Interaction potential between extended bodies

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For a $1/r$ potential, the outer multipole moments of a source body are calculated analytically from its inner multipole moments. For certain geometries, this makes it possible to calculate the interaction potential between extended bodies to any required accuracy without the need for numerical integrations.

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I. INTRODUCTION

In experimental gravitation, forces and torques on test masses due to external mass distributions are frequently calculated in terms of the inner and outer multipole moments, respectively [1–3]. These moments, in turn, are usually determined by numerical integrations. The external mass distribution can often be approximated by the superposition of elementary shapes. In this Brief Report, the outer moments of an individual source body are obtained analytically from its inner moments. For certain geometries, this makes it possible to calculate the gravitational torque or force between extended bodies without the need for numerical integrations. All calculations are equally valid for other $1/r$ potentials such as the electrostatic one.

II. CONVERSION OF INNER TO OUTER MULTIPOLES

The gravitational potential between two bodies, a ''test'' body and a ''source'' body, can be written as

$$W=-4\pi G\sum_{L=0}^{\infty}\frac{1}{2L+1}\sum_{M=-L}^{L}q_{LM}Q_{LM}. \tag{1}$$

Here, the inner multipole moments q_{LM} are defined by

$$q_{LM}=\int \rho(\vec{r})r^{L}Y_{LM}^{*}(\theta,\phi)d^{3}r \tag{2}$$

and the integral extends over the volume of the test body. The outer multipole moments Q_{LM} are defined by

$$Q_{LM}=\int \rho(\vec{r})r^{-(L+1)}Y_{LM}(\theta,\phi)d^{3}r \tag{3}$$

and the integral extends over the volume of the source body.

In order for the gravitational potential to be given by Eq. (1), both inner and outer multipole moments have to be evaluated in the same reference frame. For convenience, the origin of this reference frame is often chosen to coincide with the center of mass of the test body.

Given the inner moments q_{LM} of a body, the outer moments Q_{LM} generated when this body is translated by an arbitrary vector $\vec{a}=(a,\theta_a,\phi_a)$ (see Fig. 1) can be calculated. Bearing in mind that $\vec{r}=\vec{a}+\vec{r'}=\vec{r'}-(-\vec{a})$, the expression $r^{-(L+1)}Y_{LM}(\theta,\phi)$ can be expanded as [4]

$$r^{-(L+1)}Y_{LM}(\theta,\phi)=\sum_{l,l'=0}^{\infty}\sqrt{\frac{4\pi(2l)!}{(2L)!(2l'+1)!}}\frac{r'^{l'}}{a^{l+1}}\delta_{L,l-l'}\times\sum_{m',m}C(l',m',l,m,L,M)Y_{l'm'}(\theta',\phi')\times Y_{lm}(\theta_a,\phi_a) \tag{4}$$

where C is a Clebsch-Gordan coefficient and the sum over m (m') extends over $\pm l$ ($\pm l'$). Convergence of the series requires that the translation a be larger than the scale of the object r'.

The original expansion [4], with an additional factor of $(-1)^l$, has been used to express the outer multipoles in one reference frame in terms of those in another, translated frame [5]. In this work, no need for such a factor has been found and the original work appears to be correct.

With Eqs. (4) and (3), the following expression for the outer multipole moments is obtained:

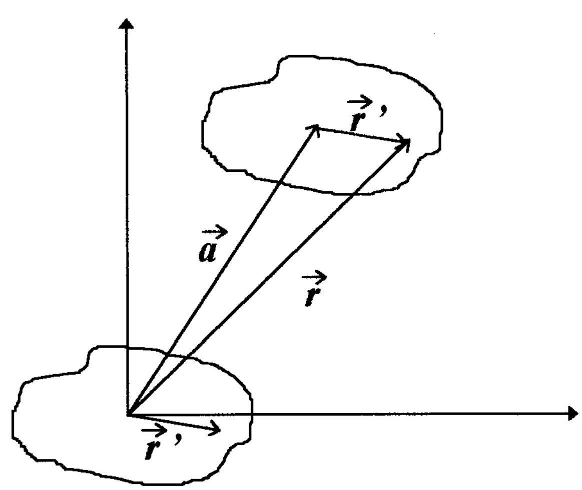

FIG. 1. Translation of a body by $\vec{a}$ to convert its inner multipoles (as calculated in the $\vec{r}\,'$ frame) into its outer multipoles (as calculated in the $\vec{r}$ frame).

$$Q_{LM}=\sum_{l,m,l',m'}\sqrt{\frac{4\pi(2l)!}{(2L)!(2l'+1)!}}\frac{Y_{lm}(\theta_a,\phi_a)}{a^{l+1}}\delta_{L,l-l'}\times(-1)^{m'}C(l',-m',l,m,L,M)q_{l'm'}. \tag{5}$$

For any given L, successive terms in Eq. (5) scale like $(r'/a)^l$. The convergence is therefore fastest if the displacement a is large compared to the typical scale of the object r'. The first term, Q_{00}, is simply the expansion of the gravitational field outside a mass distribution in terms of its inner moments, while the higher order terms correspond to the higher gravity gradients.

III. APPLICATIONS

Expressing the outer multipole fields of a body in terms of its inner moments makes it possible, for certain shapes, to calculate gravitational torques and forces to any desired accuracy. This is of particular interest if the interacting bodies cannot be treated as point masses.

For instance, consider a cylinder of uniform density, of mass M, radius R and length L. In a frame where the z axis is aligned with the symmetry axis of the cylinder and the origin coincides with its center of mass, only the inner moments with l even and $m=0$ are non-zero. They are given in closed form by [6]

$$q_{l0}=M\sqrt{\frac{2l+1}{4\pi}}\sum_{k=0}^{l/2}\frac{(-1)^k l!}{2^l k!(k+1)![l-2k+1]!}L^{l-2k}R^{2k}. \tag{6}$$

A rotation of the cylinder leads to new inner moments, which again can be determined in closed form [4]. The outer moments of this cylinder when translated by $\vec{a}$ are given by Eq. (5). If a second cylinder is now placed at the origin, its inner moments can be calculated according to Eq. (6). In other words, the gravitational potential in this system comprising two extended and uniform cylinders can, in principle, be determined to any required accuracy. In practice, the number of terms will depend strongly on how close the two cylinders are compared to their dimensions.

In a recent determination of the gravitational constant G, the source mass and the proof mass were modeled as being composed of 60 and 100 cylinders, respectively [7]. For calculation purposes, it was further assumed that each of them was uniform in density. The practical difficulties of carrying out the numerical integrations are pointed out. The geometry of the experiment is such that the expressions given above could be applied.

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[1] E.G. Adelberger *et al.*, Phys. Rev. D **42**, 3267 (1990).
[2] Y. Su *et al.*, Phys. Rev. D **50**, 3614 (1994).
[3] J.H. Gundlach *et al.*, Phys. Rev. Lett. **78**, 2523 (1997).
[4] D.A. Varshalovich *et al.*, *Quantum Theory of Angular Momentum* (World Scientific, Singapore, 1988).
[5] C. D'Urso and E.G. Adelberger, Phys. Rev. D **55**, 7970 (1997).
[6] N. Lockerbie *et al.*, Class. Quantum Grav. **11**, 1575 (1994).
[7] J.P. Schwarz, Ph.D. thesis, University of Colorado at Boulder, 1998.