Remarks on the quantum modes of the scalar field on AdS_{d+1} spacetime

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The problem of the quantum modes of the scalar free field on anti–de Sitter (AdS) backgrounds with an arbitrary number of space dimensions is considered. It is shown that this problem can be solved by using the same quantum numbers as those of the nonrelativistic oscillator and two parameters which give the energy quanta and, respectively, the ground-state energy. This last one is known to be just the conformal dimension of the boundary field theory of the AdS-conformal field theory (CFT) conjecture. The obtained normalized energy eigenfunctions represent a new result which corrects that of Burgess and Lütken. [S0556-2821(99)08520-3]

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The recent interest in the propagation of quantum scalar fields on anti–de Sitter (AdS) spacetime is due to the discovery of the AdS conformal field theory correspondence [1]. One of the central points here is the relation between the field theory on the $(d+1)$-dimensional AdS (AdS_{d+1}) spacetime and the conformal field theory on its d-dimensional Minkowski-like boundary (M_d). There are serious arguments that the local operators of the conformal field theory on M_d correspond to the quantum modes of the scalar field on AdS_{d+1} [2]. Actually, for $d=3$ [3,4] as well as for any d [5] it is proved that the conformal dimension in boundary field theory is equal with the ground-state energy on AdS_{d+1} [2]. Moreover, it is known that the energy spectrum is discrete and equidistant [5], its quanta wavelength being just the hyperboloid radius of AdS_{d+1}.

In these conditions the scalar field on AdS_{d+1} can be seen as the relativistic correspondent of the nonrelativistic harmonic oscillator in d-space dimensions. This means that the quantum modes of the relativistic field may be labeled by the same quantum numbers as those of the nonrelativistic oscillator, namely the radial and the angular quantum numbers. In the case of $d=3$ we know that this is true [3,6] but for $d>3$, the definition and the role of the angular quantum number are not completely elucidated. This is the motive for our comment on this subject. Our aim is to present here the definitive form of the normalized wave functions of the regular [4,5] scalar modes on AdS_{d+1}, in terms of the above-mentioned quantum numbers (and natural units with $\hbar=c=1$). Moreover, we establish the formula of the degree of degeneracy of the energy levels.

The AdS_{d+1} spacetime is the hyperboloid $\eta_{AB}Z^AZ^B=R^2$ of radius $R=1/\omega$ in the $(d+2)$-dimensional flat spacetime of coordinates Z^{-1}, Z^0, $Z^1, \ldots, Z^d$ and metric $\eta_{AB}=\mathrm{diag}(1,1,-1,\ldots,-1)$, $A,B=-1,0,1,\ldots d$. On AdS_{d+1} we consider the static chart where the coordinates x^μ, $\mu=0,1,\ldots,d$, i.e., the time $x^0=t$ and the Cartesian space coordinates $\mathbf{x}\equiv(x^1,x^2,\ldots,x^d)$, are defined such that $Z^{-1}=R\sec\omega r\cos\omega t$, $Z^0=R\sec\omega r\sin\omega t$ and $\mathbf{Z}=R\tan\omega r(\mathbf{x}/r)$, with $r=|\mathbf{x}|$. Then, in the generalized spherical coordinates, $r,\theta_1,\ldots,\theta_{d-1}$, commonly related to the Cartesian ones [8], the metric is given by the line element [3,5]

$$ds^2=\eta_{AB}dZ^AdZ^B=\sec^2\omega r\left(dt^2-dr^2-\frac{1}{\omega^2}\sin^2\omega r\,d\theta^2\right), \tag{1}$$

where $d\theta^2$ is the usual line element on the sphere S^{d-1}. Since $r\in D_r=[0,\pi/2\omega)$, the whole space domain is $D=D_r\times S^{d-1}$. We remind the reader that the time of AdS_{d+1} must satisfy $t\in[-\pi/\omega,\pi/\omega)$ while $t\in(-\infty,\infty)$ defines the universal covering spacetime of AdS_{d+1} $(CAdS_{d+1})$ [3].

The one-particle quantum modes of the scalar quantum field ϕ of mass M, minimally coupled with the gravitational field, are given by the particular solutions of the Klein-Gordon equation

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\,g^{\mu\nu}\partial_\nu\phi)+M^2\phi=0,\quad g=|\det(g_{\mu\nu})|. \tag{2}$$

These may be either square integrable functions or distributions on D. In both cases they must be orthonormal (in the usual or generalized sense) with respect to the relativistic scalar product [7]

$$\langle\phi,\phi'\rangle=i\int_D d^dx\,\sqrt{g}g^{00}\phi^*\overleftrightarrow{\partial_0}\phi'. \tag{3}$$

The spherical variables of Eq. (2) can be separated by using generalized spherical harmonics, $Y^{d-1}_{l\,(\lambda)}(\mathbf{x}/r)$. These are normalized eigenfunctions of the angular Lalpace operator [8],

$$-\Delta_S Y^{d-1}_{l\,(\lambda)}(\mathbf{x}/r)=l(l+d-2)\,Y^{d-1}_{l\,(\lambda)}(\mathbf{x}/r), \tag{4}$$

corresponding to eigenvalues depending on the *angular* quantum number l which takes the values $0,1,2,\ldots$ [8]. The notation (λ) stands for a collection of quantum numbers giving the multiplicity of these eigenvalues [8],

$$\gamma_l=(2l+d-2)\frac{(l+d-3)!}{l!\,(d-2)!}. \tag{5}$$

We start with (positive frequency) particular solutions of energy E,

$$\phi^{(+)}_{E,l(\lambda)}(t,\mathbf{x}) \sim (\cot \omega r)^{(d-1)/2} R_{E,l}(r) Y^{d-1}_{l\,(\lambda)}(\mathbf{x}/r) e^{-iEt}, \tag{6}$$

and we denote $\epsilon = E/\omega$ and $\mu = M/\omega$ (i.e., $\epsilon = E/\hbar\omega$ and $\mu = Mc^2/\hbar\omega$ in the usual units). Then, after a few manipulations, we find the radial equation

$$\left[-\frac{1}{\omega^2}\frac{d^2}{dr^2} + \frac{2s(2s-1)}{\sin^2 \omega r} + \frac{2p(2p-1)}{\cos^2 \omega r} \right] R_{E,l} = \epsilon^2 R_{E,l} \tag{7}$$

where

$$2s(2s-1) = \left(l + \frac{d}{2} - 1 \right)^2 - \frac{1}{4}, \quad 2p(2p-1) = \mu^2 + \frac{d^2-1}{4}. \tag{8}$$

Hereby, we obtain the radial functions

$$R_{E,l}(r) \sim \sin^{2s} \omega r \cos^{2p} \omega r \times F\left(s+p-\frac{\epsilon}{2}, s+p+\frac{\epsilon}{2}, 2s+\frac{1}{2}, \sin^2 \omega r \right) \tag{9}$$

in terms of the Gauss hypergeometric function [9]. $R_{E,l}$ has a good physical meaning only when F is a polynomial selected by a suitable quantization condition since otherwise F is strongly divergent for $r \to \pi/2\omega$. Therefore, we introduce the radial quantum number n_r [5] and impose

$$\epsilon = 2(n_r + s + p), \quad n_r = 0,1,2, \ldots . \tag{10}$$

In addition, we choose the boundary conditions of the regular modes [4,5] given by the positive solutions of Eqs. (8), i.e., $2s = l + (d-1)/2$ and $2p = k - (d-1)/2$, where

$$k = \sqrt{\mu^2 + \frac{d^2}{4}} + \frac{d}{2} \tag{11}$$

is just the conformal dimension of the field theory on M_d [2]. We note that Eq. (10) is the quantization condition on $C\mathrm{AdS}_{d+1}$ while the AdS_{d+1} one requires k to be an integer number, too [3].

The last step is to define the *main* quantum number, $n = 2n_r + l$, which takes the values, 0,1,2, . . . , giving the energy levels $E_n = \omega(k+n)$. If n is even then $l = 0,2,4,\ldots,n$, while for odd n we have $l = 1,3,5, \ldots, n$. In both cases we can demonstrate that the degree of degeneracy of the level E_n is

$$\gamma_n = \sum_l \gamma_l = \frac{(n+d-1)!}{n!\,(d-1)!}. \tag{12}$$

Now it remains only to express Eq. (9) in terms of Jacobi polynomials and to normalize them to unity with respect to Eq. (3). The final result is

$$\phi^{(+)}_{n,l(\lambda)}(t,\mathbf{x}) = N_{n,l} \sin^l \omega r \cos^k \omega r P^{(l+d/2-1,\, k-d/2)}_{n_r}(\cos 2\omega r) \times Y^{d-1}_{l(\lambda)}(\mathbf{x}/r) e^{-iE_n t}, \tag{13}$$

where

$$N_{n,l} = \omega^{(d-1)/2} \left[\frac{n_r!\, \Gamma(n_r+k+l)}{\Gamma(n_r+l+d/2)\Gamma(n_r+k+1-d/2)} \right]^{1/2}. \tag{14}$$

We specify that our result coincides with those of Refs. [3,4,6] for $d=3$ but in the general case of any d this is similar (up to notations) to that of Ref. [5] only for $l=0$, while for $l \neq 0$ the first index of the Jacobi polynomial and the normalization factor are different.

The parameter k we use instead of M could play an important role in the supersymmetry and shape invariance of the radial problem as well as in the structure of the dynamical algebra. The argument is that the radial problems for arbitrary d are of the same nature as that with $d=1$, for which we have recently shown that k determines the shape of the relativistic potential and, in addition, represents the minimal weight of the irreducible representation of its so(1,2) dynamical algebra [10].

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