}{2}\gamma^2-32\sqrt{2}\beta\gamma\right)\times\frac{x}{729a^2}(9a^2+x)(4\beta^2+\gamma^2)\phi^2, \tag{54}$$

which differ from GHGU [Eqs. (6.6) and (6.7)] for the magnetic moment densities. The axial-vector to vector coupling ratio is given by [GHGU, Eq. (6.9)]

$$\frac{g_{\rm A}}{g_{\rm V}}=2\left\langle NJ_z=\frac{1}{2}T_3=\frac{1}{2}\right|\int d^3r{:}\left[\frac{1}{2}\bar{\hat{\psi}}\gamma^5\gamma_3\tau_0\hat{\psi}+\hat{\sigma}\partial^z\hat{\sigma}\hat{\pi}_0-\hat{\phi}_0\partial^z\hat{\sigma}\right]{:}\left|NJ_z=\frac{1}{2}T_3=\frac{1}{2}\right\rangle, \tag{55}$$

from which we find

$$\frac{g_{\rm A}}{g_{\rm V}}=4\pi\int_0^\infty dr\,r^2\left[\left(\frac{5}{3}\alpha^2+\frac{5}{27}\beta^2+\frac{25}{27}\gamma^2+\frac{32\sqrt{2}}{27}\beta\gamma\right)\times(u^2-v^2/3)+\frac{8}{3\sqrt{3}}\alpha\beta(a+b)\frac{d\sigma}{dr}\phi\right], \tag{56}$$

which differs with GHGU [Eq. (6.10)] in the first term. Finally, the πNN coupling constant can be calculated from the pion field or the pion source term. Using the pion field form, one has

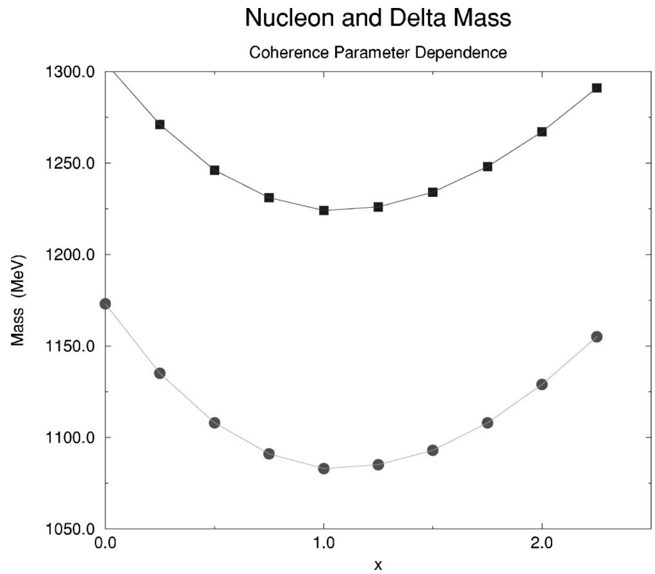

FIG. 1. Dependence of chiral nucleon and delta mass with respect to the coherence parameter x, using $g=5$ and $m_\sigma=700$ MeV.

$$\frac{g_{\pi NN}}{2M_N}=4\pi m_\pi^2\frac{2}{3\sqrt{3}}\alpha\beta(a+b)\int_0^\infty dr\,r^3\phi(r), \tag{57}$$

which is in agreement with GHGU, Eq. (6.15). Using the pion source term, one can obtain an alternative form of the πNN coupling constant,

$$\frac{g_{\pi NN}}{2M_N}=4\pi\int_0^\infty dr\,r^3g\left(\frac{10}{9}\alpha^2+\frac{10}{81}\beta^2+\frac{50}{81}\gamma^2+\frac{64\sqrt{2}}{81}\beta\gamma\right)u(r)v(r)-4\pi\lambda^2\alpha\beta\int_0^\infty dr\,r^3\left[\frac{2}{3\sqrt{3}}(a+b)[\sigma^2(r)-f_\pi^2]\phi(r)+\frac{4}{\sqrt{3}}(a^2b+2b^2a+ac^2)\phi^3(r)\right], \tag{58}$$

which differs from GHGU, Eq. (6.14), by a factor of 3 in the last term.

The σ-term was not calculated in GHGU, but is an important quantity in that it measures the degree of chiral symmetry breaking. For the linear sigma model considered here, this quantity is given by

$$\sigma_{\pi N}=4\pi f_\pi m_\pi^2\int_0^\infty dr\,r^2[\sigma(r)-f_\pi]. \tag{59}$$

For a review of this quantity, see Ref. [12].

VI. NUMERICAL RESULTS

As mentioned earlier, the chiral quark model of the nucleon has two free parameters once the pion mass and pion

TABLE I. Energy contributions to coherent-pair nucleon using $g=5$ and $m_\sigma=700$ Mev. (All values are in MeV.) Self-consistent solution evaluated using a coherent parameter value $x=1$. The details of this solution are discussed in the test.

Quantity	Nucleon	Delta
Quark eigenerergy	150	219
Quark kinetic energy	1124	975
Sigma kinetic energy	304	268
Pion kinetic energy	236	185
Quark-meson interaction energy	−675	−318
Meson interaction energy	84	114
Baryon mass	1073	1224
Nucleon-delta mass difference		140

decay constant are fixed at their experimental values. One can choose the free parameters to be the σ mass and the meson-quark coupling constant, g. For comparison with GHGU and to illustrate the systematics of the model, for the results to follow, we fix these two parameters to $m_\sigma=700$ MeV and $g=5.00$.

First consider the coherence parameter. Figure 1 shows the baryon (nucleon and delta) energy as the coherence parameter is varied. As was similarly shown in GHGU's Fig. 1, there is a clear minimum in the region of $x=1$ with a corresponding $N_\pi=0.43$. Table I shows the various energy contributions to the baryon mass. Table II gives the values of the derived parameters determined by the minimization procedure. The values found are in good agreement with that found by GHGU (self-consistent case) even though a different coherence dependence [which shows up in the d-term term in Eq. (29)] was used. This is expected, however, since the fourth-power pion self-interaction energy makes a relatively small contribution to the total energy. The nucleon-delta mass difference is found to be about 150 MeV. Figure 2 shows the quark wave functions and the meson fields for the $x=1$ case, and Fig. 3 shows the pion field and its derived kinetic-energy density function, $\phi_p(r)$. All of these quantities are little changed from those reported by GHGU. We have examined the fate of the initial ''chiral circle,'' where the sum of the squares of the meson fields is constant. For the self-consistent coherent-pair model we find a 40% downward deviation at the location of the peak of the pion field compared to a 20% downward deviation for the hedgehog model.

TABLE II. Numerical values of various parameters for the coherent-pair nuclon with a coherence parameter of $x=1$ and $g=5$, and $m_\sigma=700$ MeV after self-consistent minimization. (N/A means ''not available.'')

Quantity	This work	Goeke *et al.* [1]
Mass (GeV)	1.073	1.08
α	0.820	0.82
β	0.379	0.38
γ	0.429	0.43
Pion eigenvalue κ (GeV)	−0.173	N/A
Quark eigenvalue ϵ (GeV)	0.150	0.14
N_π	0.43	N/A

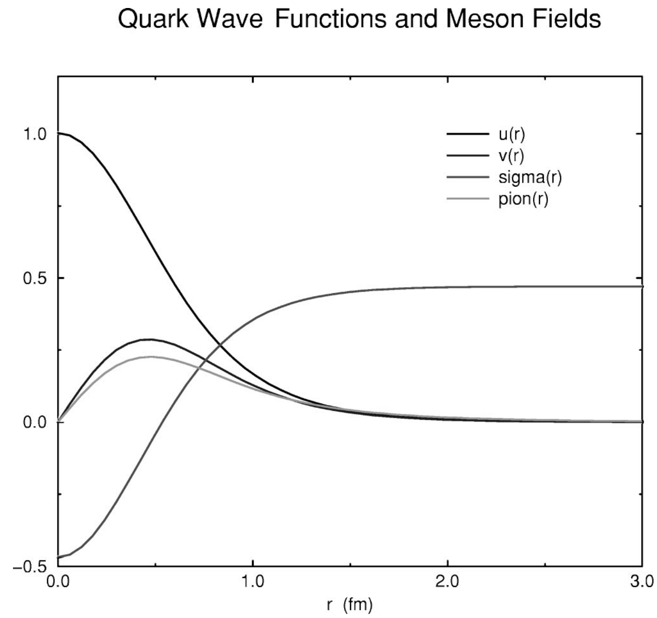

FIG. 2. Quark wave functions and meson fields using a coherence parameter of $x=1$ with $g=5$ and $m_\sigma=700$ MeV.

Next consider the nucleon physical properties. Table II shows the results for several nucleon observables compared with the projected hedgehog model of Birse [5] and with experiment. For those nucleon quantities calculated by GHGU, our results are nearly identical despite the differences in several quantities noted previously.

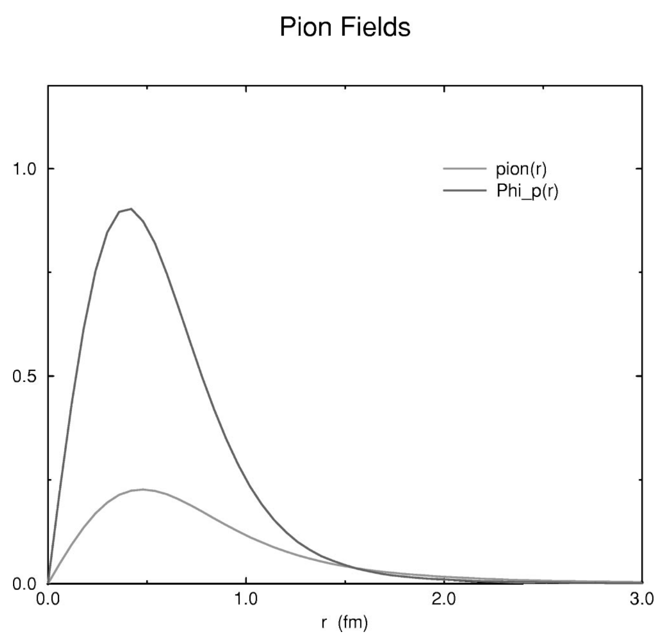

FIG. 3. Pion field shown with kinetic-energy pion field function $\Phi_p(r)$, using a coherence parameter of $x=1$ with $g=5$ and $m_\sigma=700$ MeV.

TABLE III. Observables for the coherent-pair nucleon with a coherence parameter of $x=1$ and using $m_\sigma=700$ MeV and $g=5$. Magnetic moments are in nuclear magnetons. The charge radius is in fm. For comparison, the results from the projected hedgehog model of Birse [5] are also presented.

	Coherent pair			Hedgehog [5]			
Quantity	quark	meson	total	quark	meson	total	Expt.
$\langle r^2\rangle_{\text{ch-proton}}$ (fm^2)	0.533	0.023	0.556	0.39	0.16	0.55	0.70
$\langle r^2\rangle_{\text{ch-neutron}}$ (fm^2)	0.019	−0.023	−0.004	0.09	−0.16	−0.070	−0.12
$\langle \mu\rangle_{\text{proton}}$	1.53	0.18	1.71	1.74	1.13	2.88	2.79
$\langle \mu\rangle_{\text{neutron}}$	−1.13	−0.18	−1.31	−1.16	−1.13	−2.29	−1.91
$\frac{g_A}{f_V}$	1.07	0.39	1.46	1.11	0.75	1.86	1.25
$g_{\pi NN}\frac{m_\pi}{2M}$ [Eq. (59)]			0.25			0.93	1.0
$g_{\pi NN}\frac{m_\pi}{2M}$ [Eq. (60)]	1.11	0.24	1.35	1.16	0.379	1.53	1.0
Sigma term (MeV)			88.9			94.0	∼45±5 [12]

In comparing the predictions of the coherent-pair model to those of the hedgehog model, we find that the quark contributions to each observable are roughly similar; however, the pion contributions are significantly smaller, a feature consistently seen throughout. This is somewhat surprising given the fact that both approaches attempt to solve the nucleon problem from the same starting model, albeit with very different methods and resulting different model parameters. Quantities such as the sigma commutator which depend on the sigma field are quite similar to that of the hedgehog model, both giving a sigma commutator of roughly 90 MeV. The small pion contributions to the calculated observables seems to be the coherent-pair approximation's principal phenomenological shortcoming.

Consider the nucleon charge radius squared. For the hedgehog model the pion's contribution to the proton's charge radius squared is roughly 40% that of the quarks; while for the coherent-pair approximation the pions contribution is only 4% that of the quarks. The small pionic contribution is compensated for by a slightly larger quark contribution leaving the total proton charge radius squared very close to that calculated in the hedgehog model, but still about 20% that of the experimental value.

The situation is similar for the magnetic moments. The quark contribution to the magnetic moments is about the same for both models, but for the hedgehog model the pions make a contribution to the proton magnetic moment which is 65% that of the quarks, while for the coherent-pair approximation the pionic contribution is only 12% that of the quarks. The smaller pionic contribution to the magnetic moments results in magnetic moments roughly 60% that of the empirical values.

For the case of the axial vector coupling constant, we again find that the quark contributions in the two models are similar, but the mesonic contribution in the coherent-pair model is only half that of the hedgehog model. In this case this actually helps in that the hedgehog model predicts too large a value for this quantity.

Finally we consider the πNN coupling constant calculated in two ways first using the pion field itself [Eq. (59)], and, second, using the pion source term [Eq. (60)]. In the hedgehog model Birse found that both methods gave roughly the same value, but in the coherent-pair approximation the two methods give very different results. This large difference in the two, supposedly equivalent, ways of calculating $g_{\pi\text{NN}}$ was noted already by Fiolhais *et al.* [6], who studied the generalized projected hedgehog model and compared with other models including GHGU. In Fiolhais *et al.* the difference in $g_{\pi\text{NN}}$ calculated from the pion field and the pion source terms along with the value expected from the Goldberger-Treiman relation provide a virial measure of how well the self-consistency condition in the pion sector is met. The Fiolhais *et al.* relation for the fractional virial deviation is

$$\Delta=\frac{g_{\pi NN}-g'_{\pi NN}}{g^{av}_{\pi NN}}+2\,\frac{(1.08)g^{\text{GT}}_{\pi NN}-g^{\text{av}}_{\pi NN}}{(1.08)g^{\text{GT}}_{\pi NN}+g^{\text{av}}_{\pi NN}}, \qquad (60)$$

where $g_{\pi NN}$ is the source-calculated value [Eq. (60)], $g'_{\pi NN}$ is the field-calculated value [Eq. (59)], $g^{\text{av}}_{\pi NN}$ is the average of these two, and $g^{\text{GT}}_{\pi NN}$ is the value expected from the Goldberger-Treiman relation:

$$g^{\text{GT}}_{\pi NN}=\frac{M_N}{f_\pi}\frac{g_A}{g_V}. \qquad (61)$$

The factor of 1.08 accounts for the effects of explicit chiral symmetry breaking. Using the coherent-pair approximation results of GHGU, Table III of Fiolhais *et al.* reports a fractional virial violation of 173%. This should be compared with 51% for the mean-field hedgehog model of Birse and Bannerjee, and 6% for the generalized projected hedgehog model of Fiolhais *et al.* Using the values in Table III, we find a fractional virial violation of 149% which is a little better

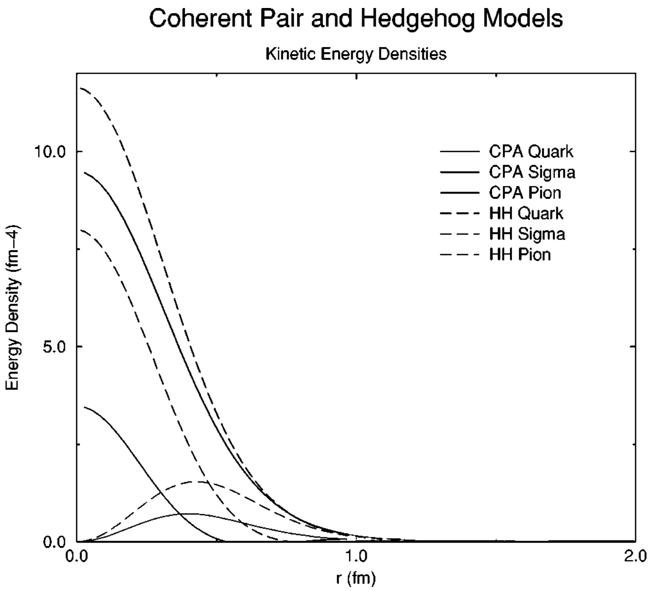

FIG. 4. Comparison of kinetic-energy densities for the hedgehog model and the coherent-pair approximation using a coherence parameter of $x=1$ with $g=5$ and $m_\sigma=700$ MeV.

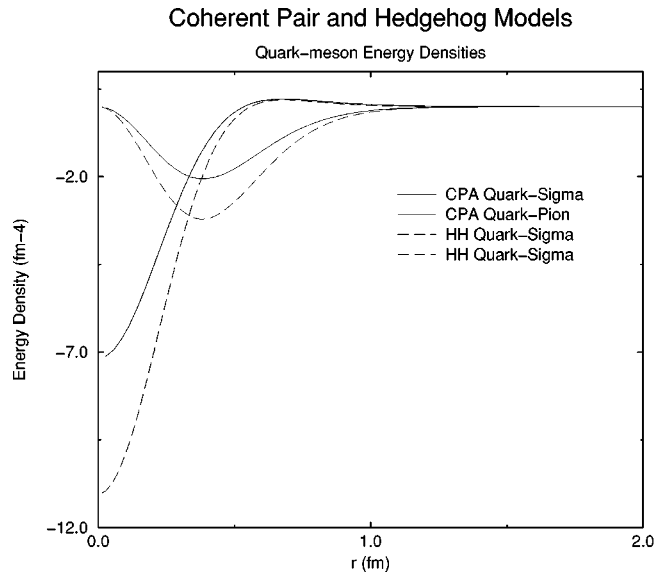

FIG. 5. Comparison of quark-meson interaction energy densities for the hedgehog model and the coherent-pair approximation using a coherence parameter of $x=1$ with $g=5$ and $m_\sigma=700$ MeV.

than that previously reported, but still shows a clear and substantial problem with self-consistency in the pion sector.

One must question why this should be so. That the two models should give such different pionic contributions is a bit puzzling, as the starting model is the same and both employ a self-consistent mean-field type approach which minimizes the energy. In comparing the quark wave functions and meson field solutions in the two models one finds, not surprisingly, that the quark wave functions and sigma field are nearly identical in the two models, but, though the pion field function in the coherent-pair approximation is smaller by a factor of about 2 from that found in the hedgehog model, this cannot be the source of the discrepancy since the pion field in the coherent-pair approximation is normalized but the magnitude of the pion field (treated as a classical field) in the hedgehog model is determined by the source terms. The magnitude of the pionic contribution to any quantity in the coherent-pair approximation is determined by the Fock space coefficients $\{\alpha,\beta,\gamma\}$ and by the various coherent functions $\{a,b,c,d\}$. With the minimized solution around $x\sim1.0$ the mean number of pions in the coherent-pair approximation case is $N_\pi=0.43$, which apparently yields the small pionic contributions to the various baryon observables. A comparison of the magnitude of the pion field resulting from the hedgehog model with the normalized field arising in the coherent-pair model would imply an effective number of pions about three times greater in the hedgehog model. A preliminary search of the parameter space to see if some other parameter set may correct this deficiency was not successful. We have examined the model up to a coherence parameter of $x=2.25$. Larger coherence parameters do indeed give rise to greater pionic contributions to the various nucleon observables, but at the expense of greater energy as well. While the issue is being investigated further, we suspect the problem with self-consistency as revealed in the large fractional virial violation to be the principal difficulty.

Another useful comparison is shown in Figs. 4 and 5, where the energy densities for various terms are shown for both the hedgehog model and the coherent-pair approximation. Again there is a systematically smaller contribution to each term in the coherent-pair model than in the hedgehog model. Yet the total energies of both solutions are roughly the same (1070 and 1120 MeV, respectively).

VII. SUMMARY AND CONCLUSIONS

In this work we reexamined a linear sigma model of the nucleon using quarks and sigma and pion mesons as the fundamental degrees of freedom. To solve this model we have employed the coherent-pair approximation following the work of Goeke *et al.*, correcting several errors. We solved the model using a Fock space ansatz, treating the sigma field as a classical field and treating the pions as quantum fields using the coherent-pair approximation of Bolsterli [9]. We neglect vacuum effects and center-of-mass corrections. Despite the several corrections to the work of Goeke *et al.*, the numerical solutions we find are very close to those presented in their original paper, as are all the calculated nucleon observables. We find that the calculated nucleon observables are reasonably close to experiment, but, in the case of electromagnetic quantities such as charge radii or magnetic moments, the pionic contributions seem too small when compared to that of other chiral nucleon models such as the Hedgehog model of Birse and co-workers [4,5]. The origin of this difference is not fully understood, but probably arises from the lack of self-consistency in pionic sector using this approach as was noted previously by Fiolhais *et al.* Therefore, at this stage we must concur with the rather disappointing conclusion of Goeke *et al.*, namely, that a better description of the pionic sector is required before a more satisfactory nucleon phenomenology can be expected in this model.

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