Note on supersymmetric Yang-Mills thermodynamics

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The thermodynamics of supersymmetric Yang-Mills theories is studied by computing the two-loop correction to the canonical free energy and to the equation of state for theories with 16, 8, and 4 supercharges in any dimension $4\leqslant d\leqslant 10$, and in two dimensions at finite volume. In the four-dimensional case we also evaluate the first nonanalytic contribution in the 't Hooft coupling to the free energy, arising from the resummation of ring diagrams. To conclude, we discuss some applications to the study of the Hagedorn transition in string theory in the context of Matrix strings and speculate on the possible physical meaning of the transition. [S0556-2821(99)08618-X]

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I. INTRODUCTION

The study of the thermal properties of supersymmetric Yang-Mills (SYM) and superstring theories has received a boost [1–6] after the D-brane revolution [7]. More recently, the conjecture by Maldacena of a duality between string theory on anti–de Sitter (AdS) backgrounds and the large-N limit of SYM theories [8,9] has further motivated the study of these issues in the hope that it could lead to a clarification of the mechanisms of confinement in non-Abelian gauge theories [10].

Beyond its purely cosmological and/or phenomenological relevance, the study of SYM thermodynamics finds interesting applications in the study of near-extremal black holes [11] and D-branes [12,13]. When looking at the effective theory of coincident D-branes there are, generically, two regimes associated with the values of the effective gauge couplings g_sQ (Q being the number of coincident D-branes or the charges of the black hole). When $g_sQ<1$ the field theory limit is well described by a perturbative SYM theory living on the world volume of the D-brane. On the other hand, taking $g_sQ>1$ the weakly coupled D-brane picture is not appropriate any more. Using Maldacena's conjecture, however, it is possible to relate the field theory limit in this nonperturbative regime (large 't Hooft coupling) with a supergravity computation on some background of the form $\mathrm{AdS}_d\times$(spheres). Corrections to the leading strong coupling result of order $\mathcal{O}(1/g_sQ)$ are then associated with higher dimensional terms (α' corrections) in the supergravity effective action.

The study of the corrections to the leading results on both sides (weak [14] and strong [15,16] coupling) has been done for the conformal $\mathcal{N}=4$SYM theory in $d=4$ and a tendency in both curves to meet was detected (see, however, [17]). In this paper we will try to achieve a twofold objective. First, to compute the two-loop free energy of SYM theories with 16, 8, and 4 supercharges in various dimensions. The first class of theories are especially interesting because of their potential application to nonconformal versions of the Maldacena conjecture [9]. Second, to obtain the first nonanalytic correction in the 't Hooft coupling, of order $\mathcal{O}[(g_{\mathrm{YM}}^2N)^{3/2}]$, to the two-loop result in four-dimensional theories arising from the resummation of ring diagrams. We will analyze also with some detail the two-dimensional case, where the infrared divergences will be handled by putting the system at finite volume.

The physics of the high temperature string gas has been a recurrent issue in string theory (for a sample of papers from the ''golden age'' of thermal strings see [18,19]). In recent years we have gained interesting insights about the physical meaning of the Hagedorn divergence, in spite of the fact that a full and detailed understanding of the problem seems to be still at large. Although this will not be the main subject of this paper, we will try to discuss some aspects of the Hagedorn transition that could be enlightened by our results on SYM thermodynamics. In particular we will use our study of the two-dimensional SYM theories to try to get some qualitative information about the Hagedorn transition using Matrix strings as a nonperturbative definition of type-IIA superstrings. In spite of being nonconclusive, we hope the discussion will be helpful in shedding some light on such a confusing issue.

The present paper is organized as follows: in the next section, the two-loop corrections to the thermal free energy of SYM theories with 16, 8, and 4 supercharges will be computed in any dimension $4\leqslant d\leqslant 10$ using dimensional reduction from the corresponding maximal $\mathcal{N}=1$ SYM theory. In Sec. II C we compute the next correction to the two-loop free energy for SYM theories in four dimensions. Section II D will be devoted to the study of the two-dimensional case at finite volume. Finally, in Sec. III we will summarize the conclusions and discuss some possible application of our results to the study of the Hagedorn transition in Matrix string theory.

II. TWO-LOOP FREE ENERGY OF SYM THEORIES

A. Supersymmetric Yang-Mills theories in various dimensions

In this section we will compute the next-to-leading contribution to the canonical free energy of supersymmetric

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Yang-Mills theories with 16, 8, and 4 supercharges. In order to keep the analysis general, we will start with $\mathcal{N}=1$ SYM in $\mathcal{D}$ dimensions, whose dynamics is governed by the action [20]

$$S=\int d^{\mathcal{D}}x\,\mathrm{Tr}\left[-\frac{1}{4g_{\mathrm{YM}}^2}F_{AB}F^{AB}+i\overline{\psi}\Gamma^A D_A\psi\right],$$

where $A,B=0,...,\mathcal{D}-1$ and both the gauge fields and spinors are in the adjoint representation of $U(N)$. We get theories with different number of supercharges by choosing the appropriate value of $\mathcal{D}$ for which that number $\#_{\mathrm{SC}}$ is maximal:

$$\#_{\mathrm{SC}}=16\rightarrow\mathcal{D}_{\mathrm{max}}=10,$$

$$\#_{\mathrm{SC}}=8\rightarrow\mathcal{D}_{\mathrm{max}}=6,$$

$$\#_{\mathrm{SC}}=4\rightarrow\mathcal{D}_{\mathrm{max}}=4.$$

In addition, different conditions on fermions have to be imposed in order to keep the number of bosonic and fermionic degrees of freedom equal. Thus, when $\mathcal{D}_{\mathrm{max}}=10$ fermions have to be taken Majorana-Weyl, while for $\mathcal{D}_{\mathrm{max}}=6$ and $\mathcal{D}_{\mathrm{max}}=4$ they satisfy Weyl condition (actually in $d=4$ we can choose the fermions to be either Majorana or Weyl, both conditions being equivalent [20]). This ensures that the number of physical bosonic and fermionic degrees of freedom will be equal to $\mathcal{D}_{\mathrm{max}}-2$.

In general, however, we will be interested in SYM theories with $\#_{\mathrm{SC}}$ supercharges in dimensions $d\leqslant\mathcal{D}_{\mathrm{max}}$. This theories can be obtained by dimensional reduction of the corresponding maximal $\mathcal{N}=1$ SYM theory in $\mathcal{D}=\mathcal{D}_{\mathrm{max}}$ [20]. Thus, we can parametrize any d-dimensional SYM theory with any number of supercharges by specifying both d and the maximal dimension $\mathcal{D}_{\mathrm{max}}$ from which it is obtained by dimensional reduction. In this way, starting with $\mathcal{D}_{\mathrm{max}}=10$ ($\mathcal{N}=1$ in $d=10$) we get $\mathcal{N}=1$ in $d=8$, $\mathcal{N}=2$ in $d=6$, $\mathcal{N}=4$ in $d=4$, and $\mathcal{N}=8$ in $d=2$. Starting instead with $\mathcal{D}_{\mathrm{max}}=6$ ($\mathcal{N}=1$ in $d=6$) we will have $\mathcal{N}=2$ in $d=4$ and $\mathcal{N}=4$ in $d=2$. Finally, if we take $\mathcal{D}_{\mathrm{max}}=4$ we can retrieve $\mathcal{N}=2$ in $d=2$ (for odd dimensions $2n-1$ we have the $\mathcal{N}$ corresponding to dimension $2n$).

B. Two-loop free energy for $d\geqslant 4$

With this in mind, we can proceed to compute the canonical free energy in perturbation theory for any supersymmetric Yang-Mills theory characterized by ($\mathcal{D}_{\mathrm{max}},d$), by writing down the contribution of vacuum Feynman diagrams of $\mathcal{N}=1$ in $\mathcal{D}=\mathcal{D}_{\mathrm{max}}$ SYM and restricting internal momentum in loops to d dimensions. That way we are able to keep track of the contribution of gauge bosons and scalars (as well as their supersymmetric partners) without having to consider a larger number of diagrams.[1] The final result, of course, will depend on $(d,\mathcal{D}_{\mathrm{max}})$.

[1]For example, using this trick one can get the result of Ref. [14] by computing, instead of ten, only four two-loop Feynman diagrams.

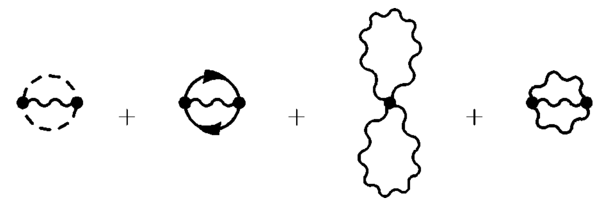

FIG. 1. Feynman diagrams contributing to the two-loop canonical free energy. Solid lines represent fermions, wavy lines gauge bosons, and dashed lines Faddeev-Popov ghosts.

As a warmup exercise, we will compute the one-loop free energy density. In the maximal $\mathcal{N}=1$ theory we have three diagrams, a bosonic loop, a fermionic loop and the ghost loop, which after multiplying by their corresponding degeneracy factor respectively give (we use the notation of Ref. [21])

$$\mathcal{F}(\beta)_{1\text{-loop}}=\tfrac{1}{2}N^2\mathcal{D}_{\mathrm{max}}\sum\!\!\!\!\!\!\int_{(P)}[d^dP]\log P^2-\tfrac{1}{2}N^2(\mathcal{D}_{\mathrm{max}}-2)\times\sum\!\!\!\!\!\!\int_{\{P\}}[d^dP]\log P^2-N^2\sum\!\!\!\!\!\!\int_{(P)}[d^dP]\log P^2,$$

where by (P) and $\{P\}$ we represent, respectively, bosonic (periodic) and fermionic (antiperiodic) boundary conditions along the Euclidean time and the factor of N^2 is due to the fact that all fields are in the adjoint representation of $U(N)$. After a straightforward computation we arrive at

$$\sum\!\!\!\!\!\!\int_{(P)}[d^dP]\log P^2\equiv\frac{1}{\beta}\sum_{(\omega_n)}\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\log(p^2+\omega_n^2)=\Lambda_0-\frac{2\Gamma(d/2)}{\pi^{d/2}}\zeta(d)\beta^{-d},$$

$$\sum\!\!\!\!\!\!\int_{\{P\}}[d^dP]\log P^2\equiv\frac{1}{\beta}\sum_{[\omega_n]}\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\log(p^2+\omega_n^2)=\Lambda_0+\frac{2\Gamma(d/2)}{\pi^{d/2}}(1-2^{1-d})\zeta(d)\beta^{-d}.$$

Λ_0 is a regularized vacuum energy that will cancel after summing all contributions. The total result for the one-loop free energy is thus

$$\mathcal{F}(\beta)_{1\text{-loop}}=-\frac{2\Gamma(d/2)}{\pi^{d/2}}\zeta(d)(1-2^{-d})(\mathcal{D}_{\mathrm{max}}-2)N^2\beta^{-d}. \tag{2.1}$$

Next we obtain the two-loop corrections with this result. Thus, we must sum the contributions of the four Feynman diagrams of Fig. 1 corresponding to $\mathcal{N}=1$ SYM in $\mathcal{D}=\mathcal{D}_{\mathrm{max}}$ where internal momenta is restricted to a d-dimensional space-time (one of whose directions is the compactified Euclidean time). Proceeding this way and after some algebra we find the contribution of each independent diagram (using the Feynman–'t Hooft gauge):

$$\mathcal{F}_1=\tfrac{1}{4}\,g_{\rm YM}^2N^3\left(\sum\!\!\!\!\!\!\int_{(P)}[d^dP]\frac{1}{P^2}\right)^2,$$

$$\mathcal{F}_2=\tfrac{1}{4}\,g_{\rm YM}^2N^3(\mathcal{D}_{\rm max}-2)\,{\rm Tr}\,\mathbf{1}\left\{\left(\sum\!\!\!\!\!\!\int_{\{P\}}[d^dP]\frac{1}{P^2}\right)^2-2\left(\sum\!\!\!\!\!\!\int_{\{P\}}[d^dP]\frac{1}{P^2}\right)\left(\sum\!\!\!\!\!\!\int_{(P)}[d^dP]\frac{1}{P^2}\right)\right\},$$

$$\mathcal{F}_3=\tfrac{1}{4}\,g_{\rm YM}^2N^3\mathcal{D}_{\rm max}(\mathcal{D}_{\rm max}-1)\left(\sum\!\!\!\!\!\!\int_{(P)}[d^dP]\frac{1}{P^2}\right)^2,$$

$$\mathcal{F}_4=-\tfrac{3}{4}\,g_{\rm YM}^2N^3(\mathcal{D}_{\rm max}-1)\left(\sum\!\!\!\!\!\!\int_{(P)}[d^dP]\frac{1}{P^2}\right)^2.$$

Here Tr $\mathbf{1}$ is the dimension of the spinors of the maximal SYM theory which in all the cases under study ($\mathcal{N}=1$ in $\mathcal{D}_{\rm max}=10,6,4$) equals $\mathcal{D}_{\rm max}-2$. Using this fact we can add all the above contributions and find the following expression for the two-loop free energy density:

$$\mathcal{F}(\beta)_{2\text{-loop}}=\tfrac{1}{4}\,g_{\rm YM}^2N^3(\mathcal{D}_{\rm max}-2)^2\times\left(\sum\!\!\!\!\!\!\int_{(P)}[d^dP]\frac{1}{P^2}-\sum\!\!\!\!\!\!\int_{\{P\}}[d^dP]\frac{1}{P^2}\right)^2. \tag{2.2}$$

The integrals appearing between brackets contain both the zero and finite temperature part of $\mathcal{F}(\beta)_{2\text{-loop}}$. They can be easily computed by performing the sum first to give

$$\sum\!\!\!\!\!\!\int_{(P)}[d^dP]\frac{1}{P^2}=\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{1}{2\omega_p}+\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{N_p}{\omega_p},$$

$$\sum\!\!\!\!\!\!\int_{\{P\}}[d^dP]\frac{1}{P^2}=\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{1}{2\omega_p}-\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{n_p}{\omega_p}, \tag{2.3}$$

where $\omega_p=p$ and N_p, n_p are the Bose-Einstein and Fermi-Dirac distribution functions, respectively,

$$N_p=\frac{1}{e^{\beta\omega_p}-1},\quad n_p=\frac{1}{e^{\beta\omega_p}+1}.$$

The first thing to be said about Eq. (2.2) is that the zero temperature (ultraviolet divergent) contribution cancels out between the bosonic and the fermionic integral. This is just a consequence of supersymmetry since the vacuum energy should not be corrected at zero temperature if supersymmetry is to be preserved by the vacuum [22]. This is the reason why this cancellation occurs not only for the conformal $\mathcal{N}=4$, SYM$_4$ ($\mathcal{D}_{\rm max}=10$) [14], but for all SYM theories under study.

Notice that although all SYM theories in dimension higher than four are nonrenormalizable, the two-loop finite temperature free energy is well behaved in the ultraviolet. This is a consequence of the fact that the ultraviolet region in the thermal integrals is effectively cutoff for momenta $p\gg T$ and therefore the temperature dependent part of the amplitudes is, to a great extent, insensitive to ultraviolet ambiguities. Of course, divergences should have been taken care of in the zero temperature sector by an appropriate cutoff in momenta Λ (although some protected observables, like the vacuum energy, will be finite due to supersymmetry). In that case, consistency will require $T<\Lambda$.

To get an analytical expression for the free energy we can evaluate the integrals appearing in Eq. (2.3) for generic values of the dimension[2]

$$\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{N_p}{\omega_p}=\frac{2^{2-d}\pi^{-(d-1)/2}}{\Gamma[(d-1)/2]}\zeta(d-2)\Gamma(d-2)\beta^{2-d},$$

$$\int\frac{d^{d-1}p}{(2\pi)^{d-1}}\frac{n_p}{\omega_p}=(1-2^{3-d})\frac{2^{2-d}\pi^{-(d-1)/2}}{\Gamma[(d-1)/2]}\times\zeta(d-2)\Gamma(d-2)\beta^{2-d}.$$

After substituting in Eqs. (2.3) and (2.2) we obtain

$$\mathcal{F}(\beta)_{2\text{-loop}}=g_{\rm YM}^2N^3\left[(\mathcal{D}_{\rm max}-2)\frac{2^{2-d}\zeta(d-2)}{\pi^{(d-1)/2}\Gamma[(d-1)/2]}\times(1-2^{2-d})\Gamma(d-2)\right]^2\beta^{4-2d},$$

which together with the one-loop contribution can be written as

$$\mathcal{F}(\beta)=-N^2\beta^{-d}\left\{\frac{2\Gamma(d/2)}{\pi^{d/2}}\zeta(d)(1-2^{-d})(\mathcal{D}_{\rm max}-2)-g_{\rm YM}^2N\left[(\mathcal{D}_{\rm max}-2)\frac{2^{2-d}\zeta(d-2)}{\pi^{(d-1)/2}\Gamma[(d-1)/2]}\times(1-2^{2-d})\Gamma(d-2)\right]^2\beta^{4-d}+\mathcal{O}[(g_{\rm YM}^2N)^2]\right\}. \tag{2.4}$$

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[2]Actually, the fermionic integral can be written in terms of the bosonic one using

$$\frac{2}{e^{2\beta\omega_p}-1}=\frac{1}{e^{\beta\omega_p}-1}-\frac{1}{e^{\beta\omega_p}+1}$$

which is just a realization of the well-known relation between the one-loop free energy of a bosonic and a fermionic quantum field, $F_{\rm fer}(\beta)=F_{\rm bos}(\beta)-2F_{\rm bos}(2\beta)$ [23].

Incidentally, the two-loop correction to the free energy is always positive for $d>1$, so it always tends to increase the (negative) one-loop contribution.

From Eq. (2.4) we see explicitly how corrections to the one-loop result come in powers of the 't Hooft coupling $g_{\rm YM}^2 N$ which is kept fixed in the large-N limit. Since in d dimensions the Yang-Mills coupling constant $g_{\rm YM}^2$ has dimension of (energy)$^{4-d}$, the condition for the perturbative expansion to be reliable is the effective dimensionless coupling at a given temperature to be small

$$g_{\rm eff}^2 = g_{\rm YM}^2 N \beta^{4-d} \ll 1.$$

Notice that for $d>4$, for which the SYM theory is nonrenormalizable, perturbative corrections to the free energy will be governed by a small parameter in the low temperature limit, corresponding to the fact that the theory is well behaved in the infrared. Actually, higher order corrections to formula (2.4) have better and better infrared behavior as we increase the order in perturbation theory. At the same time the ultraviolet divergences worsen in the zero temperature sector, while in the temperature-dependent part of the amplitude the ultraviolet behavior is smoother due to the presence of Boltzmann factors that effectively cutoff momenta beyond a scale of order[3] T.

From the two-loop canonical free energy we can obtain the corrections to the equation of state of SYM_d. We first compute the canonical entropy density $\mathcal{S}(T)$ as a function of the temperature and invert it up to terms of order $\mathcal{O}[(g_{\rm YM}N)^2]$ to obtain

$$T(\mathcal{S}) = \left(\frac{\mathcal{S}}{N^2\mathcal{F}_0 d}\right)^{1/(d-1)} + g_{\rm YM}^2 N \frac{(2d-4)\mathcal{F}_1}{d(d-1)\mathcal{F}_0} \times \left(\frac{\mathcal{S}}{N^2\mathcal{F}_0 d}\right)^{(d-3)/(d-1)} + \mathcal{O}[(g_{\rm YM}^2N)^2],$$

where the numerical coefficients $\mathcal{F}_0$, $\mathcal{F}_1$ are defined from the free energy Eq. (2.4) by $\mathcal{F} = -\mathcal{F}_0 N^2 T^d + (g_{\rm YM}^2 N) N^2 \mathcal{F}_1 T^{2d-4}$. Now we can substitute into the internal energy density $\mathcal{E} = \mathcal{F} + T\mathcal{S}$ with the result

$$\mathcal{E} = (d-1)\mathcal{F}_0 N^2 \left(\frac{\mathcal{S}}{N^2\mathcal{F}_0 d}\right)^{d/(d-1)} \left[1 + (g_{\rm YM}^2 N)\frac{\mathcal{F}_1}{(d-1)\mathcal{F}_0} \times \left(\frac{\mathcal{S}}{N^2\mathcal{F}_0 d}\right)^{(d-4)/(d-1)}\right] + \mathcal{O}[(g_{\rm YM}^2 N)^2].$$

C. The $d=4$ case

We will now concentrate our attention on the four-dimensional case. Taking $d=4$ in Eq. (2.4) we find

$$\mathcal{F}(\beta)_{d=4} = -\tfrac{1}{8} N^2 \beta^{-4} (\mathcal{D}_{\rm max}-2) \times \left[\frac{\pi^2}{6} - \frac{1}{32}(\mathcal{D}_{\rm max}-2) g_{\rm YM}^2 N\right]. \qquad (2.5)$$

As a check, we can evaluate Eq. (2.4) for the superconformal $\mathcal{N}=4$ SYM_4 theory, which is obtained by dimensional reduction of $\mathcal{N}=1$ in SYM_{10} (i.e., $\mathcal{D}_{\rm max}=10$) to give

$$\mathcal{F}(\beta)_{\mathcal{N}=4} = -N^2\beta^{-4}\left(\frac{\pi^2}{6} - \frac{1}{4} g_{\rm YM}^2 N\right),$$

which indeed agrees with the result of Ref. [14]. In the equation of state, the loop correction just results in a renormalization of the overall numerical factor (this also happens in the nonconformal cases $\mathcal{D}_{\rm max}=6,4$)

$$\mathcal{E}_{\mathcal{N}=4} = \frac{\pi^2}{2}\left(\frac{3}{2\pi^2}\right)^{4/3}\left(1 + \frac{g_{\rm YM}^2 N}{2\pi^2}\right)\mathcal{S}^{4/3} N^{-(2/3)}.$$

Generically, the next contribution to Eq. (2.4) is naively given by three-loop diagrams of order $\mathcal{O}[(g_{\rm YM}^2 N)^2]$. However in four dimensions, as it happens in QCD [24,21], at the three-loop level there are already uncanceled infrared divergences that have to be cured by summing over ring diagrams. This gives a nonanalytic (of order $\mathcal{O}[(g_{\rm YM}^2 N)^{3/2}]$) contribution to the free energy, representing a mild failure of perturbation theory due to the infrared ambiguities. The evaluation of this term is essentially equivalent to dressing the A_0^a and scalar propagators in loops by introducing the effect of Debye screening and thermal mass for the scalars. To leading order in the 't Hooft coupling, the electric (Debye) mass can be easily computed from the static limit of the one-loop self-energy to give

$$m_{\rm el}^2 \equiv \lim_{\vec{p}\to 0} \Pi_{00}^{aa}(0,\vec{p}) = \tfrac{1}{4}(\mathcal{D}_{\rm max}-2) g_{\rm YM}^2 N T^2 + \mathcal{O}[(g_{\rm YM}^2 N)^2], \qquad (2.6)$$

while for the scalars we have

$$m_\phi^2 \equiv \lim_{\vec{p}\to 0} \Pi^{aa}(0,\vec{p}) = \tfrac{1}{8}(\mathcal{D}_{\rm max}-2) g_{\rm YM}^2 N T^2 + \mathcal{O}[(g_{\rm YM}^2 N)^2], \qquad (2.7)$$

In order to obtain the $\mathcal{O}[(g_{\rm YM}^2 N)^{3/2}]$ terms in the free energy we use the technique of Ref. [21], and rewrite the original Lagrangian density as

$$\mathcal{L}_{\mathrm{SYM}_4} = \left(\mathcal{L}_{\mathrm{SYM}_4} + \frac{1}{2} m_{\rm el}^2 \,\mathrm{Tr}\, A_0^2 \delta_{p_0,0} + \frac{1}{2} m_\phi^2 \sum_{i=1}^{n_s} \mathrm{Tr}\,\phi_i^2\right) - \frac{1}{2} m_{\rm el}^2 \,\mathrm{Tr}\, A_0^2 \delta_{p_0,0} - \frac{1}{2} m_\phi^2 \sum_{i=1}^{n_s} \mathrm{Tr}\,\phi_i^2, \qquad (2.8)$$

[3] The better ultraviolet behavior of the temperature-dependent sector of the theory does not guarantee its finiteness in higher loops; for example, the free energy of SYM_{10} is ultraviolet divergent at three loops as can be seen by thermal averaging the one-loop effective action, which contains a F^4 term that scales quadratically with the ultraviolet cutoff. The resulting thermal averaged divergent part is of order $\mathcal{O}[(g_{\rm YM}^2 N)^2]$ [16], as corresponds to a three-loop contribution. I thank A. Tseytlin for pointing this out to me.

where ϕ_i are the $n_s = \mathcal{D}_{\max} - 4$ adjoint scalars in the theory and the electric mass only affects to the zero-frequency component of the A_0^a field (cf. [24]). The strategy now is to treat the last two terms as a perturbation to the Lagrangian density between brackets. This results in a reorganization of perturbation theory in which the ring-diagram contribution can be easily evaluated.

The first thing will be to compute again the one-loop free energy density, including now the effect of the masses in the Lagrangian equation (2.8) and, at the same time, adding new one-loop diagrams containing vertices associated with the counterterms. Expanding the results up to order $\mathcal{O}[(g_{\rm YM}^2 N)^2]$ we find[4]

$$\mathcal{F}(\beta)_{\text{1-loop}}^{\text{resum}} = \mathcal{F}(\beta)_{\text{1-loop}} + \frac{1}{24\pi} N^2 T [m_{\rm el}^3 + (\mathcal{D}_{\max} - 4) m_\phi^3] + \mathcal{O}[(g_{\rm YM}^2 N)^2] \tag{2.9}$$

with $\mathcal{F}(\beta)_{\text{1-loop}}$ given by Eq. (2.1). Proceeding similarly with the two-loops diagrams of Fig. 1, we obtain

$$\mathcal{F}(\beta)_{\text{2-loop}}^{\text{resum}} = \mathcal{F}(\beta)_{\text{2-loop}} - \frac{1}{8\pi} N^2 T [m_{\rm el}^3 + (\mathcal{D}_{\max} - 4) m_\phi^3] + \mathcal{O}[(g_{\rm YM}^2 N)^2].$$

So we are left with the following final result for the ''$2\frac{1}{2}$-loop'' contribution to the free energy density

$$\mathcal{F}(\beta)_{2(1/2)\text{-loop}} = -\frac{1}{12\pi} N^2 T [m_{\rm el}^3 + (\mathcal{D}_{\max} - 4) m_\phi^3] \tag{2.10}$$

where the values of the thermal masses are given by Eqs. (2.6) and (2.7). It is important to notice that this term is always negative for all $4 \leqslant \mathcal{D}_{\max} \leqslant 10$.

The only thing left now will be to add Eq. (2.10) to the two-loop result (2.5). In particular, doing so for the superconformal $\mathcal{N}=4$, SYM_4 and evaluating the numerical coefficients, we find ($\lambda^2 \equiv g_{\rm YM}^2 N$)

$$\mathcal{F}(\beta)_{\mathcal{N}=4} = -N^2 T^4 [1.645 - 0.250\lambda^2 + 0.234\lambda^3 + \mathcal{O}(\lambda^4)].$$

Next terms in the perturbative expansion in four dimensions will be of order $\mathcal{O}(\lambda^4)$ and $\mathcal{O}(\lambda^4 \log \lambda)$ and can be also evaluated using the strategy employed in [21] or up to order $\mathcal{O}(\lambda^5)$ using [25,26]. However, finite temperature perturbation theory is expected to break down at order $\mathcal{O}[(g_{\rm YM}^2 N)^3]$ [27]. Since this failure of perturbation theory is associated with the infrared sector of the theory, supersymmetry is not expected to solve the problem or even improve the situation. As in QCD [26,28] some kind of nonperturbative analysis will be needed in order to compute higher orders. Actually, the general structure of the series in λ is important in trying to decide whether there is a phase transition occurring at some intermediate value of the 't Hooft coupling that precludes the extrapolation of supergravity physics into the gauge theory domain [17].

D. SYM_2 thermodynamics on $\mathbf{S}_L^1 \times \mathbf{R}$

When $d \leqslant 3$ the analysis of the quantum corrections to the one-loop free energy has additional complications due to the hard infrared divergences that afflict super-renormalizable theories. For $d=3$, we see that expression (2.4) diverges because of a $\zeta(1)$ factor. In principle this can be cured, as usual, by computing the thermal masses and inserting them into the propagators, thus regularizing the low-momentum behavior of the Feynman integrals. However, in the three-dimensional case the computation of the electric mass has to be done with extra care, since the one-loop corrections to the propagators are already infrared divergent. Thus, the electric mass has to be evaluated self-consistently *à la* Hartree-Fock [29]. Anyway, we will not dwell in this case any further.

The two-dimensional case, on the other hand, is more interesting from several points of view. The one that will concern us here is that $\mathcal{N}=8$, SYM_2 describes the world-volume dynamics of Matrix strings [30], a nonperturbative definition of type-IIA superstrings. Naively, Eq. (2.4) is ill defined for $d=2$ due to the endemic infrared divergences of low-dimensional field theories. There are several ways in which this divergence can be regularized. Here we will get rid of the problem by putting the system in finite box[5] of length $L = 2\pi R$. We will assume that the thermal wavelengths of the fundamental fields are much smaller than the global length of the box $\beta \ll L$ and restrict our analysis to the sector without Wilson lines (the ''long strings'' that characterize the Matrix string phase). Once this is done, the only change in the computation of Feynman diagrams is that continuous space momentum is discretized in units of $1/R$ and the momentum integrals have to be replaced by discrete sums

$$\sum\!\!\!\!\!\!\int [d^2 P] \rightarrow \frac{1}{L} \sum_{n \in \mathbf{Z}} \frac{1}{\beta} \sum_{\omega_m} \tag{2.11}$$

where the second sum is, as usual, over integer or half-integer Matsubara frequencies depending on the bosonic or fermionic character of the propagating field.

In the one-loop approximation the relevant bosonic and fermionic determinants have been already computed in [31] and the resulting one-loop free energy density can be cast in terms of modular functions

[4]The trick of dimensional reduction is no longer useful here because the thermal mass distinguishes between scalar and gauge boson propagators. Thus we have to compute all diagrams separately.

[5]Actually, if we take the $d \rightarrow 2$ limit in expression (2.4) we obtain a finite result with a two-loop correction independent of the temperature. However, since dimensional regularization is not reliable in dealing with infrared divergences we will not follow this procedure.

$$\mathcal{F}(\beta,L)_{\text{1-loop}}=-\frac{1}{L\beta}N^2(\mathcal{D}_{\max}-2)\log\left[\frac{\theta_4[0|i(L/\beta)]}{\eta^3[i(L/\beta)]}\right]$$

$$\underset{\beta\ll L}{\sim}-\frac{\pi}{4}(\mathcal{D}_{\max}-2)N^2\beta^{-2}.$$

In computing the bosonic determinant, and in order to keep the argument of the logarithm dimensionless, we have added a β-independent counterterm. In the infinite volume limit $L\rightarrow\infty$ we recover the one-loop result obtained in Sec. II B.

Let us now go to the two-loop case. To compute the contribution to the free energy density we can use formula (2.2) provided we substitute the integration by the sum according to Eq. (2.11). After doing so, we find

$$\left(\sum\!\!\!\!\!\!\int_{(P)}[d^2P]\frac{1}{P^2}-\sum\!\!\!\!\!\!\int_{\{P\}}[d^2P]\frac{1}{P^2}\right)$$
$$\rightarrow\frac{1}{L\beta}\sum_{m,n}{}'(-1)^n\left[\frac{4\pi^2m^2}{L^2}+\frac{4\pi^2n^2}{(2\beta)^2}\right]^{-1}$$
$$=\frac{1}{L\beta}Z\begin{bmatrix}0&0\\0&\frac{1}{2}\end{bmatrix}(2)_{\Phi},\tag{2.12}$$

where we have made use of the Epstein zeta function [32] and $\Phi(m,n)$ represents the quadratic form between the square brackets. It is interesting to notice here that the resulting *regular* zeta function arises as the difference of two *singular* zeta functions with $\zeta(1)$-like divergences which cancel out. This is again due to the nonrenormalization of the vacuum energy for supersymmetric theories.

Actually, the zeta function in Eq. (2.12) can be written itself in terms of ordinary modular functions (see the first article in [32]), so at the end we can write

$$\mathcal{F}(\beta,L)_{\text{2-loops}}=\frac{1}{4\pi^2}g_{\text{YM}}^2N^3(\mathcal{D}_{\max}-2)^2\log^2\left[\frac{\theta_2(0|iL/2\beta)}{\eta(iL/2\beta)}\right]$$
$$\underset{\beta\ll L}{\sim}\frac{N^2}{576}g_{\text{YM}}^2N(\mathcal{D}_{\max}-2)^2\frac{L^2}{\beta^2}\tag{2.13}$$

in such a way that one arrives at the following expression for the $(1+2)$-loop free energy density

$$\mathcal{F}(\beta,L)=-\frac{1}{L\beta}N^2\left\{(\mathcal{D}_{\max}-2)\log\left[\frac{\theta_4(0|iL/\beta)}{\eta^3(iL/\beta)}\right]-\frac{1}{4\pi^2}(\mathcal{D}_{\max}-2)^2(g_{\text{YM}}^2N)(L\beta)\log^2\left[\frac{\theta_2(0|iL/2\beta)}{\eta(iL/2\beta)}\right]+\mathcal{O}[(g_{\text{YM}}^2N)^2]\right\}$$

$$\underset{\beta\ll L}{\sim}-\frac{\pi}{4}(\mathcal{D}_{\max}-2)N^2\beta^{-2}\left[1-\frac{1}{144\pi}(\mathcal{D}_{\max}-2)g_{\text{YM}}^2NL^2\right].$$

According to this expression, the natural effective dimensionless coupling in the large L limit is now $g_{\text{eff}}^2=(g_{\text{YM}}^2N)L^2$. The analysis with be reliable when $1\gg(g_{\text{YM}}^2N)L^2\gg(g_{\text{YM}}^2N)\beta^2$. Again, the equation of state in the two-dimensional case can be computed when $\beta\ll L$, with the result

$$\mathcal{E}=\frac{\mathcal{S}^2}{\pi N^2(\mathcal{D}_{\max}-2)}\left[1+\frac{(\mathcal{D}_{\max}-2)}{144\pi}g_{\text{YM}}^2NL^2\right].\tag{2.14}$$

III. CONCLUSIONS AND OUTLOOK: HAGEDORN TRANSITION FROM SYM THERMODYNAMICS?

In the present paper, the thermodynamics of supersymmetric Yang-Mills theories with 16, 8, and 4 supercharges was studied in any dimension $d\geqslant 4$. We computed the two-loop correction to the free energy for these theories and found that it always has opposite sign to the leading (negative) one-loop result. In the four-dimensional case we also evaluated the correction to the free energy arising from the resummation of the ring diagrams, using the technique of Ref. [21], and found it to be negative. For lower dimensional ($d\leqslant 3$) SYM theories, the computation is plagued with infrared divergences that have to be regularized somehow. We studied in detail the two-dimensional case at finite volume (to regularize these infrared divergences) in the high temperature limit. Again we found a positive two-loop correction which scales as T^2 with an effective dimensionless coupling given by $g_{\text{YM}}^2NL^2$.

Before closing, let us make some remarks on the potential use of SYM thermodynamics in clarifying the issue of the Hagedorn transition. On general grounds, one can expect two possible resolutions to the Hagedorn problem: either nonperturbative effects drive the critical temperature to a maximum reachable temperature for the system or new fundamental degrees of freedom appear at high energies, thus providing a picture for a phase transition (or a smooth crossover, depending on the details of the dynamics). Although at present there are no clear evidences as to which one of the two alternatives is physically realized in string and/or *M*-theory, some results [3,5] and our still incomplete knowledge of the theory seem to hint in the direction of the second one.

D-instanton corrections to the thermodynamical potentials have been studied in [2] with the result that they do not modify the critical behavior at the Hagedorn temperature. More recently, the authors of Refs. [5] have included nonperturbative semiclassical ingredients in the analysis of the physics of the Hagedorn transition at finite volume through the Horowitz-Polchinski correspondence principle [33], getting a picture in which the Hagedorn phase is bounded at high energies by a black hole phase. A similar situation occurs for a string gas on AdS backgrounds where, in the canonical ensemble, the Hagedorn transition is ''screened'' by the formation of an AdS black hole [4].

A second approach to the problem would start with a nonperturbative formulation of string theory in terms of *M*-theoretic degrees of freedom, as it has been proposed in [3]. Let us momentarily adhere ourselves to this latter path and, starting with the nonperturbative definition of the type-IIA superstring provided by Matrix strings [30], study the world-volume thermodynamics of type-IIA strings in the microcanonical ensemble. The world-volume theory is governed by $\mathcal{N}=8$ SYM_2 with the Yang-Mills coupling constant given by $g_{\mathrm{YM}}^2=1/(g_s^2\alpha')$, with g_s the string coupling constant. On the other hand, free field configurations are determined by the overall scale α'. In the infrared, $E\ll g_{\mathrm{YM}}$, the physics is dominated by ''long string'' excitations along the flat directions. It is in this regime in which Matrix strings reproduce, in the large-*N* limit, the multistring type-IIA ensemble [30,6]. If the energy is increased, the system will begin to be excited along nonflat directions as well. At energies $E\gg g_{\mathrm{YM}}$ the potential terms in the $\mathcal{N}=8$ SYM_2 theory will behave as a small perturbation and the system will enter a perturbative regime. Thermodynamics there is well defined, as we have seen from the previous analysis.

It is tempting to try to make some connection between these two world-sheet regimes and the low and high energy regimes in the target space string theory. At low energies we have perturbative type-IIA superstring theory that, in ten open space-time dimensions, we know is characterized at high energies by a negative specific heat phase. This negative specific heat phase is viewed as a breakdown of equipartition in energy, in the sense that most of the energy of the string ensemble is stored into one (or a small number) of highly excited strings [18,19]. From the philosophy of *M*-theory it seems quite reasonable to expect that if too much energy is stored into a single string some transition to nonperturbative (maybe eleven-dimensional) physics should take place, putting an end to the negative specific phase. Alternatively, a black hole could be formed before the system leaves the string regime [5]. In any case, the final conclusion would be that the Hagedorn phase will be bounded by a new phase into which the system will decay either via a smooth crossover or a phase transition.

In the case at hand, however, it is not clear how to connect the world-volume theory with some kind of target picture. One of the difficulties lies in the fact that Matrix strings are formulated in the light-cone gauge, in which the space-time interpretation is rather obscure. Nonetheless, one can naively argue that the negative specific heat phase at intermediate energies $\alpha'^{-1/2}<E\ll g_s^{-1}\alpha'^{-1/2}$ is bounded at high energy $E\gg g_s^{-1}\alpha'^{-1/2}$ by a new phase with regular thermodynamics (i.e., positive specific heat) effectively described by a perturbative two-dimensional $U(N)$ supersymetric Yang-Mills theory with sixteen supercharges in the large-*N* limit. If this were so, the transition between the low energy string phase and the new high energy phase would be through a first-order phase transition across the unstable (negative specific heat) phase (cf. Carlitz in [18]). The critical points would be determined by the Maxwell rule for the entropy, provided the complete profile of the microcanonical temperature $T(E)$ is known.

The space-time interpretation of such a phase is far from being straightforward. In the SYM_2 perturbative regime [or directly in the free limit $g_{\mathrm{YM}}^2\sim(g_s^2\alpha')^{-1}\rightarrow 0$] the two-dimensional action is that of sigma model in a ''noncommutative'' target space with matrix coordinates $X^\mu\in\mathrm{Adj}[U(N)]$. Whether this indicates that the Hagedorn transition corresponds[6] to the nucleation of noncommutative bubbles in a commutative space-time is something that it is difficult to decide with our present knowledge of the theory. One of the problems to be clarified will be, for example, how the target space volume dependence of the extensive quantities emerges as a function of *N*. In any case, we stress that this extrapolation of the world-sheet picture to space-time physics is very speculative, and should be tested by a detailed computation. We hope to report on this elsewhere.

In a sense, this picture can be regarded as dual to the one proposed in [3]. There, the Hagedorn transition is linked to the condensation of D0-branes and their low-energy dynamics will be $U(N)$ super quantum mechanics with sixteen supercharges. Both descriptions could in principle be related by performing a *T*-duality along the ninth dimension and interchanging its role with the *M*-theory circle.

Note added. After this paper appeared in the LANL hep-th archive, I learned directly from S.-J. Rey of his parallel and independent work on SYM thermodynamics, part of which overlaps with the results presented here and that has later appeared in [34]. I would like also to thank A. Nieto and A. Tseytlin for their interesting remarks on the first version of the article.

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[6]At least in those cases in which it is not preceded by the formation of black holes due to the corresponding principle. Actually, we can tune the string coupling constant g_s, the volume and the total energy in such a way that the system avoids the correspondence line and thus we prevent the formation of black holes.

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