

Path integral evaluation of D-brane amplitudes

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We extend Polchinski's evaluation of the measure for the one-loop closed string path integral to open string tree amplitudes with boundaries and crosscaps embedded in D-branes. We explain how the non-Abelian limit of near-coincident D-branes emerges in the path integral formalism. We give a careful path integral derivation of the cylinder amplitude including the modulus dependence of the volume of the conformal Killing group. [S0556-2821(99)01322-3]

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In this paper note we clarify some important issues stemming from the moduli dependence of the measure in the open string path integral. We will be primarily interested in open string tree amplitudes with multiple boundaries and crosscaps on D-branes. It will suffice for the discussion in this paper to consider the simplest configuration of $N+1$, spatially separated, parallel, and static, Dp -branes, arrayed in a spatial direction X^{25} transverse to the brane volume. Upon setting the branes in relative motion in an orthogonal spatial direction, X^{24} , also transverse to the brane volume, the spacetime infrared limit of the amplitude has an interpretation as the exchange interaction of $N+1$ Dp -branes [2]. In supersymmetric theory, its low energy effective description is given by supergravity.

The spacetime ultraviolet limit of D-brane amplitudes remains a puzzle in many respects. The low energy effective theory in this limit is known to be a $U(1)^{N+1}$ Abelian gauge theory, with Goldstone bosons arising from the breaking of X^{25} translational invariance. These appear at the first massive level in the open string spectrum. In the limit of N near-coincident D-branes, a non-Abelian structure is found to emerge [1,3–5], and the massless open string multiplet is enhanced from $N+1$ singlets, each transforming as a Lorentz vector on the $(p+1)$ -dimensional Minkowskian brane volume, to a singlet plus the N^2 -dimensional adjoint multiplet of $U(N)$. The additional massless gauge bosons arise from an enhanced background of zero length Dirichlet open strings stretched between pairs of *distinct* near-coincident D-branes. The open strings have definite orientation, giving a total of $N(N-1)$ distinct states. The non-Abelian states play an important role in studies of weak/strong coupling duality in string theory. It would be helpful to have a description of their dynamics *directly* in the world-sheet formalism, beyond that given by the low energy effective theory. In what follows, we will explain briefly how the non-Abelian limit of near-coincident D-branes emerges in the world-sheet formalism. We will also clarify issues relating to the moduli dependence of the measure in the path integral. A more detailed discussion of the physics of the ultraviolet limit of open string theory is reserved for future work [8].

We begin by clarifying the measure in the open string path integral. The result follows rather simply from Polchinski's evaluation of the one-loop closed string path integral, a sum over surfaces with the topology of a torus [11]. For surfaces with negative Euler character, with no conformal

Killing vectors, it is convenient to use uniformization theory [7] to choose a constant curvature gauge slice in the space of world-sheet metrics, and a parametrization of the moduli space of metrics that leaves the measure for moduli in its simplest form [6]. This procedure will be explained in detail for open string tree amplitudes in an accompanying paper [8]. For surfaces with the topology of a cylinder, the conformal Killing volume is a matter of concern. We follow the procedure adopted in [11] and retain the moduli dependence in the fiducial world-sheet metric, as opposed to the world-sheet coordinate intervals. A careful derivation of the moduli dependence of the conformal Killing volume then follows for surfaces of cylindrical topology. This procedure correctly recovers both the annulus amplitude with Neumann boundaries, and the exchange amplitude for a pair of static D-branes. The extension to the supersymmetric exchange amplitude will be reported upon elsewhere [9].

Let us begin by expressing the exchange amplitude between N static bosonic D-branes as a conformally invariant path integral over world sheets. Surfaces $\mathcal{M}_h^{(b,c)}$ of type (b,c) with b boundaries, c crosscaps, and h handles, have Euler characteristic $\chi = 2 - 2h - (b+c)$, and are weighted by a factor $g_{open}^{2\chi}$ in open string perturbation theory. To keep things simple, we will for the most part focus on the sum over orientable world sheets with boundaries, but no handles and/or crosscaps. In this case, the number of D-branes, $N = b+c$, equals the number of boundaries, b . We follow the method of [10] and [11], extending to string world sheets with boundaries embedded in D-branes. The separation of parallel Dp -branes in a single spatial dimension orthogonal to their world volume breaks the $SO(1,d)$ Lorentz invariance of the Minkowskian spacetime background to

$$SO(1,p) \times SO(0,d-p).$$

Let \mathbf{y}_I be a $(d-p)$ -dimensional vector giving the location of the I th D-brane. We have the boundary conditions:

$$X_{(I)}^m = y_I^m, \quad I = 1, \dots, b; \quad m = p+1, \dots, d \quad (1)$$

$$n_{(I)}^a \partial_a X_{(I)}^\mu = 0, \quad \mu = 0, \dots, p,$$

where \hat{n}_I is an inward pointing normal to the I th boundary circle, C_I , $I = 1, \dots, b$. The relative locations of the D-branes are assumed to be fixed. In the T-dual picture, the

\mathbf{y}_I correspond to Wilson lines wrapped around the compact dual coordinates, $\tilde{\mathbf{X}}$, of the critical bosonic open string theory compactified on a $(d-p)$ -dimensional torus [2].

The exchange amplitude can be expressed as a Polyakov path integral [10] over orientable Riemann surfaces, \mathcal{M} , with boundaries, C_I , $I=1, \dots, b$, embedded in parallel Dp -branes:

$$\mathcal{A} = \int \frac{[d\delta X][d\delta g]}{\text{Vol}[\text{Diff} \times \text{Weyl}]} \exp \left(- \left[\frac{1}{4\pi\alpha'} \int_{\mathcal{M}} d^2\xi \sqrt{g} g^{ab} \partial_a X \partial_b X + S_{\text{ren}}[g; \nu_0, \mu_0, \lambda_0^{(I)}] \right] \right), \quad (2)$$

on which we must impose the boundary conditions in Eq. (1). The bare action includes all local, renormalizable, terms necessary to ensure exact conformal invariance of the path integral. Thus, S_{ren} includes bulk, and boundary, cosmological constants, plus a term proportional to the Euler characteristic of the surface:

$$S_{\text{ren}} = \frac{\nu_0}{2\pi} \left[\frac{1}{2} \int_{\mathcal{M}} d^2\xi \sqrt{g} R_g + \sum_{I=1}^b \oint_{C_I} ds \kappa \right] + \mu_0 \int_{\mathcal{M}} d^2\xi \sqrt{g} + \sum_{I=1}^b \lambda_0^{(I)} \oint_{C_I} ds. \quad (3)$$

The renormalization constants, $[\nu_0, \mu_0; \lambda_0^{(I)}]$, will be cancelled in the critical spacetime dimension by divergent counterterms originating in the Weyl anomaly of the measure. The factor in square brackets in Eq. (3) is a topological invariant equal to the Euler characteristic, $2\pi\chi$, from the Gauss-Bonnet theorem, and $\hat{\kappa}$ is the geodesic curvature on the boundary. The gauge fixed path integral in Eq. (2) is required to be exactly invariant under conformal reparametrizations of the surface including boundaries. We will use this as a guiding principle in determining the measure for the path integral.

Let C_I denote the I th boundary circle with parametrization unspecified; we refer to this as a *physical* boundary circle. Recall that an arbitrary boundary metric, also known as an einbein, $e(\lambda)$, can be brought to a constant by a reparametrization, $\lambda \rightarrow f(\lambda)$, where λ is the circle variable, $0 \leq \lambda \leq 1$, and the only coordinate invariant property of the circle, C_I , is its physical length, $l_I = \int_0^1 d\lambda e(\lambda)$. Let $\hat{g}_{ab}(\xi; \tau)$ denote a world-sheet metric with fixed conformal class characterized by moduli, τ , with corresponding einbein, $\hat{e}_I(\lambda; l_I) = \sqrt{\hat{g}}(\xi; \tau)|_{C_I}$, on the I th boundary circle. Physics requires us to sum over all world sheets linking the physical boundary circles in the path integral. We will make this gauge fixing procedure on world-sheet metrics explicit below.

We begin with the integration over coordinate embeddings, $X^M(\xi)$, with a fixed fiducial metric, $\hat{g}_{ab}(\xi; \tau_i)$, on the world sheet, and fixed einbeins on the boundary circles, $\hat{e}_I = \sqrt{\hat{g}}|_{C_I}$, for all $I=1, \dots, b$. The metric in the tangent space

for small variations, $(\delta X^m, \delta X^\mu)$, $m=p+1, \dots, d$, $\mu=0, \dots, p$, is required to be reparametrization invariant. Normalizing the Gaussian integral over infinitesimal variations to unity, we can factor out and perform the integration over constant modes $\delta X_0^\mu = \delta X^\mu - \delta X'^\mu$, $\mu=0, \dots, p$, where the primes denote nonconstant modes [11]. Since we fix the location of the branes in the Dirichlet directions, there are no constant modes to be integrated over. We regulate the infrared divergence coming from the integration over $p+1$ noncompact directions parallel to the brane volume by putting the system in a box of volume V_{p+1} :

$$\prod_{\mu=0}^p \int [d\delta \mathbf{X}_0] = V_{p+1}. \quad (4)$$

For the Dirichlet directions, we change basis to the $b-1$ distances, $|\mathbf{y}_I - \mathbf{y}_J|$, $I \neq J$, and the center of mass, $\mathbf{y}_{\text{c.m.}}$, each a $(d-p)$ -dimensional vector. For spatially separated branes, we must include the contribution to the classical action, $T_{(IJ)}$, for every $I \neq J$, from open strings stretched between the (IJ) th pair of branes. Thus, the static exchange amplitude only depends upon the distances between branes. We can integrate over the center of mass of the brane configuration to obtain the volume of the orthogonal $(d-p)$ -dimensional Dirichlet space, V_{d-p} .

Performing the Gaussian integrations over zero modes in the Neumann directions gives the normalization of the measure for nonconstant modes:

$$\prod_{\mu=0}^p \int [d\delta X'^\mu] e^{-(1/4\pi\alpha')|\delta X'|^2} = \left[(4\pi^2\alpha')^{-1} \int d^2\xi \sqrt{\hat{g}} \right]^{(p+1)/2}, \quad (5)$$

with the analogous integration in the Dirichlet directions normalized to unity [11]. We can expand the infinitesimal nonconstant variations, $\delta X'^M$, in a complete set of harmonic functions satisfying the appropriate boundary condition on the C_I . Then the Gaussian integrations over coordinate embeddings can be straightforwardly performed with the result

$$V_{d+1} (4\pi^2\alpha')^{-(p+1)/2} \exp \left(- \sum_{I \neq J} T_{IJ} \right) \left(\int d^2\xi \sqrt{\hat{g}} \right)^{(p+1)/2} \times (\text{Det}' \Delta_0)^{-(d-p)/2} (\text{Ndet}' \Delta_0)^{-(p+1)/2}, \quad (6)$$

where Det and Ndet denote the functional determinants evaluated, respectively, with Dirichlet and Neumann boundary condition on the C_I . On the cylinder these functional determinants are identical.

The integration over world-sheet metrics is treated as in Polchinski's evaluation of the measure for the one-loop path integral on the torus [11], except that we specialize to a fiducial metric with constant curvature. The number of moduli, n_m , and conformal Killing vectors, n_c , are as given by the Riemann-Roch theorem: $3\chi = n_c - n_m$. A specific choice of basis for quadratic differentials on the Riemann surface corresponds to a specific parametrization, $[\tau_i]$, of

conformally inequivalent metrics $\hat{g}(\xi; \tau_i)$. The real parameters τ_i , $i=1, \dots, n_m$, are the moduli of the Riemann surface, b of which can be related to the lengths, l_I , $I=1, \dots, b$ of the boundaries as measured by the fiducial boundary metric, $\sqrt{\hat{g}}(\xi; \tau_i)|_{C_I} = \hat{e}_I(\lambda; l_I)$. The remaining $2(b-3)$ moduli have a simple interpretation in terms of lengths and angles parametrizing the shape of internal geodesics on the constant curvature surface. The nonorientable surface, $\mathcal{M}_{(b,c)}$, can be obtained from the orientable surface, $\mathcal{M}_{(b+c,0)}$, with $b+c$ boundaries, by “plugging” c holes with crosscaps. The counting of moduli is straightforward in this gluing. Since the boundary of the crosscap is identified with the boundary of the hole up to a relative twist, we lose the free length parameter for the geodesic boundary of the hole, gaining an angle from the relative twist of crosscap to hole. Thus, for surfaces of negative Euler character with both boundaries and crosscaps, $n_m = b + c + 2(b + c - 3)$. Making the appropriate extensions to the analysis of [11] we obtain the result:

$$\begin{aligned} \int \frac{[d\delta g]}{\text{Vol}[\text{Diff} \times \text{Weyl}]} \rightarrow (2\pi)^{(n_c - n_m)/2} \prod_{i=1}^{n_m} \int d\tau_i (\text{Det}' \Delta_1)^{1/2} \\ \times \left(\int d^2 \xi \sqrt{\hat{g}} \right)^{n_m/2} (\det Q_{ab})^{-1/2} \\ \times [\det((\zeta_k)_{ef}(\zeta_l)^{ef})]^{1/2}. \end{aligned} \quad (7)$$

The notation for the various factors in the Jacobian is as follows. Δ_1 is the Laplacian acting on vector fields on the Riemann surface, related to the scalar Laplacian by the identity: $(\Delta_1)_c^d = -\delta_c^d \Delta_0 - \nabla^d \nabla_c + \nabla_c \nabla^d$. $(\det Q_{ab})^{-1/2}$ is the contribution to the Jacobian from the constant modes of the vector Laplacian. The matrices, ζ_i , $i=1, \dots, n_m$ are determined by the moduli dependence of the fiducial metric, $(\zeta_i)_{ab} = \hat{g}_{ab,i} - \frac{1}{2} \hat{g}_{ab} \hat{g}^{cd} \hat{g}_{cd,i}$. Thus, given an explicit form for the constant curvature metric with dependence on n_m real moduli parameters manifest, the measure for the path integral is completely determined. In the critical spacetime dimension, the integration over the volume of the group of reparametrizations continuously connected to the identity, $\text{Diff}_0 \times \text{Weyl}$, can be straightforwardly performed leaving the result on the left-hand side of Eq. (8). For surfaces with $(b+c) \geq 3$, the contribution from the conformal Killing vectors is to be dropped from this expression since $n_c = 0$.

Thus, the expression for the gauge fixed path integral takes the form

$$\begin{aligned} \mathcal{A} = \int [d\tau] \exp \left(- \sum_{I \neq J} T_{IJ} \right) (\text{Det}' \Delta_1)^{1/2} (\text{Det}' \Delta_0)^{-(d-p)/2} \\ \times (\text{N det}' \Delta_0)^{-(p+1)/2}, \end{aligned} \quad (8)$$

where the normalized measure for moduli, $[d\tau]$, is given by

$$\begin{aligned} [d\tau] = V_{d+1} (2\pi)^{(n_c - n_m)/2} (4\pi^2 \alpha')^{-(p+1)/2} \\ \times \prod_{i=1}^{n_m} d\tau_i \left(\int d^2 \xi \sqrt{\hat{g}} \right)^{(n_m + (p+1))/2} (\det Q_{ab})^{-1/2} \\ \times [\det((\zeta_k)_{ef}(\zeta_l)^{ef})]^{1/2}. \end{aligned} \quad (9)$$

To proceed further, we need an explicit parametrization of the Riemann surface following global uniformization to some region in the complex plane. An explicit parametrization of this region, and the formulation of the eigenfunction problem, is a challenging problem for Riemann surfaces with $b+c \geq 3$ boundaries and/or crosscaps. We reserve that discussion for future work [8]. The disk is a special case all by itself because it uniformizes to the unit circle with positive constant curvature metric; it also has three conformal Killing vectors. Surfaces with cylindrical topology are of course easiest in this respect since they can be conformally mapped to a rectangle in the flat complex plane, and the eigenfunction problems with either choice of boundary condition have an explicit solution. The cylinder also has a conformal Killing vector so it is helpful to treat it separately, as we do now. We will recover both the annulus diagram of open string theory with Neumann boundaries and the exchange amplitude between a pair of static D-branes [2].

An arbitrary cylinder can be uniformized to a rectangle in the complex plane with area:

$$A = \int d^2 \sigma \sqrt{\hat{g}} = t. \quad (10)$$

We choose a parametrization such that the rectangle is bounded by the unit intervals, $0 \leq \sigma^1 \leq 1$, $0 \leq \sigma^2 \leq 1$. Then the moduli dependence is restricted to the flat world-sheet metric:

$$ds^2 = (d\sigma^1)^2 + t^2 (d\sigma^2)^2, \quad (11)$$

and t also corresponds to the length of the cylinder. The complete set of eigenfunctions of the scalar Laplacian on this domain are simply the circular functions of two variables, (σ^1, σ^2) . The length of the cylinder, t , is a Euler-Lagrange parameter appearing in the effective action. Note that for the special case of the cylinder, renormalizations of the bulk and boundary cosmological constants are not independent. The cylinder has a single conformal Killing vector, η_0 , which contributes to the path integral a 1×1 matrix with determinant, $(2\pi/\det Q_{ab})^{1/2} = (2\pi/t)^{1/2}$. The measure for moduli can be computed from the matrix, $(\zeta)_{ab} = \hat{g}_{ab,t} - \frac{1}{2} \hat{g}_{ab} \hat{g}^{cd} \hat{g}_{cd,t}$. It takes diagonal form on the cylinder, $\zeta_{11} = -1/t$, $\zeta_{22} = t$, and has determinant, $(\det \zeta^{ab} \zeta_{ab})^{1/2} = 1/t$.

By a reparametrization of the world sheet, the unit normal and unit tangent vectors at both boundary circles, C_1 , C_2 , can be chosen to lie along the (σ^2, σ^1) grid. The Neumann determinant is composed from the basis of periodic functions nonvanishing on the boundary, $\sigma^2 = 0, 1$:

$$\psi_{(n_1, n_2)}^{(1)}(\sigma^1, \sigma^2) = e^{2n_1 \pi i \sigma^1} \cos(n_2 \pi \sigma^2), \quad (12)$$

with $-\infty \leq n_1 \leq \infty$, and $n_2 > 0$. The Dirichlet determinant is composed from the orthogonal basis of periodic functions vanishing on the boundary:

$$\psi_{(n_1, n_2)}^{(2)}(\sigma^1, \sigma^2) = e^{2n_1 \pi i \sigma^1} \sin(n_2 \pi \sigma^2), \quad (13)$$

once again, with $-\infty \leq n_1 \leq \infty$, and $n_2 > 0$. Either choice of basis satisfies a completeness relation on the cylinder.

Substituting $n_c = n_m = 1$ in Eq. (10) for the measure, $[d\tau]$, we recover the familiar result:

$$\int [d\tau] = \int \frac{dt}{t} V_{d+1} (4\pi^2 \alpha')^{-(p+1)/2} t^{(p+1)/2}, \quad (14)$$

and the gauge fixed path integral takes the form

$$\begin{aligned} \mathcal{A}(y) &= \int \frac{dt}{t} V_{d+1} (4\pi^2 \alpha')^{-(p+1)/2} t^{(p+1)/2} \\ &\times e^{-(1/4\pi\alpha')(y^2/t)} [\text{Det}' \Delta_0]^{-12}, \end{aligned} \quad (15)$$

where we have substituted $d-1=24$ for the critical string, and the relation between vector and scalar Laplacians, $(\text{Det}' \Delta_1)^{1/2} = \text{Det}' \Delta_0$, which holds on the cylinder. The classical contribution to the action is from open strings stretched between the branes.

The functional determinants on the cylinder can be evaluated directly following the analysis in the appendix of Ref. [11]. The eigenmodes of the scalar Laplacian with zero Dirichlet boundary condition, $\delta\eta(\sigma^1)|_{\sigma^2=0,1}=0$, are composed from the basis functions in Eq. (14), and the unregulated determinant is therefore the infinite product:

$$\text{Det}' \Delta_0 = \prod_{n_1, n_2} \left[\frac{4\pi^2}{t^2} \left(n_1^2 t^2 + \frac{n_2^2}{4} \right) \right], \quad (16)$$

with the restrictions, $-\infty \leq n_1 \leq \infty$, $n_2 > 0$. Equivalently, we could compute the product over eigenmodes of the vector Laplacian with zero Dirichlet condition, i.e., the variations $\eta^a(\sigma^1)|_{\sigma^2=0,1}=0$. This gives the unrestricted product, $-\infty \leq n_1 \leq \infty$, $-\infty \leq n_2 \leq \infty$, with the $n_1 = n_2 = 0$ term subtracted out. The required scalar determinant is obtained by taking its square root [11,12]. The result for the determinant of the scalar Laplacian with Dirichlet condition on the cylinder is [12]

$$\begin{aligned} (\text{Det}' \Delta_0)^{1/2} &= (2t)^{1/2} e^{-2\pi t/12} \prod_{n=1}^{\infty} (1 - e^{-4n\pi t}) \\ &= (2t)^{1/2} \eta(2it) = \eta(i/2t), \end{aligned} \quad (17)$$

where we have used a modular transformation in the variable t to obtain the second equality. From Eqs. (13) we have an identical answer for the Neumann determinant. Substituting in Eq. (16), the static amplitude between parallel Dp -branes is given by the expression

$$\begin{aligned} \mathcal{A}(y) &= \int_0^\infty \frac{dt}{t} V_{d+1} (4\pi^2 \alpha')^{-(p+1)/2} t^{(p+1)/2} \\ &\times e^{-(1/4\pi\alpha')(y^2/t)} [\eta(i/2t)]^{-24}. \end{aligned} \quad (18)$$

A change of variables, $s=1/2t$, in the integral gives the equivalent form:

$$\begin{aligned} \mathcal{A}(y) &= \int_0^\infty \frac{ds}{s} V_{d+1} (8\pi^2 \alpha' s)^{-(p+1)/2} \\ &\times e^{-(1/2\pi\alpha')(y^2 s)} [\eta(is)]^{-24}, \end{aligned} \quad (19)$$

which can be compared with the expression for the exchange amplitude between static bosonic Dp -branes obtained in the operator formalism [2]. The variable s corresponds to the length of the boundary. By setting $p=d$, we recover the cylinder amplitude of open string theory with Neumann boundaries in a single Chan-Paton state, as originally obtained in [13]. This is also consistent with the result from the method of images [14], which can be used to relate the cylinder and torus amplitudes.

We now return to the puzzle mentioned in the introduction—the emergence of non-Abelian structure in the world-sheet picture, in the limit of near-coincident D-branes. A simple non-Abelian configuration with an interesting low energy description is that of a single probe-D-brane, spatially distant from a pair of near-coincident D-branes. It can be given a world-sheet description as follows. Consider a *pants* surface: an orientable Riemann surface with boundaries on three D-branes, with boundaries C_2 , C_3 on near-coincident D-branes. The low-energy effective theory on the brane volume is a $[\text{SU}(2) \times \text{U}(1)] \times \text{U}(1)$ non-Abelian gauge theory.

A generic pants surface can be mapped to a simply connected domain as follows. Pick a base point on the surface, z_0 , and cut along paths joining z_0 to the three boundaries. The resulting surface can be mapped to a -9χ -sided polygon, a simply connected domain in the complex plane. For a constant curvature pants surface, with $\hat{R} = -1$, this gives a polygon in the upper half plane with hyperbolic metric [7], and the joining paths can be chosen as geodesics on the surface. Going around the polygon once, we can label the edges:

$$(r_1, s_1, \tilde{s}_2, r_2, s_2, \tilde{s}_3, r_3, s_3, \tilde{s}_1), \quad (20)$$

where the r_i are the images of the C_i upon mapping to the complex plane. The images of the base point lie at the intersections of adjacent pairs of joining curves, (s_i, \tilde{s}_{i+1}) , $i = 1, \dots, 3$, taken in cyclic order. Notice that the location of the base point *within* the surface is arbitrary, but the limit of the mapping when the base point approaches any boundary is *singular*; the domain contains a cusp on its boundary, and the ordinary properties of function theory on the domain have to be suitably modified to take into account the cusp.

What is the shape of the world-sheets that contribute to the singular limit of this mapping? Consider the case that z_0 lies on boundary C_3 , and the pair, (s_3, \tilde{s}_3) , is shrunk to zero

length. Now the boundary lengths, l_i , correspond to open string proper times. Recall that the inverse length, $l_{[12]} = 1/l$, of a cylinder can be interpreted, through open-closed string world-sheet duality, as the proper time of a propagating closed string exchanged between a pair of D-branes [2]. A singular pants surface can be given a surprisingly simple interpretation in the language of closed string “proper times.” The singular world sheet can be visualized as a closed string emitted by D-brane \mathbf{D}_1 , which propagates smoothly towards D-brane \mathbf{D}_2 . In addition, \mathbf{D}_3 emits a closed string which propagates for a vanishingly short proper time, $l_{[23]}$, before being absorbed by D-brane \mathbf{D}_2 . Consider the limit when boundary C_2 approaches boundary C_3 , as with

near-coincident branes. Such a pants surface can be mapped to the complex plane by cutting along only *two* joining curves, with the base point z_0 on the boundary C_3 . The apparent mismatch in the counting of moduli is accounted for by the presence of the cusp, in agreement with the Gauss-Bonnet and Riemann-Roch theorems.

Function theory on the resulting domain has to be modified by extending the analysis of the eigenfunction problem to domains with a cusp, a particular instance of a domain with an isolated singular point [15]. We will reserve that discussion for future work [8].

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- [1] J. Polchinski, Phys. Rev. Lett. **75**, 4724 (1995); J. Dai, R. Leigh, and J. Polchinski, Mod. Phys. Lett. A **4**, 2073 (1989).
 - [2] J. Polchinski, *String Theory* (Cambridge University Press, Cambridge, England, 1998), Vol. I, especially Chaps. 7 and 8.
 - [3] J. Polchinski, Phys. Rev. D **50**, 6041 (1994).
 - [4] J. Polchinski and E. Witten, Nucl. Phys. **B460**, 525 (1996).
 - [5] E. Witten, Nucl. Phys. **B460**, 335 (1996).
 - [6] E. D’Hoker and D.H. Phong, Nucl. Phys. **B269**, 205 (1986).
 - [7] L. Bers, Bull. London Math. Soc. **4**, 257 (1972).
 - [8] S. Chaudhuri, J. High Energy Phys. **08**, 003 (1999); and (work in progress).
 - [9] S. Chaudhuri and E. Novak (work in progress).
 - [10] A.M. Polyakov, Phys. Lett. **103B**, 207 (1981).
 - [11] J. Polchinski, Commun. Math. Phys. **104**, 37 (1986).
 - [12] A. Cohen, G. Moore, P. Nelson, and J. Polchinski, Nucl. Phys. **B267**, 143 (1986); **B281**, 127 (1987).
 - [13] G. Frye and L. Susskind, Phys. Lett. **31B**, 589 (1970).
 - [14] H. Nielsen, NORDITA report (1969); D. Fairlie and H. Nielsen, Nucl. Phys. **B20**, 637 (1970); C. Lovelace, Phys. Lett. **32B**, 703 (1970); V. Alessandrini, Nuovo Cimento A **2**, 321 (1971); V. Alessandrini and D. Amati, *ibid.* **4**, 793 (1971).
 - [15] An intuitive discussion of boundary value problems on domains with isolated singular points can be found in the appendix to Chap. VI, R. Courant and D. Hilbert, *Mathematical Methods of Physics* (Interscience, New York, 1953), Vol. 2.