On the uniqueness of the expected stress-energy tensor in renormalizable field theories

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It is argued that the ambiguity introduced by the renormalization in the effective action of a four-dimensional renormalizable quantum field theory is at most a local polynomial action of canonical dimension four. The allowed ambiguity in the expected stress-energy tensor of a massive scalar field is severely restricted by this fact. [S0556-2821(99)05920-2]

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In a recent paper [1], it has been rightly noted that Wald axioms [2] allow for a much larger ambiguity in the expectation value of the renormalized stress-energy tensor for massive scalar particles than for massless ones. It was previously assumed [2] that, as in the massless case [3], in the massive case the Wald axioms were also sufficient to completely determine the expected stress-energy tensor as a functional of the state and background metric (up to the usual two parameter ambiguity) without the need of further physical input. The paper [1] clears up this misconception and so it is relevant to the foundations of the point-splitting procedure in the massive case (see also [4]).

In [1], the expected stress-energy tensor was computed for minimally coupled massive scalar fields in a nearly flat spacetime using two different formalisms, but no enlargement in the ambiguity was actually found as compared to the massless case. In this Brief Report, we want to note that this result should not be surprising since, in addition to Wald axioms, there are additional criteria which can be used to drastically reduce the ambiguity.

The problem of the ultraviolet ambiguities in quantum field theory appears in flat as well as in curved spacetime. It seems sensible to assume that the ultraviolet sector will depend only on local properties of the spacetime manifold and will not be affected by global topological aspects [5]. Therefore we can restrict ourselves to spacetimes which are topologically equivalent to R^n (this certainly covers the case of nearly flat spaces). The possibility of particle creation induced by a local curved region or even a different in and out vacuum due to different asymptotic metric tensors is not specific to curved spacetimes since these phenomena can also occur in the presence of suitable external fields in flat spacetime [6]. Another issue is that of the appropriate measure for the functional integration on the configuration space of the field to be quantized. General covariance requires it to depend on the metric [7]; however, this is also not specific to curved spaces; for instance, the axial anomaly for Dirac fermions implies that the fermion measure should depend on the background vector and axial field configurations [8]. Therefore, for the assumed spacetime topologies, we can regard the quantum field theory as one on a flat spacetime with the metric $g_{\mu\nu}(x)$ playing the role of a particular kind of background field and general covariance as a particular symmetry of the action (under simultaneous transformations of the quantum and background fields).

The problem of the appropriate functional measure is in fact completely equivalent to that of the ultraviolet ambiguities introduced by the renormalization. On the one side, the measure is not entirely well-defined because of the ultraviolet divergences and their necessary renormalization, but on the other side the measure cannot be completely arbitrary, since that would amount to end up with a completely arbitrary action, unrelated to the original action of the theory. A practical point of view is to use perturbation theory (or other expansions) in order to isolate the ultraviolet divergent pieces. All contributions which are finite without regularization are naturally postulated to be free from ambiguities and all acceptable renormalization prescriptions are required to reproduce those finite values. On the other hand, contributions which are divergent under any expansion are subject to renormalization by subtraction of appropriate counterterms. Since finiteness does not fix the counterterms uniquely, the renormalized contributions become finite but not unique. The requirement of finiteness allows for very general renormalizations. The class of allowed regularizations can naturally be restricted by demanding that they should be independent on the background fields and parameters of the theory, otherwise the regularization would mask the dependence of the effective action and expectation values on these background fields and parameters (e.g., we require a cutoff not to change under variations of the mass of the quantum field or the metric tensor).

For definiteness, let us consider a single quantum scalar field $\phi(x)$ with Lagrangian density $\mathcal{L}(x)$. More general cases are discussed below. In general $\mathcal{L}(x)$ will depend on some parameters λ_i (such as masses, coupling constants and spectroscopic factors) and background fields, $L_i(x)$. Without loss of generality we can include the parameters λ_i into the set of $L_i(x)$ since they can be seen as scalar background fields which happen to take a constant configuration. Moreover, we can assume that all background fields (including the parameters) couple linearly to local operators depending on ϕ and its derivatives. For instance, the Lagrangian density of a minimally coupled scalar field

$$\mathcal{L}(x)=\tfrac{1}{2}\sqrt{-g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-m^2\phi^2) \quad (1)$$

can be seen as a particular case of the more general theory

$$\mathcal{L}(x)=\tfrac{1}{2}H^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\tfrac{1}{2}G\phi^2 \quad (2)$$

where $H^{\mu\nu}(x)$ and $G(x)$ are arbitrary independent external fields; at the end of the calculation they can be particularized to their values in Eq. (1).

It is standard to describe the properties of the quantum field $\phi(x)$ using the effective action $\Gamma[\phi,L]$ [9]. The effective action is a finite quantity from which derive all renormalized Green functions, i.e., all expectation values. It will be sufficient for us to use the quantity $W[L]$ defined as $\Gamma[\phi,L]$ evaluated at its minimum with respect to ϕ. This is equivalent to the logarithm of the partition functional in the absence of external currents, $\int[d\phi(x)]e^{iS[\phi]}$. The quantity $W[L]$ is also frequently referred to as the effective action and provides the expectation values of operators coupled to the external fields. For instance, the expectation value of the stress-energy tensor $T_{\mu\nu}$ is given by the functional variation of W with respect to $g_{\mu\nu}$. Conservation of $T_{\mu\nu}$ follows if W is a scalar under general coordinate transformations [10]. Let us remark that the renormalization of the effective action introduces ambiguities in this quantity but no further ambiguities appear in the extraction of the expectation values since this is done through differentiation or minimization (diagrammatically this only involves tree level graphs).

After these general considerations, let us come to the issue of the allowed ambiguity introduced by the renormalization. We begin by considering the scalar field without self-interaction in Eq. (2). The Lagrangian density is quadratic and, after integration by parts, it can be rewritten as $\mathcal{L}(x)=\frac{1}{2}\phi A\phi$, where A is a second order differential operator depending on the external fields. As is well known, the effective action is formally given by $W[H,G]=\frac{1}{2}i\,\mathrm{Tr}\log(A)$. This is a formal expression which is ultraviolet divergent and so requires renormalization. To do perturbation theory, the Lagrangian is separated as

$$\mathcal{L}(x)=\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}m_0^2\phi^2+\frac{1}{2}\delta H^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}\delta G\phi^2 \tag{3}$$

Diagrammatically, $\frac{1}{2}i\,\mathrm{Tr}\log(A)$ corresponds to one-loop graphs where the field ϕ runs in the loop with mass m_0^2 and with any number of insertions of the external fields $\delta H^{\mu\nu}(x)$ and $\delta G(x)$. In order to find the dependence on the field δG of the allowable ambiguities, some standard diagrammatic results can be applied [11]. All diagrams with a sufficiently large number of insertions of δG (depending on the space-time dimension) are ultraviolet convergent. This is because each new insertion implies a new propagator line thereby decreasing the degree of divergence of the graph. This implies that all pieces in the effective action with a sufficiently large power of δG are ultraviolet finite. Therefore, the divergent part is a polynomial in the external field δG. Likewise, taking derivatives with respect to the external momenta carried by $\delta G(x)$ also increases the convergence and eventually the graphs become convergent. This implies that their divergent component is a polynomial in the external momenta of $\delta G(x)$. Equivalently, the divergent contribution to the effective action functional contains a finite number of partial derivatives of the $\delta G(x)$. (More precisely, it is a polynomial in the operators ∂_μ acting on δG.)

The polynomial character of the dependence on δG of the divergences also follows easily from the formal expression $\frac{1}{2}i\,\mathrm{Tr}\log(A)$, without resorting to perturbation theory, since it is clear that after a sufficient number of functional derivatives with respect δG, there will be a sufficient number of powers of A in the denominator so that the expression becomes fully ultraviolet convergent. The argument also applies to the dependence on m_0^2, since taking a derivative with respect to the mass also increases the convergence. (Note that this kind of arguments assumes that the regularization does not depend on the parameters to be varied.)

From straightforward power counting [11] it follows that, as a function of m_0^2 and δG, the divergent component in the effective action functional is a local polynomial constructed with m_0^2, $\delta G(x)$ and its derivatives, of canonical dimension at most four (or more generally, of canonical dimension at most D in D-dimensional field theories). The canonical dimension refers to that carried by m_0^2, and the field $\delta G(x)$ and its derivatives. The total mass dimension (of the effective action density ambiguity) is four and comes, in addition, from the derivatives of $H^{\mu\nu}(x)$ and from possible further dimensionful parameters (with non-negative mass dimension, as we will see) introduced by the renormalization procedure. The divergent components are canceled by adding new terms to the action. After such a renormalization, the effective action is finite, but not unique. The renormalization leaves an ambiguity which is again a local polynomial in m_0^2 and $\delta G(x)$ and its derivatives of canonical dimension at most four. Therefore, two mathematically acceptable versions of the renormalized effective action will differ at most by a local polynomial of dimension four. This is necessary if they are to coincide in their ultraviolet finite components. This latter requirement is natural since such ultraviolet finite components can be computed without any regularization and hence have a unique and well-defined value.

The previous discussion implies that ambiguities of the form considered in [1], such as very general functions of R/m^2 (R being the scalar curvature), although allowed by Wald axioms, cannot actually appear. Only polynomials in m^2 may appear. This is because taking n derivatives with respect to m^2 in the effective action yields (the connected part of) the expectation value of $[\int dx^4\sqrt{-g}\,\phi^2(x)]^n$, which is a completely ultraviolet finite quantity when $n>2$ and hence free from ambiguities. However, two versions of the effective actions differing by an arbitrary function of m^2 would yield different result for $\langle[\int dx^4\sqrt{-g}\,\phi^2(x)]^n\rangle_c$ and so at most one these effective actions could be acceptable [12].

As it stands, the unique requirement of having a consistent ultraviolet finite component permits a rather large ambiguity in the effective action. This allows in particular adding terms containing new parameters or even external fields not present in the original Lagrangian. For instance, one can choose to renormalize the theory in such a way that the effective action depends on m_0^2 and $\delta G(x)$ as independent variables, although the Lagrangian was only a function of $G(x)=m_0^2+\delta G(x)$, or introduce a mass scale even if the underlying theory is massless [13]. Usually, the choice of renormalization is restricted in order to preserve the symmetries in the Lagrangian. When no such choice is available the symmetry is anomalously broken by the quantization.

Next, let us analyze the dependence of the allowed ambiguities on the field $H^{\mu\nu}(x)$. The difference with the previous case of the field δG is that $H^{\mu\nu}$ is coupled to an operator containing two derivatives of $\phi(x)$. In the Feynman rules in momentum space, this translates to two momenta for each insertion of $H^{\mu\nu}$ and so, higher orders in $H^{\mu\nu}$ are no longer necessarily ultraviolet convergent [14]. It is still true, however, that taking a sufficient number of derivatives with respect to the external momenta carried by $H^{\mu\nu}$ yields a ultraviolet convergent integral, therefore, the allowed ambiguity in the effective action will contain no more than four derivatives of $H^{\mu\nu}$ (more precisely, the ambiguity will be a polynomial, at most of degree four, on the operator ∂_μ acting on $H^{\mu\nu}$).

Since covariance under general coordinate transformations is a symmetry free from anomalies it is natural to renormalize the theory imposing this symmetry. This restricts the class of effective actions so that they are scalars under such transformations, and thus the possible ambiguities within this restricted class of effective actions are also scalars. Consequently the ambiguity in the effective action will depend on $H^{\mu\nu}(x)$ only through the Riemann tensor (as follows, for instance, from using Riemann coordinates). More precisely, the allowed ambiguities will be of the form $\sqrt{-g}$ times a function of the Riemann tensor and the scalar $G(x)/\sqrt{-g}$, and their covariant derivatives. The dependence on $G/\sqrt{-g}$ has already been discussed, and is a local polynomial of dimension at most four. The dependence on the Riemann tensor is highly restricted by the fact that there can be at most four derivatives of the metric tensor [15]. For instance, for the massive scalar field of Eq. (1), and assuming that no new dimensional parameters are introduced by the renormalization, the ambiguity will contain only the terms [9] m^4, m^2R, R^2, $R^2_{\mu\nu}R^{\mu\nu}$, and the Gauss-Bonnet density (which being a topological term does not contribute to the stress-energy expectation value nor to the semiclassical field equations). More generally, if the renormalization introduces new dimensionful parameters, dimensional counting allows further terms of the form M^4, m^2M^2, and M^2R [9].

The previous considerations completely cover the case of scalar fields without self-interaction. They can be extended to the case of arbitrary renormalizable theories with scalars, spin-1/2 fermions and gauge fields, as follows.

As is well-known, the effective action $W[L]$ comes from adding all one-particle-irreducible graphs with any number of insertions of the external fields and without external legs (that is, legs of the quantum fields). The quantum fields run on the propagators corresponding to internal lines in the graph. Each of these legless Feynman graphs has a superficial degree of divergence as well as possible subdivergences. It is a standard result of renormalization theory [11] that the renormalization procedure is such that all subdivergences have already been removed due to counterterms of lower order than the graph under consideration. Thus, only the superficial or overall divergence needs to be canceled. (This holds in fact for all Feynman graphs, with or without legs.)

When the theory is perturbatively renormalizable there is only a finite number of operators in the renormalized Lagrangian, coming from operators in the bare Lagrangian plus possibly new operators introduced as counterterms. (In fact, and up to symmetries, all local polynomial operators of canonical dimension at most four will appear.) The corresponding parameters, such as masses and coupling constants, are not fixed by the renormalization procedure but in principle they can be fixed to their experimental values. Thus we will not count them for the possible ambiguity in the effective action. (Note that for case considered before of quantum fields without self-interaction these parameters need not be renormalized.)

Thus when computing the legless graphs of the effective action $W[L]$ in a renormalized theory, it is only necessary to cancel the overall divergence. Again, power-counting arguments show that for a renormalizable theory the corresponding counterterms are local polynomial operators of canonical dimension at most four, constructed with the external fields and their derivatives. This holds for all external fields except $g_{\mu\nu}(x)$ because the metric tensor couples to the kinetic energy operator [and so new insertions of $g_{\mu\nu}(x)$ do not increase the convergence]. The dependence on the metric tensor is restricted by general covariance, which requires a dependence only through the Riemann tensor and its covariant derivatives and with canonical dimension at most four.

For the effective action $\Gamma[\phi,L]$, the previous arguments hold as well. The effective action comes from graphs which may contain external legs of the quantum fields. After the theory has been renormalized only the overall divergence needs to be cured and for renormalizable theories this introduces terms which are local polynomials and of canonical dimension at most four, constructed with the quantum fields, the external fields (or the Riemann tensor) and their covariant derivatives.

Two final comments are in order. First, in self-interacting theories there can be ambiguities of purely nonperturbative origin which would not leave a trace at the perturbative level and so they are not covered by the present analysis. This is the case of the renormalon ambiguities introduced by Borel resummation of the perturbative series in nonasymptotically free theories such as $\lambda\phi^4$ or quantum electrodynamics [16]. Second, it should be noted that the stress-energy tensor defined as the variation of the effective action is the consistent one. In fermionic theories it is also useful to consider new stress-energy tensors obtained from the consistent one by addition of local polynomial terms (in order not to modify its finite, unambiguous, part) which do not necessarily come as the variation of local polynomial terms in the effective action. This allows to obtain the covariant stress-energy tensor, which is not consistent but it is covariant under local Lorentz-frame transformations [17].

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[1] W. Tichy and E.E. Flanagan, Phys. Rev. D **58**, 124007 (1998).

[2] R.M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (University of Chicago Press, Chicago, IL, 1994); Commun. Math. Phys. **54**, 1 (1977); Phys. Rev. D **17**, 1477 (1978).

[3] G.T. Horowitz, Phys. Rev. D **21**, 1445 (1980).

[4] V. Moretti, Commun. Math. Phys. **201**, 327 (1999); J. Math. Phys. **40**, 3843 (1999).

[5] An example of the independence of the ultraviolet sector on the global spacetime topology is provided by the axial anomaly at finite temperature. In the imaginary time formalism, the finite temperature is introduced through a S^1 compactification in the time direction. The axial anomaly, which comes solely from ultraviolet divergences, is known to be temperature independent. A. Gómez Nicola and R.F. Álvarez-Estrada, Int. J. Mod. Phys. A **9**, 1423 (1994).

[6] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).

[7] D.J. Toms, Phys. Rev. D **35**, 3796 (1987).

[8] K. Fujikawa, Phys. Rev. Lett. **42**, 1195 (1979); **44**, 1733 (1980).

[9] L.S. Brown and J.C. Collins, Ann. Phys. (N.Y.) **130**, 215 (1980); S.L. Adler, Rev. Mod. Phys. **54**, 729 (1982).

[10] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

[11] J. Collins, *Renormalization* (Cambridge University Press, Cambridge, Great Britain, 1984).

[12] Note that the effective action itself needs not be a polynomial or even analytical as a function of m^2, only the allowed ambiguities should be polynomial. This allows, for instance, for terms of the form $m^4\log(m^2/\mu^2)$ in the effective action, where μ is a mass scale introduced by the renormalization.

[13] There are instances in which the introduction of new parameters or fields comes in naturally due to further requirements in the theory. For example, for two-dimensional Dirac fermions in the presence of vector and axial fields, $\overline{\psi}[\gamma^\mu(i\partial_\mu+V_\mu+\gamma_5 A_\mu)-m]\psi$, the identity $\gamma_\mu\gamma_5=\epsilon_{\mu\nu}\gamma^\nu$, implies that the vector and axial fields only actually appear in the action though the two combinations V_0-A_1 and V_1-A_0. However, the axial anomaly only takes its minimal Bardeen form after adding suitable counterterms depending on V_μ and A_μ separately. Another example is provided by quark models with vector mesons when they are required to reproduce the QCD anomaly. J. Bijnens and J. Prades, Phys. Lett. B **320**, 130 (1994); E. Ruiz Arriola and L.L. Salcedo, Nucl. Phys. **A590**, 703 (1995).

[14] For a coupling of the form $L_\mu(x)\phi(x)\partial_\mu\phi(x)$, each new insertion of $L_\mu(x)$ decreases the degree of divergence and so the required counterterms are still local polynomials in $L_\mu(x)$ of canonical dimension at most four.

[15] Let us emphasize that this counting refers to the total number of derivatives. Thus for instance, R^n contains $2n$ derivatives.

[16] D. Espriu and R. Tarrach, Phys. Lett. B **383**, 482 (1996).

[17] W.A. Bardeen and B. Zumino, Nucl. Phys. **B244**, 421 (1984); L. Álvarez-Gaumé and P. Ginsparg, Ann. Phys. (N.Y.) **161**, 423 (1985); **171**, 233(E) (1986).