Scattering in anti–de Sitter space and operator product expansion

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We develop a formalism to evaluate generic scalar exchange diagrams in anti–de Sitter $(\mathrm{AdS})_{d+1}$ relevant for the calculation of four-point functions in AdS conformal field theory (CFT) correspondence. The result may be written as an infinite power series of functions of cross ratios. Logarithmic singularities appear in all orders whenever the dimensions of involved operators satisfy certain relations. We show that the AdS_{d+1} amplitude can be written in a form recognizable as the conformal partial wave expansion of a four-point function in CFT_d and identify the spectrum of intermediate operators. We find that, in addition to the contribution of the scalar operator associated with the exchanged field in the AdS_{d+1} diagram, there are also contributions of some other operators which may possibly be identified with two-particle bound states in AdS_{d+1}. The CFT_d interpretation also provides a useful way to ''regularize'' the logarithms appearing in AdS_{d+1} amplitude.
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I. INTRODUCTION

There has been a recent revival of interest in the connection between large-N Yang-Mills theory [1] and string theory [2] following the conjecture [3] that there is an exact correspondence [4,5] between type-IIB superstring theory on anti–de Sitter $(\mathrm{AdS}_5)\times S_5$ and $\mathcal{N}=4$ super-Yang-Mills theory in four dimensions (see also [6]).

Under this proposal, correlation functions of $\mathcal{N}=4$ super-Yang-Mills (SYM) theory with gauge group SU (N) in the large-N and large 't Hooft coupling limit can be obtained by evaluating scattering amplitudes of type-IIB supergravity on $\mathrm{AdS}_5\times S_5$. Some ''model'' and ''realistic'' two-point and three-point functions have been computed in [7–15]. Since the structures of two- and three-point functions are severely restricted by conformal symmetry, in many cases the computations amount to fixing the overall constants. Four-point functions can be arbitrary functions of cross ratios and thus encode more dynamical information. Recently some efforts have been made in this direction [16–22] aiming to understand more about the nonperturbative dynamics of $\mathcal{N}=4$ SYM theory.

Considering a d-dimensional conformal field theory,[1] (CFT_d), we shall assume that there exists a closed operator algebra, which is a strong version of the Wilson operator product expansion,

$$O_i(x)O_j(0)=\sum_k C_{ij}^k(x)O_k(0). \tag{1.1}$$

Here the summation is over all operators and their coordinate derivatives, and C_{ij}^k are c-number functions. From Eq. (1.1), a four-point function may be expanded in terms of conformal partial waves, e.g., when x_{12}, $x_{34}\to 0$, as an s-channel exchange,

$$\langle O_{i_1}(x_1)O_{i_2}(x_2)O_{i_3}(x_3)O_{i_4}(x_4)\rangle$$
$$=\sum_j C^j_{i_1i_2}(x_{12})C^j_{i_3i_4}(x_{34})\langle O_j(x_2)O_j(x_4)\rangle. \tag{1.2}$$

Alternatively, we can also write the four-point function in terms of t- or u-channel exchanges in the limit x_{14}, $x_{23}\to 0$ or x_{13}, $x_{24}\to 0$. If the algebra (1.1) is complete and associative, all channels of exchange are equivalent.

We would like to examine whether a four-point function calculated from the scattering amplitude in AdS_{d+1} can be written in the form of Eq. (1.2) as we take the corresponding limits in cross ratios. A positive answer would be a confirmation of the assumption of a closed algebra (1.1), which hitherto has been only known to hold in two dimensions. And we could further extract important nonperturbative information about CFT_d by identifying the spectra of intermediate operators in each channel. In the case of $\mathcal{N}=4$ SYM theory in the large-N and large-g^2N limit, knowledge of four-point functions would help us answer questions such as [16–18]:

(1) Does $\mathcal{N}=4$ SYM theory in the large-N and large-g^2N limit have a closed algebra (1.1)?

(2) If yes, what is the spectrum of operators? In particular, do chiral operators, which are in one-to-one correspondence with type-IIB supergravity modes on $\mathrm{AdS}_5\times S_5$, form a complete set?

In this paper, we shall make some preliminary progress in answering these questions. In particular, we shall find indications that there are operators in the spectrum which correspond to two-particle bound states in AdS_{d+1}.

One of the obstacles in the computation of realistic four-point functions in $\mathcal{N}=4$ SYM theory has been the difficulty in evaluating exchange diagrams [5] in AdS space [see Fig. 1(a)], which involve very complicated integrals. Here we present a formalism to address this problem (see also [19]),

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[1]Though our prime interest is $\mathcal{N}=4$SYM theory, most of our discussion will apply to any d-dimensional conformal field theory appearing in AdS_{d+1}-CFT_d correspondence.

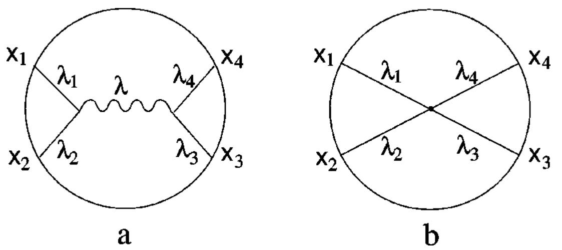

FIG. 1. Exchange and contact diagrams in AdS_{d+1}: λ and λ_i are dimensions of the conformal operators corresponding to the fields in the diagrams.

providing explicit formulas for AdS integrals involving scalar fields of arbitrary mass needed to evaluate generic four-point functions. The result is written as a single inverse Mellin integral so that the analytic properties of the amplitudes become transparent. In particular, for the exchange diagram of Fig. 1(a), in the limit $x_{12}, x_{34}\to 0$ the scattering amplitude can be written as a contour integral

$$S_\lambda=\int_C ds\,\Gamma\left(\frac{\lambda_1+\lambda_2}{2}-s\right)\Gamma\left(\frac{\lambda_3+\lambda_4}{2}-s\right)\times\Gamma\left(\frac{\lambda}{2}-s\right)H(s,\eta,\xi), \tag{1.3}$$

where ξ and η are independent cross ratios and H is a function of complex variable s and ξ, η. In Eq. (1.3) we have only explicitly written down the Γ functions which generate poles inside the contour. S_λ can then be evaluated by the calculus of residues and written as a sum of residues of the integrand at three infinite pole sequences (see Fig. 2). We find that logarithms of cross ratios, first found [18] in leading-order expansion of some contact diagrams, arise generically whenever the poles in Eq. (1.3) merge into double poles or triple poles, i.e., when

$$\frac{\lambda_1+\lambda_2-\lambda_3-\lambda_4}{2}\ \text{or}\ \frac{\lambda_1+\lambda_2-\lambda}{2}\ \text{or}\ \frac{\lambda_3+\lambda_4-\lambda}{2}=\text{integer}. \tag{1.4}$$

They appear in all orders of the series. In particular, the contribution from a triple pole will contain a part proportional to

$$\left(\ln\left|\frac{x_{12}x_{34}}{x_{13}x_{24}}\right|\right)^2.$$

Similarly, it can be shown in contact diagrams [see Fig. 1(b)] logarithms occur [18] when

$$\frac{\lambda_1+\lambda_2-\lambda_3-\lambda_4}{2}=\text{integer}. \tag{1.5}$$

In $\mathcal{N}=4$ SYM theory, the dimensions of the chiral fields are protected by supersymmetry and take integer values. Since in type-IIB supergravity on $\mathrm{AdS}_5\times S_5$, all vertices are SU (4) singlets, the scattering diagrams associated with four-point functions will in general satisfy Eqs. (1.4) or (1.5). This implies that logarithms are universally present.[2]

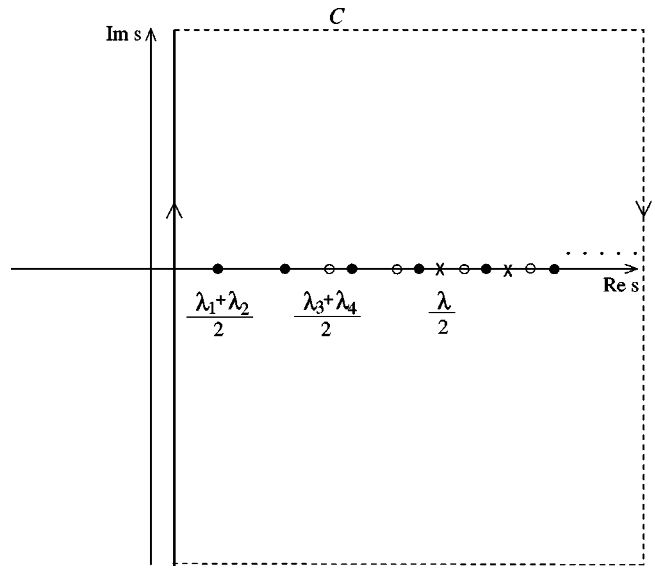

FIG. 2. Poles and the contour of Eq. (1.3). There are three sequences of poles: (1) $s=(\lambda_1+\lambda_2)/2+n$ (represented by solid circles); (2) $s=(\lambda_3+\lambda_4)/2+n$ (circles); (3) $s=\lambda/2+n$ (crosses), where $n=0,1,2,\ldots$.

We then proceed to investigate whether the amplitude we find can be written as the conformal partial wave expansion (CPWE) Eq. (1.2). The Mellin integral representation (1.3), in which our results are presented, turns out to be particularly convenient to identify them with s-channel operator product expansion (OPE) exchanges in CFT_d. The contribution of each pole sequence in Eq. (1.3) can be identified with the CPWE Eq. (1.2) of a conformal operator: the value of a pole corresponds to the scale dimension of a spin-0 descendant[3] (we shall call it a subprimary), while the residue at the pole may be identified with the CPWE contribution of a subset of descendants associated with the subprimary. The pattern may be presented diagrammatically as in Fig. 3.

The first diagram on the right-hand side corresponds to the exchange of a scalar primary operator of dimension λ, which may be interpreted as the operator (we shall call it O_λ) related to the exchanged field in AdS_{d+1} by AdS-CFT correspondence. This result was expected earlier in [17] on the basis of indirect considerations. Here we identify the contributions of all the descendents of O_λ and show that their relative OPE couplings (1.1) are consistent with those required by conformal symmetry. The second and third diagram on the right-hand side correspond to the exchanges of operators of dimensions $\lambda_1+\lambda_2$ and $\lambda_3+\lambda_4$, respectively (which we shall call O_{12} and O_{34}). However, in these cases, there are some mismatches in the identifications. In Fig. 3 we have used dotted lines in intermediate states to distinguish

[2]It might still happen that when we add up all the diagrams contributing to a four-point function, logarithms will cancel.

[3]Here we mean an SO(d,2) descendant. A spin-0 descendant takes the form $(\partial^2)^n O$, where O is the primary and ∂^2 is the Laplacian.

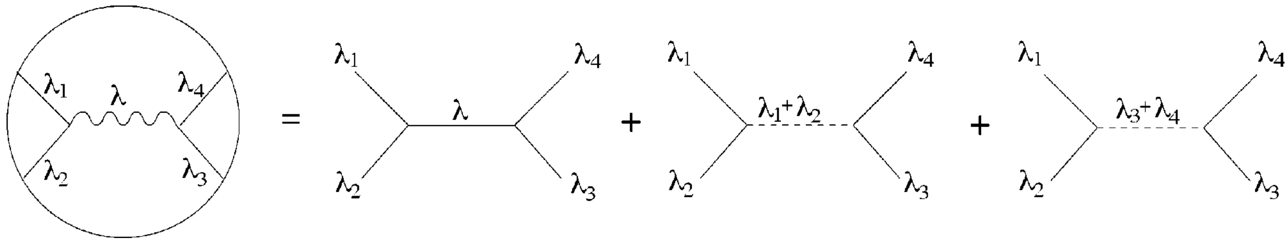

FIG. 3. *s*-channel OPE interpretation of an exchange diagram.

them from the first diagram. Although we have found contributions from operators having the same quantum numbers as the complete set of descendants of a primary operator of dimension $\lambda_1+\lambda_2$ (and $\lambda_3+\lambda_4$), the relative OPE couplings (1.1) between the primary and descendants seem to be inconsistent with those required by conformal symmetry.[4] The OPE couplings of these descendant operators have a peculiar pattern suggesting the mismatch may be due to some mixing among different operators. But we have not been able to make it precise in this paper.

Similarly the contact diagram Fig. 1(b) may be represented in terms of *s*-channel exchanges as in Fig. 4 and the identifications are also not complete in the sense as described in exchange case.

The identification of AdS_{d+1} diagrams with CPWE also sheds light on the appearance of logarithms. Conditions (1.4) and (1.5) are satisfied precisely when the quantum numbers (spin, scale dimensions, etc.) of certain descendants of O_λ or O_{12} or O_{34} become identical to one another. The OPE couplings in Figs. 3 and 4 determined from Eq. (1.3) fall into the following pattern: when the quantum numbers of descendants of different operators are degenerate, their contributions to the conformal partial wave expansion become identical and cancel one another. The results are given by their derivatives, which contain logarithms. Thus by moving infinitesimally away from the degeneracy points (1.4) and (1.5) in parameter space, we see that the relations in Figs. 3 and 4 provide a physically meaningful way to ''regularize'' logarithms.

Although our present analysis based on generic diagrams would not give a conclusive answer to the questions listed earlier, it nevertheless provides a starting point for further study. The relation we find here between an arbitrary exchange diagram in AdS_{d+1} and CPWE appears to be *universal* and should be helpful for understanding the general structure of AdS-CFT correspondence. Before having a complete calculation of realistic four-point functions in a specific theory, it is probably premature to speculate about the relevance of operators O_{12} and O_{34} and the mismatches in their CFT_d identification. However, if their contributions are indeed present in a realistic amplitude, it should imply the existence of new operators in the spectrum not seen in the Lagrangian of supergravity. In $\mathcal{N}=4$ SYM theory they may be written as double-trace operators, while in AdS_5 supergravity they may be identified with two-particle bound states.

The plan of the paper is as follows. In Secs. II and III we discuss the evaluation of scattering diagrams in AdS_{d+1}. In Sec. IV we review, for the convenience of comparing with AdS_{d+1} results, the conformal partial wave expansion in CFT_d. In Sec. V we discuss the CFT_d interpretation of AdS_{d+1} amplitude. We have included a number of appendixes. In Appendix A we describe briefly the subtleties in evaluation of integrals using Mellin transform and analytic continuation. Appendixes B and C are devoted to detailed evaluation of some integrals in the main text.

II. SCALAR EXCHANGE IN ANTI–DE SITTER SPACE

We consider tree-level scattering of four scalar fields in AdS_{d+1} with masses m_i, $i=1,\dots,4$ by exchanging a scalar field of mass m. According to AdS-CFT correspondence, a scalar field ϕ_i of mass m_i in AdS_{d+1} corresponds to a scalar operator Φ_{λ_i} in CFT_d, with conformal dimension $\lambda_i=d/2+\sqrt{m_i^2+d^2/4}=d/2+\nu_i$. The scattering amplitude describes in CFT_d the contribution of Φ_λ to the four-point function of scalar operators Φ_{λ_i}, $i=1,\dots,4$.

In this section, we shall take the interacting vertices to be of the form

$$\mathcal{L}=\phi_1\phi_2\phi+\phi_3\phi_4\phi.$$

Scattering amplitudes resulting from more complicated vertices involving derivatives and contact vertices will be discussed in next section.

As in [5] we use the Euclidean (half-space) metric,

$$ds^2=g_{\mu\nu}du^\mu du^\nu=\frac{1}{u_0^2}(du_0^2+du_i^2),\quad i=1,2,\dots,d. \tag{2.1}$$

The AdS_{d+1} bulk indices will be denoted by $\mu,\nu,\dots$ and will take values $0,1,\dots,d$. The points in the bulk are labeled by $u,v,\dots$, while those on the boundary by $x,y,\dots$. We also use shorthand notations $u=(u_0,\vec{u})$, $x=(\vec{x})$ and $x_{ij}^2=|\vec{x}_i-\vec{x}_j|^2$, $|u-x_i|^2=u_0^2+|\vec{u}-\vec{x}_i|^2$.

The scattering amplitude can then be written as

[4] In a CFT_d, the OPE coupling (1.1) of a descendant is uniquely determined by that of the primary.

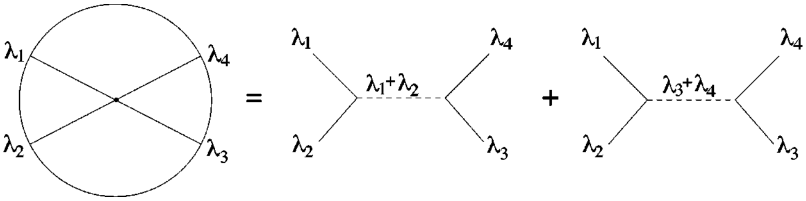

FIG. 4. *s*-channel OPE interpretation of a contact diagram.

$$S_\lambda(x_1,x_2,x_3,x_4) = \int \frac{du_0 d^d u}{u_0^{d+1}} \frac{dv_0 d^d v}{v_0^{d+1}} \mathcal{K}_{\lambda_1}(u,x_1)\mathcal{K}_{\lambda_2}(u,x_2) \times G(u,v)\mathcal{K}_{\lambda_3}(v,x_3)\mathcal{K}_{\lambda_4}(v,x_4), \tag{2.2}$$

where $\mathcal{K}_{\lambda_i}(u,x_i)$, $i=1,\ldots,4$ is the bulk-to-boundary propagator [5] for field ϕ_i,

$$\mathcal{K}_{\lambda_i}(u,x_i) = c_{\lambda_i}\left(\frac{u_0}{|u-x_i|^2}\right)^{\lambda_i}, \quad c_{\lambda_i} = \frac{\Gamma(\lambda_i)}{\pi^{d/2}\Gamma(\nu_i)} \tag{2.3}$$

and $G(u,v)$ is the AdS bulk scalar propagator [23],

$$G(u,v) = r t^{-\lambda} F\left(\lambda, \nu+\frac{1}{2}; 2\nu+1, t^{-1}\right). \tag{2.4}$$

In Eq. (2.4) F is a hypergeometric function and

$$r = \frac{\Gamma(\lambda)}{2^{2\lambda+1}\pi^{d/2}} \frac{1}{\Gamma(\nu+1)}, \quad t = \frac{(u_0+v_0)^2 + (\vec{u}-\vec{v})^2}{4u_0 v_0}.$$

To evaluate Eq. (2.2), first we would like to get rid of the cross term of u_0 and v_0 in t in Eq. (2.4), which complicates the integrals. This can be achieved by a quadratic transformation[5] of the hypergeometric function in Eq. (2.4), after which the bulk propagator becomes

$$G(u,v) = \frac{\Gamma(\lambda)}{2^{\lambda+1}\pi^{d/2}} \frac{1}{\Gamma(\nu+1)} q^{-\lambda} F\left(\frac{\lambda+1}{2}, \frac{\lambda}{2}; \nu+1; \frac{1}{q^2}\right), \tag{2.5}$$

where

$$q = \frac{u_0^2 + v_0^2 + |\vec{u}-\vec{v}|^2}{2u_0 v_0}.$$

Now we use the Mellin-Barnes representation of a hypergeometric function

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \times \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)} \Gamma(-s)(-z)^s \tag{2.6}$$

in Eq. (2.5) and plug it into Eq. (2.2). This gives us

$$S_\lambda = C_1 \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} ds \times \frac{\Gamma[(\lambda+1)/2+s]\Gamma(\lambda/2+s)}{\Gamma(\nu+1+s)} \Gamma(-s)(-1)^s J(s) \tag{2.7}$$

with

$$J(s) = \int \frac{du_0 d^d u}{u_0^{d+1}} \frac{dv_0 d^d v}{v_0^{d+1}} \left(\frac{u_0}{|u-x_1|^2}\right)^{\lambda_1} \times \left(\frac{u_0}{|u-x_2|^2}\right)^{\lambda_2} \left(\frac{2u_0 v_0}{u_0^2+v_0^2+|\vec{u}-\vec{v}|^2}\right)^{\lambda+2s} \times \left(\frac{v_0}{|v-x_3|^2}\right)^{\lambda_3} \left(\frac{v_0}{|v-x_4|^2}\right)^{\lambda_4} \tag{2.8}$$

and

$$C_1 = \frac{1}{4\pi^{(d+1)/2}} \Pi_{i=1}^4 c_{\lambda_i}.$$

$J(s)$ still involves quite complicated integrals. We present its detailed evaluation in Appendix B. The result can be written in terms of the cross ratios of the boundary points as an inverse Mellin-type integral (for notations see Appendix B),

$$J(s) = C_2 \frac{1}{2\pi i} \int_C ds_1 \xi^{-s_1-\Delta_{34}/2} \Gamma\left(\frac{\lambda_{12}}{2}-s_1\right) \Gamma\left(\frac{\lambda_{34}}{2}-s_1\right) F\left(\frac{\Delta_{34}}{2}+s_1, \frac{\Delta_{12}}{2}+s_1; 2s_1; 1-\frac{\eta}{\xi}\right) \times \frac{\Gamma(\lambda/2+s-s_1)}{\Gamma[(\lambda_{12}+\tilde{\epsilon}_{34})/2+s-s_1]} \frac{\Gamma(\Delta_{34}/2+s_1)\Gamma(\Delta_{43}/2+s_1)\Gamma(\Delta_{12}/2+s_1)\Gamma(\Delta_{21}/2+s_1)}{\Gamma(2s_1)}, \tag{2.9}$$

[5]The one we use here is $F(a,b;2b;z) = (1-z/2)^{-a} F(1/2a, 1/2(a+1); b+1/2; z^2(2-z)^{-2})$.

where η, ξ are cross ratios defined by

$$\eta=\frac{|x_{13}|^2|x_{24}|^2}{|x_{12}|^2|x_{34}|^2},\quad \xi=\frac{|x_{14}|^2|x_{23}|^2}{|x_{12}|^2|x_{34}|^2}, \tag{2.10}$$

and

$$C_2=\frac{\pi^d}{4}\frac{\Gamma[(\lambda_{12}+\lambda_{34}-d)/2]\Gamma(s+\tilde{\epsilon}_{12}/2)\Gamma(s+\tilde{\epsilon}_{34}/2)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)\Gamma(\lambda+2s)}\times\frac{2^{\lambda+2s}}{|x_{12}|^{\lambda_{12}+\Delta_{34}}|x_{14}|^{\Delta_{12}-\Delta_{34}}|x_{24}|^{\Delta_{21}-\Delta_{34}}|x_{34}|^{2\lambda_3}}.$$

The path of integration $\mathcal{C}$ in Eq. (2.9) (see the last paragraph of Appendix B for a more precise description) is taken to be parallel to the imaginary s_1 axis and is deformed if necessary to separate the poles of ascending sequences [e.g., those of $\Gamma(\lambda_{12}/2-s_1)$] from the poles of descending sequences [e.g., those of $\Gamma(\Delta_{12}/2+s_1)$] of the integrand.

Plugging the expression for J into Eq. (2.7), using the duplication formula for Γ functions

$$\Gamma(\lambda+2s)=\frac{1}{2\pi^{1/2}}2^{\lambda+2s}\Gamma\left(\frac{\lambda+1}{2}+s\right)\Gamma\left(\frac{\lambda}{2}+s\right),$$

and regrouping the terms in the integrand, we find,

$$S_\lambda=C_3\frac{1}{2\pi i}\int_C ds_1\xi^{-s_1-\Delta_{34}/2}\Gamma\left(\frac{\lambda_{12}}{2}-s_1\right)\Gamma\left(\frac{\lambda_{34}}{2}-s_1\right)F\left(\frac{\Delta_{34}}{2}+s_1,\frac{\Delta_{12}}{2}+s_1;2s_1;1-\frac{\eta}{\xi}\right)\times\frac{\Gamma(\Delta_{34}/2+s_1)\Gamma(\Delta_{43}/2+s_1)\Gamma(\Delta_{12}/2+s_1)\Gamma(\Delta_{21}/2+s_1)}{\Gamma(2s_1)}I_1 \tag{2.11}$$

with

$$I_1=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}ds\,\Gamma(-s)(-1)^s\times\frac{\Gamma(\tilde{\epsilon}_{12}/2+s)\Gamma(\tilde{\epsilon}_{34}/2+s)\Gamma(\lambda/2+s-s_1)}{\Gamma[s-s_1+(\lambda_{12}+\tilde{\epsilon}_{34})/2]\Gamma(\nu+1+s)}. \tag{2.12}$$

I_1 is nothing but the Mellin-Barnes representation of the generalized hypergeometric function ${}_3F_2$ [24], which leads to[6]

$$I_1=\frac{\Gamma(\tilde{\epsilon}_{12}/2)\Gamma(\tilde{\epsilon}_{34}/2)}{\Gamma(\nu+1)}\frac{\Gamma(\lambda/2-s_1)}{\Gamma[-s_1+(\lambda_{12}+\tilde{\epsilon}_{34})/2]}\times{}_3F_2\left(\frac{\tilde{\epsilon}_{12}}{2},\frac{\tilde{\epsilon}_{34}}{2},\frac{\lambda}{2}-s_1;\frac{\lambda_{12}+\tilde{\epsilon}_{34}}{2}-s_1,\nu+1;1\right). \tag{2.13}$$

When the parameters of a generalized hypergeometric function ${}_3F_2(a,b,c;e,f;z)$ satisfy the relation $e+f=a+b+c+1$, the series will be said to be Saalschutzian.[7] It is easy to check that the hypergeometric series in Eq. (2.13) is Saalschutzian. Saalschutz's theorem states that ${}_3F_2(a,b,c;e,f;z)$ satisfies

$${}_3F_2(a,b,c;e,f;1)=\frac{\Gamma(e)\Gamma(1+a-f)\Gamma(1+b-f)\Gamma(1+c-f)}{\Gamma(1-f)\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)}, \tag{2.14}$$

provided $e+f=a+b+c+1$ and a, b, or c is a negative integer.

From Eqs. (2.11) and (2.13), we reach the final expression for S_λ,

[6] If necessary, the integration path in Eq. (2.12) should be deformed to separate the poles $s=0,1,\ldots$ from those poles in descending series.

[7] The hypergeometric series ${}_3F_2(a,b,c;e,f;z)$ converges when $|z|<1$, also when $z=1$ provided that $\mathrm{Re}(e+f-a-b-c)>0$. Thus we see a Saalschutzian series is convergent at $z=1$.

$$S_\lambda = C\frac{1}{2\pi i}\int_C ds_1 \xi^{-s_1}\Gamma\left(\frac{\lambda_{12}}{2}-s_1\right)\Gamma\left(\frac{\lambda_{34}}{2}-s_1\right)\Gamma\left(\frac{\lambda}{2}-s_1\right)F\left(\frac{\Delta_{34}}{2}+s_1,\frac{\Delta_{12}}{2}+s_1;2s_1;1-\frac{\eta}{\xi}\right)$$
$$\times\frac{\Gamma(\Delta_{34}/2+s_1)\Gamma(\Delta_{43}/2+s_1)\Gamma(\Delta_{12}/2+s_1)\Gamma(\Delta_{21}/2+s_1)}{\Gamma[(\lambda_{12}+\tilde{\epsilon}_{34})/2-s_1]\Gamma(2s_1)}\,{}_3F_2\left(\frac{\tilde{\epsilon}_{12}}{2},\frac{\tilde{\epsilon}_{34}}{2},\frac{\lambda}{2}-s_1;\frac{\lambda_{12}+\tilde{\epsilon}_{34}}{2}-s_1,\nu+1;1\right) \quad (2.15)$$

with

$$C=\frac{1}{8\pi^{3/2d}}\frac{\Gamma[(\lambda_{34}+\lambda_{12}-d)/2]\Gamma(\tilde{\epsilon}_{12}/2)\Gamma(\tilde{\epsilon}_{34}/2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)\Gamma(\nu+1)}\frac{1}{|x_{12}|^{\lambda_{12}}|x_{14}|^{\Delta_{12}}|x_{24}|^{\Delta_{21}-\Delta_{34}}|x_{23}|^{\Delta_{34}}|x_{34}|^{\lambda_{34}}}.$$

Thus we have been able to reduce Eq. (2.2) to a single inverse Mellin-type integral in Eq. (2.15), which may be evaluated by choosing the appropriate contour in the complex s_1 plane and the calculus of residues.

Let us consider the *s*-channel OPE limit where x_{12}, x_{34} are much smaller than other distances, i.e., $\xi,\eta \gg 1$ and $1-\eta/\xi \ll 1$. In this case, we can take the integration path $\mathcal{C}$ over a contour enclosing the right half plane and the integral is given by the sum of the residues of the integrand at the poles of ascending sequences.

On the right half plane we have three pole series which come from $\Gamma(\lambda_{12}/2-s_1)$, $\Gamma(\lambda_{34}/2-s_1)$ and $\Gamma(\lambda/2-s_1)$, respectively,

$$(1)\ s_1=\frac{\lambda}{2}+n,\quad m=0,1,2,\ldots;$$

$$(2)\ s_1=\frac{\lambda_{12}}{2}+n,\quad m=0,1,2,\ldots; \quad (2.16)$$

$$(3)\ s_1=\frac{\lambda_{34}}{2}+n,\quad m=0,1,2,\ldots.$$

We first consider the case that no pole series in the above coincide with one another, i.e., none of $\epsilon_{12}/2$, $\epsilon_{34}/2$, and $(\lambda_{12}-\lambda_{34})/2$ is an integer. Then we can write S_λ as

$$S_\lambda=\sum_{n=0}^{\infty}S_n^{(\lambda)}+\sum_{n=0}^{\infty}S_n^{(\lambda_{12})}+\sum_{n=0}^{\infty}S_n^{(\lambda_{34})},$$

where $S_n^{(\lambda_{12})}$, $S_n^{(\lambda_{12})}$, and $S_n^{(\lambda_{34})}$ are the contributions from the nth pole in each series.

A. Series 1

Let us first look at the pole series $s_1=\lambda/2+n$. In this case the third parameter in ${}_3F_2(\tilde{\epsilon}_{12}/2,\tilde{\epsilon}_{34}/2,-s+\lambda/2;-s+(\lambda_{12}+\tilde{\epsilon}_{34})/2,\nu+1;1)$ becomes a negative integer and we can use the Saalschutz's theorem (2.14) to get

$$\begin{aligned}&{}_3F_2\left(\frac{\tilde{\epsilon}_{12}}{2},\frac{\tilde{\epsilon}_{34}}{2},-n;-s+\frac{\lambda_{12}+\tilde{\epsilon}_{34}}{2},\nu+1;1\right)\\&=\frac{\Gamma[(\lambda_{12}+\lambda_{34}-d)/2-n]\Gamma(\epsilon_{12}/2)\Gamma(\epsilon_{34}/2)\Gamma(-n-\nu)}{\Gamma(-\nu)\Gamma(\epsilon_{12}/2-n)\Gamma(\epsilon_{34}/2-n)\Gamma[(\lambda_{12}+\lambda_{34}-d)/2]}\\&=(-1)^n\frac{\Gamma(1+\nu)\Gamma(\epsilon_{12}/2)\Gamma(\epsilon_{34}/2)}{\Gamma[(\lambda_{12}+\lambda_{34}-d)/2]}\frac{\Gamma[(\lambda_{12}+\lambda_{34}-d)/2-n]}{\Gamma(\epsilon_{12}/2-n)\Gamma(\epsilon_{34}/2-n)\Gamma(1+n+\nu)},\end{aligned} \quad (2.17)$$

where in the second identity we have used the relation $\Gamma(x)\Gamma(1-x)=\pi/\sin\pi x$. Now plugging Eq. (2.17) into Eq. (2.15), we get

$$S_n^{(\lambda)}=A^{(\lambda)}\frac{\xi^{-n}}{n!}\frac{\Gamma(\delta_1/2+n)\Gamma(\delta_2/2+n)\Gamma(\delta_3/2+n)\Gamma(\delta_4/2+n)}{\Gamma(\lambda+2n)\Gamma(\nu+n+1)}F\left(\frac{\delta_3}{2}+n,\frac{\delta_1}{2}+n;\lambda+2n;1-\frac{\eta}{\xi}\right) \quad (2.18)$$

with

$$A^{(\lambda)}=\frac{1}{8\pi^{3/2d}}\frac{\Gamma(\epsilon_{12}/2)\Gamma(\epsilon_{34}/2)\Gamma(\overline{\epsilon_{12}}/2)\Gamma(\overline{\epsilon_{34}}/2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)}\frac{1}{|x_{12}|^{\epsilon_{12}}|x_{14}|^{\delta_1}|x_{24}|^{\Delta_{21}-\Delta_{34}}|x_{23}|^{\delta_3}|x_{34}|^{\epsilon_{34}}}.$$

Note that $\Gamma(-s)$ has residue $(-1)^{n-1}/n!$ at its pole $s=n$.

B. Series 2

In this case we have $s_1=\lambda_{12}/2+n$, and

$$S_n^{(\lambda_{12})}=A^{(\lambda_{12})}\frac{(-1)^n}{n!}\xi^{-n}G_nF\left(\frac{\lambda_{12}+\Delta_{34}}{2}+n,\lambda_1+n;\lambda_{12}+2n;1-\frac{\eta}{\xi}\right)$$
$$\times\Gamma\left(\frac{\lambda_{34}-\lambda_{12}}{2}-n\right)\frac{\Gamma[(\lambda_{12}+\Delta_{34})/2+n]\Gamma[(\lambda_{12}-\Delta_{34})/2+n]\Gamma(\lambda_1+n)\Gamma(\lambda_2+n)}{\Gamma(\lambda_{12}+2n)} \tag{2.19}$$

with

$$A^{(\lambda_{12})}=\frac{1}{8\pi^{3/2d}}\frac{\Gamma[(\lambda_{34}+\lambda_{12}-d)/2]}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)}\frac{1}{|x_{14}|^{2\lambda_1}|x_{24}|^{\Delta_{21}-\Delta_{34}}|x_{23}|^{\lambda_{12}+\Delta_{34}}|x_{34}|^{\lambda_{34}-\lambda_{12}}} \tag{2.20}$$

and

$$G_n=\frac{\Gamma(\tilde{\epsilon}_{12}/2)\Gamma(\tilde{\epsilon}_{34}/2)\Gamma(-\epsilon_{12}/2-n)}{\Gamma(\tilde{\epsilon}_{34}/2-n)\Gamma(\nu+1)}\,{}_3F_2\left(\frac{\tilde{\epsilon}_{12}}{2},\frac{\tilde{\epsilon}_{34}}{2},-\frac{\epsilon_{12}}{2}-n;\frac{\tilde{\epsilon}_{34}}{2}-n,\nu+1;1\right)$$
$$=\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\Gamma(\tilde{\epsilon}_{12}/2+m)\Gamma(\tilde{\epsilon}_{34}/2+m)\Gamma(m-\epsilon_{12}/2-n)}{\Gamma(\tilde{\epsilon}_{34}/2-n+m)\Gamma(\nu+1+m)}. \tag{2.21}$$

By a transformation of ${}_3F_2$ [24], Eq. (2.21) can be written in a form symmetric under $\lambda\to\tilde{\lambda}$ and given by a terminating series

$$G_n=-\frac{1}{(\tilde{\epsilon}_{12}/2)(\epsilon_{12}/2)}\,{}_3F_2\left(\frac{\lambda_{12}+\lambda_{34}-d}{2},1,-n;1+\frac{\tilde{\epsilon}_{12}}{2},1+\frac{\epsilon_{12}}{2};1\right)$$
$$=-\frac{\Gamma(\tilde{\epsilon}_{12}/2)\Gamma(\epsilon_{12}/2)}{\Gamma[\lambda_{12}+\lambda_{34}-d/2]}\sum_{m=0}^{n}(-1)^m\frac{n!}{(n-m)!}\frac{\Gamma[(\lambda_{12}+\lambda_{34}-d)/2+m]}{\Gamma(1+\tilde{\epsilon}_{12}/2+m)\Gamma(1+\epsilon_{12}/2+m)}. \tag{2.22}$$

The contribution from the poles in series (3) can be obtained from Eqs. (2.19)–(2.21) by exchanging 1, 2, and 3, 4.

C. Coinciding poles

When $\epsilon_{12}/2$, $\epsilon_{34}/2$ or $(\lambda_{12}-\lambda_{34})/2$ become integers, the poles from different series in Eq. (2.16) may merge into double or triple poles. For example, when $(\lambda_{12}-\lambda_{34})/2$ is an integer, apart from a finite number of them, all poles in series two and three in Eq. (2.16) will merge into double poles, while the poles in the first series remain untouched. The contribution from a double pole is given by the derivative of the integrand of Eq. (2.15). The expressions are quite complicated and we do not explicitly write them down here. We simply note that there will be terms proportional to $\ln\xi$ as a result of $\partial\xi^{-s}/\partial s=-\ln\xi\xi^{-s}$. If all three parameters are integers, then apart from a finite number of simple and double poles all poles may merge into triple poles and their contributions are given by the second derivative of the integrand of Eq. (2.15). In these cases, among other things, we will have terms proportional to $(\ln\xi)^2$ from the second derivative of ξ^{-s}.

We caution that in a certain range of parameters, ${}_3F_2$ in Eq. (2.15) may develop zeros at the poles and the pole structure may be different from what we naively read from Eq. (2.15). This happens when $\epsilon_{12}/2$ or $\epsilon_{34}/2$ is a positive integer.[8] As an example let us take $\epsilon_{12}/2=k+1$ with $k\geqslant 0$ an integer. By a transformation [24] of generalized hypergeometric functions, ${}_3F_2$ in Eq. (2.15) can be rewritten as

$${}_3F_2\left(\frac{\tilde{\epsilon}_{12}}{2},\frac{\tilde{\epsilon}_{34}}{2},\frac{\lambda}{2}-s_1;\frac{\lambda_{12}+\tilde{\epsilon}_{34}}{2}-s_1,\nu+1;1\right)$$
$$=\frac{\Gamma[(\lambda_{12}+\tilde{\epsilon}_{34})/2-s_1]}{\Gamma(\lambda_{12}/2-s_1)\Gamma(1+\tilde{\epsilon}_{34}/2)}\,{}_3F_2\left(\frac{\tilde{\epsilon}_{34}}{2},1+\frac{\lambda-d}{2}+s_1,1-\frac{\epsilon_{12}}{2};1+\frac{\tilde{\epsilon}_{34}}{2},\nu+1;1\right). \tag{2.23}$$

Since $1-\epsilon_{12}/2=-k$, ${}_3F_2(\tilde{\epsilon}_{34}/2,1+(\lambda-d)/2+s_1,1-\epsilon_{12}/2;1+\tilde{\epsilon}_{34}/2,\nu+1;1)$ on the right-hand side of Eq. (2.23) is given by a terminating series,

---

[8]The following discussion is partly motivated by the results in [25], where simplifications in some expressions in this range of parameters have been observed. I would like to thank D. Freedman for correspondence regarding this issue.

$$
\begin{aligned}
&{}_3F_2\left(\frac{\tilde{\epsilon}_{34}}{2},1+\frac{\lambda-d}{2}+s_1,1-\frac{\epsilon_{12}}{2};1+\frac{\tilde{\epsilon}_{34}}{2},\nu+1;1\right)\\
&={}_3F_2\left(\frac{\tilde{\epsilon}_{34}}{2},1+\frac{\lambda-d}{2}+s_1,-k;1+\frac{\tilde{\epsilon}_{34}}{2},\nu+1;1\right)\\
&=\frac{\tilde{\epsilon}_{34}}{2}\frac{\Gamma(\nu+1)}{\Gamma[1+(\lambda-d)/2+s_1]}\sum_{m=0}^{k}\frac{(-1)^m k!}{m!(k-m)!}\frac{\Gamma[1+(\lambda-d)/2+s_1+m]}{(\tilde{\epsilon}_{34}/2+m)\Gamma(\nu+1+m)}
\end{aligned}
\tag{2.24}
$$

from which we can see that it is convergent and has no poles inside the contour in Eq. (2.15).

Plugging Eq. (2.23) into Eq. (2.15), we get

$$
S_\lambda=\frac{C}{\Gamma(1+\tilde{\epsilon}_{34}/2)}\frac{1}{2\pi i}\int_c ds_1\xi^{-s_1}\Gamma\left(\frac{\lambda_{34}}{2}-s_1\right)\Gamma\left(\frac{\lambda}{2}-s_1\right)H(\lambda,s_1)\,{}_3F_2\left(\frac{\tilde{\epsilon}_{34}}{2},1+\frac{\lambda-d}{2}+s_1,1-\frac{\epsilon_{12}}{2};1+\frac{\tilde{\epsilon}_{34}}{2},\nu+1;1\right),
\tag{2.25}
$$

where C is given below Eq. (2.15) and $H(\lambda,s_1)$ is defined by

$$
H(\lambda,s_1)=\frac{\Gamma(\Delta_{34}/2+s_1)\Gamma(\Delta_{43}/2+s_1)\Gamma(\Delta_{12}/2+s_1)\Gamma(\Delta_{21}/2+s_1)}{\Gamma(2s_1)}F\left(\frac{\Delta_{34}}{2}+s_1,\frac{\Delta_{12}}{2}+s_1;2s_1;1-\frac{\eta}{\xi}\right).
$$

Naively, we may expect from Eq. (2.15) that there are double poles at $s_1=\lambda_{12}/2+n$, $n=0,1,\ldots$. However, Eq. (2.25) indicates that they are actually simple poles. This result may also be seen indirectly from Eqs. (2.18) and (2.19)–(2.22): there is no singularity developed in either Eq. (2.18) or (2.19) when $\epsilon_{12}/2$ approaches a positive integer. In fact it can be checked that the residue of the integrand of Eq. (2.25) at a pole $s_1=\lambda_{12}/2+n$ is equal to the sum of Eqs. (2.18) and (2.19) at the corresponding pole. If further $\epsilon_{34}/2$ is an integer, then from Eq. (2.25) the pole at $s_1=\lambda/2+k+1+m=\lambda_{12}/2+m=\lambda_{34}/2+n$ (m and n non-negative integers) is a double pole instead of a triple pole. In particular, there are no terms proportional to $(\ln\xi)^2$ here [25]. A similar analysis can be applied when $\epsilon_{34}/2$ is a positive integer.

The appearance of the logarithm in coinciding pole cases can be summarized as follows:

(i) Only one of $\epsilon_{12}/2$, $\epsilon_{34}/2$ or $(\lambda_{12}-\lambda_{34})/2$ is an integer:

(ii) $(\lambda_{12}-\lambda_{34})/2$ is an integer: $\ln\xi$ associated with the double poles at $s_1=\lambda_{34}/2+n$,

(iii) $\epsilon_{12}/2$ or $\epsilon_{34}/2$ is a positive integer: all poles are simple poles, no logarithm,

(iv) $\epsilon_{12}/2$ or $\epsilon_{34}/2$ is zero or a negative integer: $\ln\xi$ associated with the double poles at $s_1=\lambda/2+n$.

(v) $\epsilon_{12}/2$, $\epsilon_{34}/2$, and $(\lambda_{12}-\lambda_{34})/2$ are all integers:

(vi) At least one of $\epsilon_{12}/2$ and $\epsilon_{34}/2$ is positive: except for a finite number of simple poles, all poles are double poles with $\ln\xi$.

(vii) $\epsilon_{12}/2$ and $\epsilon_{34}/2$ are zero or negative: except for a finite number of them, all poles are triple poles with $(\ln\xi)^2$.

III. SCATTERING AMPLITUDES FROM GENERIC VERTICES AND CONTACT TERMS

Scattering amplitudes from more complicated interaction vertices such as $\phi\partial_\mu\phi_1\partial^\mu\phi_2$ and $\phi D_\nu\partial_\mu\phi_1 D_\nu\partial^\mu\phi_2$ can be reduced to Eq. (2.2) and contact-type interactions by integration by part [17] or field redefinitions [14]. For example, the amplitude resulting from vertices $\phi\partial\phi_1\partial\phi_2$ and $\phi\phi_3\phi_4$ can be written as [$d\alpha$ and $d\beta$ denote the integration measures as in Eq. (2.2)]

$$
\begin{aligned}
&\int d\alpha\, d\beta\, \partial\mathcal{K}_1\,\partial\mathcal{K}_2\,G(x,y)\,\mathcal{K}_3\mathcal{K}_4\\
&=\frac{1}{2}\int d\alpha\, d\beta\,[\partial^2(\mathcal{K}_1\mathcal{K}_2)-\partial^2\mathcal{K}_1\mathcal{K}_2-\partial^2\mathcal{K}_2\mathcal{K}_1]\,G(x,y)\,\mathcal{K}_3\mathcal{K}_4\\
&=-\frac{1}{2}\int d\alpha\, d\beta\,\mathcal{K}_1\mathcal{K}_2\mathcal{K}_3\mathcal{K}_4+\frac{1}{2}(m^2-m_1^2-m_2^2)\int d\alpha\, d\beta\,\mathcal{K}_1\mathcal{K}_2\,G(x,y)\,\mathcal{K}_3\mathcal{K}_4.
\end{aligned}
\tag{3.1}
$$

Note the coefficient of the second term $1/2(m^2-m_1^2-m_2^2)$ is precisely the ratio between coefficients of $\langle\Phi_\lambda\Phi_{\lambda_1}\Phi_{\lambda_2}\rangle$ calculated from two types of interactions $\phi\partial\phi_1\partial\phi_2$ and $\phi\phi_1\phi_2$ [10].

In general we can consider the following Lagrangian of scalar fields:

$$\mathcal{L}=\frac{1}{2}(\partial\phi_i)^2+\frac{1}{2}m_i^2\phi_i^2+A_{ijk}^{(0)}\phi_i\phi_j\phi_k+A_{ijk}^{(1)}\phi_iD^\mu\phi_jD_\mu\phi_k+\cdots+A_{ijk}^{(n)}\phi_iD^{(n)}\phi_jD^{(n)}\phi_k, \tag{3.2}$$

where $D^{(m)}\phi_i$ is defined by

$$D^{(m)}\phi_i=D_{\{\mu_1}D_{\mu_2}\cdots D_{\mu_m\}}\phi_i. \tag{3.3}$$

The $\{\}$ in Eq. (3.3) denotes that the indices are symmetrized and traces are removed.[9] For the purpose of tree-level four-particle scattering we can eliminate those vertices with derivatives by a field redefinition

$$\phi_i=\phi_i'+B_{ijk}^{(0)}\phi_j'\phi_k'+\cdots+B_{ijk}^{(n-1)}D^{(n-1)}\phi_j'D^{(n-1)}\phi_k'. \tag{3.4}$$

B's in Eq. (3.4) can found by plugging Eq. (3.4) into Eq. (3.2) and setting to zero the coefficients of the cubic derivative vertices. The resulting Lagrangian can be written as

$$\mathcal{L}=\frac{1}{2}(\partial\phi_i')^2+\frac{1}{2}m_i^2(\phi_i')^2+\lambda_{ijk}\phi_i'\phi_j'\phi_k' + \text{contact vertices of quartic or higher order}. \tag{3.5}$$

For example for $n=2$ in AdS_{d+1}, B's can be found to be [14],

$$B_{ijk}^{(1)}=\frac{1}{2}A_{ijk}^{(2)},\quad B_{ijk}^{(0)}=\frac{1}{2}A_{ijk}^{(1)}+\frac{1}{4}A_{ijk}^{(2)}(m_i^2-m_j^2-m_k^2+2d)$$

and

$$\lambda_{ijk}=A_{ijk}^{(0)}+B_{ijk}^{(0)}(m_i^2-m_j^2-m_k^2)-\frac{2}{d+1}m_j^2m_k^2B_{ijk}^{(1)}. \tag{3.6}$$

Thus for generic interactions, the scattering amplitude can be written as

$$A_\lambda=S_\lambda+S^{(4)}, \tag{3.7}$$

where S_λ is given by Eq. (2.15) with normalized vertices (3.6) and $S^{(4)}$ is given by quartic vertices in Eq. (3.5).

[9]We can use $D^2\phi=m^2\phi+\cdots$ to reduce terms containing traces of indices to lower order terms. Similarly the commutators of derivatives $[D_\mu,D_\nu]\propto R$ also reduce to lower order terms, where R is the constant curvature.

Let us now look at the contribution from contact terms. We observe that by repeatedly using the identity $[J_{\mu\nu}(x)=\delta_{\mu\nu}-2x^\mu x^\nu/|x|^2]$,

$$\begin{aligned}D^\mu\mathcal{K}_{\lambda_i}(u,x_i)D_\mu\mathcal{K}_{\lambda_j}(u,x_j)&=c_{\lambda_i}c_{\lambda_j}u_0^2\partial_\mu\left(\frac{u_0}{|u-x_i|^2}\right)^{\lambda_i}\partial_\mu\left(\frac{u_0}{|u-x_j|^2}\right)^{\lambda_j}\\&=\lambda_i\lambda_j\mathcal{K}_{\lambda_i}(u,x_i)\mathcal{K}_{\lambda_j}(u,x_j)J_{\mu0}(u-x_i)J_{\mu0}(u-x_j)\\&=\lambda_i\lambda_j\mathcal{K}_{\lambda_i}(u,x_i)\mathcal{K}_{\lambda_j}(u,x_j)\\&\quad-2\nu_i\nu_jx_{ij}^2\mathcal{K}_{\lambda_i+1}(u,x_i)\mathcal{K}_{\lambda_j+1}(u,x_j),\end{aligned} \tag{3.8}$$

and $D^2\mathcal{K}_{\lambda_i}=m_i^2\mathcal{K}_{\lambda_i}$ we can put a generic quartic contribution into a sum of terms without derivatives,

$$\begin{aligned}S_c&=\int\frac{du_0d^du}{u_0^{d+1}}\mathcal{K}_{\lambda_1}(u,x_1)\mathcal{K}_{\lambda_2}(u,x_2)\mathcal{K}_{\lambda_3}(u,x_3)\mathcal{K}_{\lambda_4}(u,x_4)\\&=\Pi_ic_{\lambda_i}\int\frac{du_0d^du}{u_0^{d+1}}\left(\frac{u_0}{|u-x_1|^2}\right)^{\lambda_1}\left(\frac{u_0}{|u-x_2|^2}\right)^{\lambda_2}\\&\quad\times\left(\frac{u_0}{|u-x_3|^2}\right)^{\lambda_3}\left(\frac{u_0}{|u-x_4|^2}\right)^{\lambda_4}.\end{aligned} \tag{3.9}$$

Thus it is enough to look at Eq. (3.9).

Contact contributions (3.9) in AdS_{d+1} have been discussed before in [8] and [18] (see also [20]). In particular in [18] it was pointed out that when $\lambda_{12}=\lambda_{34}$ the leading term in the short distance limit $x_{12}, x_{34}\to 0$ is given by a logarithmic contribution. Here we give a more thorough analysis of the analytic properties of Eq. (3.9), presenting the result in a way suitable for our later discussion of its CFT_d interpretation.

In Appendix C, we show that similarly to the exchange amplitude, the contact contribution (3.9) can also be written as an inverse Mellin integral,

$$\begin{aligned}S_c=C_c\frac{1}{2\pi i}\int_C ds\,\xi^{-s}\Gamma\left(\frac{\lambda_{12}}{2}-s\right)\Gamma\left(\frac{\lambda_{34}}{2}-s\right)\\\times F\left(\frac{\Delta_{34}}{2}+s,\frac{\Delta_{12}}{2}+s;2s;1-\frac{\eta}{\xi}\right)\\\times\frac{\Gamma(\Delta_{34}/2+s)\Gamma(\Delta_{43}/2+s)\Gamma(\Delta_{12}/2+s)\Gamma(\Delta_{21}/2+s)}{\Gamma(2s)}\end{aligned} \tag{3.10}$$

with

$$C_c = \frac{1}{2\pi^{3d/2}} \frac{\Gamma[(\lambda_{12}+\lambda_{34}-d)/2]}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)} \times \frac{1}{|x_{12}|^{\lambda_{12}}|x_{14}|^{\Delta_{12}}|x_{24}|^{\Delta_{21}-\Delta_{34}}|x_{23}|^{\Delta_{34}}|x_{34}|^{\lambda_{34}}},$$

where the integration path $\mathcal{C}$ should be understood in the same sense as that in Eq. (2.9) [see the remark below Eq. (2.9)]. Thus in the s-channel limit $\eta, \xi \gg 1$, Eq. (3.10) can be written as

$$S_c = \sum_{n=0}^{\infty} S_{cn}^{(\lambda_{12})} + \sum_{n=0}^{\infty} S_{cn}^{(\lambda_{34})},$$

where

$$S_{cn}^{(\lambda_{12})} = 4A^{(\lambda_{12})}\frac{(-1)^n}{n!}\xi^{-n}F\left(\frac{\lambda_{12}+\Delta_{34}}{2}+n,\lambda_1+n;\lambda_{12}+2n;1-\frac{\eta}{\xi}\right) \times\Gamma\left(\frac{\lambda_{34}-\lambda_{12}}{2}-n\right)\frac{\Gamma[(\lambda_{12}+\Delta_{34})/2+n]\Gamma[(\lambda_{12}-\Delta_{34})/2+n]\Gamma(\lambda_1+n)\Gamma(\lambda_2+n)}{\Gamma(\lambda_{12}+2n)}, \tag{3.11}$$

and $S_{cn}^{(\lambda_{34})}$ can be obtained from Eq. (3.11) by taking 1, 2 $\to$3,4. Note $A^{(\lambda_{12})}$ in the above is given by Eq. (2.20) and except for the extra G_n in Eq. (2.19), Eq. (3.11) is almost identical to Eq. (2.19).

When $(\lambda_{12}-\lambda_{34})/2$ is an integer, except for a finite number of poles, the two ascending simple-pole sequences of the integrand in Eq. (3.10) will merge into a double-pole sequence. Again as in the case of exchange amplitude, the double-pole contribution will contain $\ln\xi$. In particular, when $\lambda_{12}=\lambda_{34}$ all ascending poles become double poles and the leading contribution contains a $\ln\xi$.

Note that since Eq. (3.9) is symmetric under exchanges of its four boundary propagators, its expansion in the u-channel limit x_{13}, $x_{24}\to 0$ can be obtained by exchanging 2 and 3 in Eqs. (3.10) and (3.11) and $\xi\to\xi/\eta$ and $\eta\to 1/\eta$.

IV. FOUR-POINT FUNCTIONS AND CONFORMAL PARTIAL WAVE EXPANSION IN CFT

To seek a CFT_d interpretation of the AdS_{d+1} amplitudes discussed in the last two sections, in this section we review the conformal partial wave expansion (CPWE) approach to the calculation of four-point functions in CFT [26,27] (for a review see [28,29], see also [30] for some recent discussions[10]).

In CFT_d, the states generated by acting by a product of the conformal operators on the vacuum can be decomposed into a direct sum of irreducible representations of the conformal group

$$\Phi_1(x_1)\Phi_2(x_2)|0\rangle = \sum_k \int d^dx\, Q_{12k}(x|x_1,x_2)|k,x\rangle, \tag{4.1}$$

where k sums over all the irreducible representations in the Hilbert space and states $|k,x\rangle=\Phi_k(x)|0\rangle$ span the space of an irreducible representation of the conformal group. Equation (4.1) can be further lifted into an operator equation,

$$\Phi_1(x_1)\Phi_2(x_2) = \sum_k \int d^dx\, Q_{12k}(x|x_1,x_2)\Phi_k(x), \tag{4.2}$$

understood as a relation between correlation functions. The summation in Eq. (4.2) is over primary fields (nonderivatives) only and the integration over all space effectively incorporated the contribution of their SO(d,2) descendants (fields with derivatives). The short-distance OPE can be obtained from Eq. (4.2) in small $|x_{12}|$ limit by expanding the integrand in terms of x_1-x_2. When Φ's are orthogonal to each other, it can be seen from Eq. (4.1) that the Q's are given by the amputated three-point functions.

Applying Eq. (4.1) to a four-point function we find

$$\begin{aligned} W_{1234}(x_1,x_2,x_3,x_4) &= \langle 0|\Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\Phi_4(x_4)|0\rangle \\ &= \sum_k \int d^dx\, d^dy\, Q_{12k}(x_1,x_2|x) \\ &\quad \times W_k(x-y)\, Q_{k34}(y|x_3,x_4), \end{aligned} \tag{4.3}$$

where $W_k(x-y)=\langle 0|\Phi_k(x)\Phi_k(y)|0\rangle$.

In the following, we shall look at the contribution of an intermediate scalar operator Φ_λ (with dimension λ) to the four-point function of four scalar operators Φ_{λ_i}, $i=1,\ldots,4$ (with dimensions λ_i respectively),

$$S_\lambda = \int d^dx\, d^dy\, Q_{\lambda\lambda_1\lambda_2}(x_1,x_2|x)W_\lambda(x-y)Q_{\lambda\lambda_3\lambda_4}(y|x_3,x_4). \tag{4.4}$$

In the Euclidean signature, the two- and three-point functions are given by

[10]I would like to thank A. Petkou for bringing these references to my attention.

$$G_\lambda(x-y)=\langle\Phi_\lambda(x)\Phi_\lambda(y)\rangle=\frac{c}{|x-y|^{2\lambda}},$$

$$G_{\lambda\lambda_1\lambda_2}(x,x_1,x_2)=\langle\Phi_\lambda(x)\Phi_{\lambda_1}(x_1)\Phi_{\lambda_2}(x_2)\rangle$$
$$=f_{\lambda\lambda_1\lambda_2}A_{\lambda\lambda_1\lambda_2}(x,x_1,x_2),$$

$$G_{\lambda\lambda_3\lambda_4}(x,x_3,x_4)=\langle\Phi_\lambda(x)\Phi_{\lambda_3}(x_3)\Phi_{\lambda_4}(x_4)\rangle$$
$$=f_{\lambda\lambda_3\lambda_4}A_{\lambda\lambda_3\lambda_4}(x,x_3,x_4),$$

where function $A_{abc}(x,y,z)$ is defined by

$$A_{abc}(x,y,z)=\frac{1}{|x-y|^{a+b-c}|z-y|^{c+b-a}|x-z|^{a+c-b}}.$$

The normalization constants c and f will be taken to be those given by AdS calculations [10],[11] i.e.,

$$c=\frac{\Gamma(\lambda)}{\pi^{d/2}\Gamma(\nu)}(2\lambda-d),$$

$$f_{\lambda\lambda_1\lambda_2}=\frac{\Gamma(\epsilon_{12}/2)\Gamma(\tilde{\epsilon}_{12}/2)\Gamma(\delta_1/2)\Gamma(\delta_2/2)}{2\pi^d\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu)}, \tag{4.5}$$

and a similar expression for $f_{\lambda\lambda_3\lambda_4}$ obtained from $f_{\lambda\lambda_1\lambda_2}$ by taking 1, $2\to3,4$.

In the Minkowski signature, due to the spectrality condition, it is more convenient to work in momentum space, where the two- and three-point functions are given by

$$W(p)=-i\,\mathrm{Disc}\,G(p)|_{p^d=-ip^0}$$
$$=\frac{2\pi}{\Gamma(1+\nu)\Gamma(-\nu)}\theta(p^0)\theta(-p^2_{\mathrm{Min}})G(p)|_{p^d=-ip^0}, \tag{4.6}$$

$$W(p|x_1,x_2)=-i\,\mathrm{Disc}(p|x_1,x_2)|_{p^d=-ip^0}, \tag{4.7}$$

where $G(p)$ and $G(p|x_1,x_2)$ are Euclidean two- and three-point functions in momentum space

$$G(p)=c\int d^dy\, e^{-i\mathbf{p}\cdot\mathbf{y}}\frac{1}{|y|^{2\lambda}}=c\,\frac{\pi^{d/2}\Gamma(-\nu)}{2^{2\nu}\Gamma(\lambda)}p^{2\nu},$$

$$G_{\lambda\lambda_1\lambda_2}(p|x_1,x_2)=\int d^dy e^{-ip\cdot y}G_{\lambda\lambda_1\lambda_2}(y,x_1,x_2)=f_{\lambda\lambda_1\lambda_2}\frac{2\pi^{d/2}}{\Gamma(\delta_1/2)\Gamma(\delta_2/2)}\frac{1}{x_{12}^{\epsilon_{12}}}\left(\frac{p^2}{4x_{12}}\right)^{\nu/2}$$
$$\times\int_0^1 du\, u^{\Delta_{12}/2+d/4-1}(1-u)^{\Delta_{21}/2+d/4-1}e^{-ip\cdot[ux_1+(1-u)x_2]}K_\nu[\sqrt{u(1-u)p^2x_{12}^2}]. \tag{4.8}$$

The amputated three-point function Q can then be found to be

$$Q_{\lambda\lambda_1\lambda_2}(p|x_1,x_2)=W_\lambda^{-1}(p)W_{\lambda\lambda_1\lambda_2}(p|x_1,x_2)=c^{-1}f_{\lambda\lambda_1\lambda_2}\frac{2^\nu\Gamma(\lambda)\Gamma(1+\nu)}{\Gamma(\delta_1/2)\Gamma(\delta_2/2)}\frac{p^{-\nu}}{x_{12}^{\lambda_{12}-d/2}}$$
$$\times\int_0^1 du\, u^{\Delta_{12}/2+d/4-1}(1-u)^{\Delta_{21}/2+d/4-1}e^{-ip\cdot[ux_1+(1-u)x_2]}I_\nu[\sqrt{u(1-u)p^2x_{12}^2}]. \tag{4.9}$$

In Eqs. (4.8) and (4.9) K_ν and I_ν are modified Bessel functions.

Plugging Eqs. (4.6) and (4.9) into Eq. (4.4), we have, in momentum space with the Minkowskian signature,

$$W_\lambda=\frac{1}{2\pi^d}\int d^dp\,Q^*_{\lambda\lambda_1\lambda_2}(p|x_1,x_2)W_\lambda(p)Q_{\lambda\lambda_3\lambda_4}(p|x_3,x_4). \tag{4.10}$$

The integrals in Eq. (4.10) were explicitly computed in [26] and the result can be written as an inverse Mellin integral,

[11] Here the normalization for three-point functions is given by the interaction vertex $\phi_1\phi_2\phi_3$ in AdS_{d+1}. When considering more complicated vertices, an additional normalization factor (3.6) should be taken into account.

$$W_\lambda = c^{-1} f_{\lambda\lambda_1\lambda_2} f_{\lambda\lambda_3\lambda_4} \frac{\Gamma(\lambda)\Gamma(1+\nu)}{\Gamma(\delta_1/2)\Gamma(\delta_2/2)\Gamma(\delta_3/2)\Gamma(\delta_4/2)} \frac{1}{|x_{12}|^{\epsilon_{12}}|x_{14}|^{\delta_1}|x_{24}|^{\Delta_{21}-\Delta_{34}}|x_{23}|^{\delta_3}|x_{34}|^{\epsilon_{34}}}$$
$$\times \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} ds(-\xi)^{-s}\Gamma(-s)\frac{\Gamma(\delta_1/2+s)\Gamma(\delta_2/2+s)\Gamma(\delta_3/2+s)\Gamma(\delta_4/2+s)}{\Gamma(\lambda+2s)\Gamma(\nu+s+1)} F\left(\frac{\delta_3}{2}+s, \frac{\delta_1}{2}+s;\lambda+2s;1-\frac{\eta}{\xi}\right), \tag{4.11}$$

where ξ and η are cross ratios defined in Eq. (2.10) and the Mellin integral should be understood in the same sense as the ones in the previous sections. Again when $\eta\,\xi>1$, Eq. (4.11) can be written as an expansion:

$$W_\lambda = \frac{1}{8\pi^{3/2d}} \frac{\Gamma(\epsilon_{12}/2)\Gamma(\epsilon_{34}/2)\Gamma(\tilde{\epsilon}_{12}/2)\Gamma(\tilde{\epsilon}_{34}/2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)} \frac{1}{|x_{12}|^{\epsilon_{12}}|x_{14}|^{\delta_1}|x_{24}|^{\Delta_{21}-\Delta_{34}}|x_{23}|^{\delta_3}|x_{34}|^{\epsilon_{34}}}$$
$$\times \sum_{n=0}^{\infty}\frac{1}{n!}\xi^{-n}\frac{\Gamma(\delta_1/2+n)\Gamma(\delta_2/2+n)\Gamma(\delta_3/2+n)\Gamma(\delta_4/2+n)}{\Gamma(\lambda+2n)\Gamma(\nu+n+1)} F\left(\frac{\delta_3}{2}+n, \frac{\delta_1}{2}+n;\lambda+2n;1-\frac{\eta}{\xi}\right). \tag{4.12}$$

We notice that Eq. (4.12) agrees precisely with Eq. (2.18) including the numerical coefficient.

V. CFT$_d$ INTERPRETATION OF AdS$_{d+1}$ AMPLITUDES

In previous sections, we have managed to express all our results as inverse Mellin integrals and when $\xi,\eta>1$ write them in terms of inverse power series of ξ, η as a sum of residues of the integrand. When the pole sequences in Eqs. (2.16) and (3.10) do not coincide with one another, in all cases [see Eqs. (2.18), (2.19), and (3.11)] the contribution from a pole sequence can be written in a similar pattern as the CPWE expression (4.12)

$$\sum_{n=0}^{\infty} a_n H(\Lambda+2n), \tag{5.1}$$

where each term in the summation is given by the residue at the pole $1/2+n$. In Eq. (5.1), a_n are numerical coefficients and H is a function defined by

$$H(\alpha) = \frac{1}{|x_{12}|^{\lambda_{12}-\alpha}|x_{14}|^{\alpha+\Delta_{12}}|x_{24}|^{\Delta_{21}-\Delta_{34}}|x_{23}|^{\alpha+\Delta_{34}}|x_{34}|^{\lambda_{34}-\alpha}}$$
$$\times \frac{\Gamma(\Delta_{12}/2+\alpha/2)\Gamma(\Delta_{21}/2+\alpha/2)\Gamma(\Delta_{34}/2+\alpha/2)\Gamma(\Delta_{43}/2+\alpha/2)}{\Gamma(\alpha)} F\left(\frac{\Delta_{34}}{2}+\frac{\alpha}{2}, \frac{\Delta_{12}}{2}+\frac{\alpha}{2};\alpha;1-\frac{\eta}{\xi}\right). \tag{5.2}$$

In Eq. (4.12),

$$a_n = \frac{1}{8\pi^{(3/2)d}} \frac{\Gamma[(\lambda_{12}-\Lambda)/2]\Gamma[(\lambda_{34}-\Lambda)/2]\Gamma[(\lambda_{12}+\Lambda-d)/2]\Gamma[(\lambda_{34}+\Lambda-d)/2]}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)} \frac{1}{\Gamma(n+1)\Gamma(\Lambda-d/2+n+1)}. \tag{5.3}$$

The contact amplitude (3.9) may be written as

$$S_c = S_c^{(\lambda_{12})} + S_c^{(\lambda_{34})}, \tag{5.4}$$

where $S_c^{(\lambda_{12})}$, $S_c^{(\lambda_{34})}$ are of the form of Eq. (5.1) with $\Lambda=\lambda_{12}$, λ_{34}. For $S_c^{(\lambda_{12})}$,

$$a_n = \frac{1}{2\pi^{3d/2}} \frac{\Gamma[(\lambda_{12}+\lambda_{34}-d)/2]}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)} \frac{(-1)^n}{\Gamma(n+1)}\Gamma\left(\frac{\lambda_{34}-\lambda_{12}}{2}-n\right) \tag{5.5}$$

with those of $S_c^{(\lambda_{34})}$ given by Eq. (5.5) with $\lambda_{12}\leftrightarrow\lambda_{34}$. Similarly, the exchange amplitude (2.2) can be written in terms of Eq. (5.1) with $\Lambda=\lambda$, λ_{12}, λ_{34},

$$S_{ex} = S^{(\lambda)} + S^{(\lambda_{12})} + S^{(\lambda_{34})}, \tag{5.6}$$

where $a_n^{(\lambda)}$ are the same as those of CPWE (5.3) and

$$a_n^{(\lambda_{12})}=\frac{1}{8\pi^{3d/2}}\frac{\Gamma[(\lambda_{12}+\lambda_{34}-d)/2]}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)}\frac{(-1)^n}{\Gamma(n+1)}\Gamma\left(\frac{\lambda_{34}-\lambda_{12}}{2}-n\right)G_n \tag{5.7}$$

with G_n given by Eq. (2.21) or (2.22). $a_n^{(\lambda_{34})}$ may be obtained from Eq. (5.7) with 1, 2 and 3, 4 exchanged.

To have a more precise picture of what we have found so far, we would like to understand the physical meaning of the poles and the residues of each pole. For this purpose, let us go back to the contribution of a primary operator Φ_Λ to the operator algebra (4.2),

$$\Phi_{\lambda_1}(x_1)\Phi_{\lambda_2}(x_2)=\int d^dx\, Q_{\Lambda\lambda_1\lambda_2}(x|x_1,x_2)\Phi_\Lambda(x), \tag{5.8}$$

where

$$Q_{\Lambda\lambda_1\lambda_2}(x|x_1,x_2)=\int d^dp\, e^{ip\cdot x}Q_{\Lambda\lambda_1\lambda_2}(p|x_1,x_2), \tag{5.9}$$

and $Q_{\Lambda\lambda_1\lambda_2}(p|x_1,x_2)$ is given by Eq. (4.9). For simplicity we will look at the analytic continuation of Eq. (4.9) to Euclidean space.[12] Plugging Eqs. (4.9) and (5.9) into Eq. (5.8), we find that [31]

$$\begin{aligned}\Phi_{\lambda_1}(x_1)\Phi_{\lambda_1}(x_2)&=\sum_{n=0}^{\infty} b_n\frac{1}{x_{12}^{\lambda_{12}-\Lambda-2n}}\int_0^1 du\, u^{\delta_1/2+n-1}(1-u)^{\delta_2/2+n-1}e^{ux_{12}\cdot\partial}\Phi_\Lambda^{(n)}(x_2)\\&=\sum_{n=0}^{\infty} b_nF_n,\end{aligned} \tag{5.10}$$

where b_n are numerical factors and we have defined operators,

$$\Phi_\Lambda^{(n)}(x_2)=(\partial^2)^n\Phi_\Lambda(x_2), \quad n=0,1,2,\ldots, \tag{5.11}$$

which we will call subprimary operators ($n=0$ is primary). To reach Eq. (5.10), we have used the series expansion for the Bessel function. F_n in Eq. (5.10) denotes the contribution to OPE from a subprimary operator $\Phi_\Lambda^{(n)}$ and its diagonal descendants. By diagonal descendants we mean the states in weight diagram diagonally generated from a subprimary (see Fig. 5). We may now interpret that each pole in Eq. (4.11) represents the dimension of a subprimary and the residue at the pole corresponds to the contribution of the subprimary and its diagonal descendants.

By comparing Eqs. (5.6) and (5.4) with CPWE (5.1) and (5.3), we see that for the case of noncoinciding poles:

(i) $S^{(\lambda)}$ in exchange amplitude (5.6), which arises from the first pole sequence $\lambda/2+n$ in Eq. (2.16) agrees precisely with the result from conformal partial wave expansion (4.12), including the overall numerical coefficient. This indicates that $S^{(\lambda)}$ has a CFT$_d$ interpretation in terms of exchange of a primary operator of dimension λ.

[12]This is not completely the right thing to do, because the Minkowski signature integration over momenta involves the spectrality conditions $p_0>0$, $p^2<0$, in which case the expressions are rather complicated. For illustrative purpose, we shall use a Euclidean expression.

(ii) In both exchange and contact amplitudes, the contributions from a pole at $\lambda_{12}/2+n$ (or $\lambda_{34}/2+n$) agree exactly with that of a subprimary operator with dimension $\lambda_{12}+2n$ ($\lambda_{34}+2n$) and its diagonal descendants. But the numerical coefficients (5.7) and (5.5) at each pole are different from that predicted by CPWE (5.3), in particular, in the exchange amplitude the coefficient involves a somewhat complicated factor G_n Eq. (2.21).

The above identification of Eqs. (5.4) and (5.6) in terms

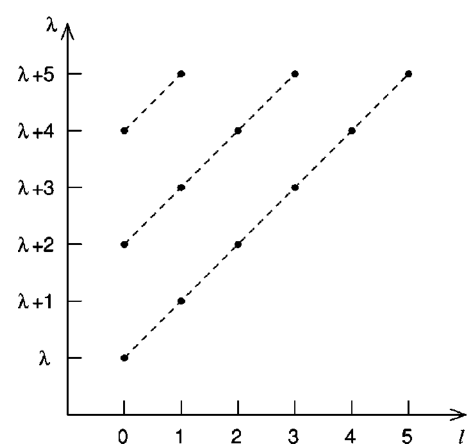

FIG. 5. Weight diagram for SO(d,2) representation $D(\lambda,0)$. The horizontal axis l represents the set of indices for SO(d) subgroup, and the perpendicular axis represents the scale dimension. A subprimary is a state lying on the line with $l=0$. The states connected to a subprimary by dotted lines are descendants diagonally generated from the subprimary.

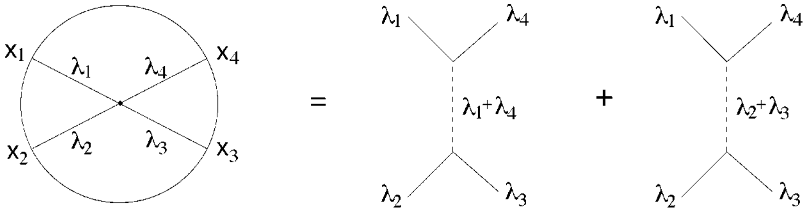

FIG. 6. *t*-channel OPE interpretation of a contact diagram.

of exchanges of subprimary operators also cast light on the conditions (1.4) and (1.5) for the occurrence of logarithms, which are satisfied precisely when subprimaries of different operators become degenerate.

For example, consider a contact diagram with $(\lambda_{12}-\lambda_{34})/2=k$ and k a positive integer. The dimension $\lambda_{12}+2n$ of a subprimary operator $O_{12}^{(n)}$ of O_{12} will then be the same as that of the subprimary $O_{34}^{(k+n)}$. In addition, the quantum numbers of all the diagonal descendants generated from $O_{12}^{(n)}$ and $O_{34}^{(k+n)}$ will be identical (see Fig. 5). To see this more explicitly, let us move slightly off the degeneracy point, i.e., consider, $\lambda_{12}=\lambda_{34}+2k+2\epsilon$ where $0<\epsilon<1$. Then Eq. (5.5) may be written as (below we will omit the overall constant),

$$a_n^{12}=(-1)^k\frac{\pi}{\sin\epsilon\pi}\,\frac{1}{\Gamma(n+1)\Gamma(1+n+k+\epsilon)},$$

where we have used $\Gamma(x)\Gamma(1-x)=\pi/\sin x\pi$. Equation (5.4) may be written as

$$S_c=\sum_{n=0}^{k-1}\frac{(-1)^n}{n!}\Gamma(k-n+\epsilon)H(\lambda_{34}+2n)+(-1)^k\frac{\pi}{\sin\epsilon\pi}\sum_{m=0}^{\infty}\left[\frac{H(\lambda_{34}+2m+2k+2\epsilon)}{\Gamma(m+1)\Gamma(1+m+k+\epsilon)}-\frac{H(\lambda_{34}+2m+2k)}{\Gamma(m+1-\epsilon)\Gamma(1+m+k)}\right]. \tag{5.12}$$

As we take ϵ to zero, the conflicting contributions from $O_{12}^{(m)}$ and $O_{34}^{(m+k)}$ in the square bracket become degenerate and the result is given by their derivatives over ϵ, which contain logarithms. The above discussion suggests that by turning on a very small ϵ at degeneracy points, Eq. (5.12) provides a useful way to ‘‘regularize’’ logarithms.

Since a contact diagram is symmetric under exchanging its external legs, its *t*- or *u*-channel expansion can be simply obtained by taking $2\leftrightarrow4$ or $2\leftrightarrow3$ in Eqs. (5.4) and (5.5). For example, it can be represented as a *t*-channel exchange in CFT_d as in Fig. 6.

VI. DISCUSSIONS

We note that conformal symmetry imposes strong restrictions on the coefficients C_{ij}^k in Eq. (1.1); that of the primary determines those of their descendants. The structure of Eqs. (5.10) and (4.12), including the numerical coefficients a_n in Eq. (5.3) and b_n in Eq. (5.10) in the summation, are uniquely fixed up to an overall constant by the fact that Φ_Λ and its derivatives fill an irreducible representation of the conformal group. Although we have found in Eqs. (5.4) and (5.6) the contributions from a complete set of subprimary operators of dimensions λ_{12} and λ_{34}, their relative OPE coefficients (1.1) are not consistent with those required by conformal symmetry, in other words, these subprimaries do not seem to fill the same irreducible multiplets.

It is probably not surprising that we do not find a complete CFT_d identification in Eqs. (5.4) and (5.6). After all, we are only looking at a generic diagram in AdS_{d+1}, which hardly makes too much sense before we specify a particular theory and add up all the diagrams contributing to a realistic amplitude. The encouraging message seems to be that we are indeed able to find a relation between an arbitrary scattering diagram and OPE, which indicates some kind of universality between a theory in AdS_{d+1} and CFT_d.

It is not clear at the present time how much we see here will survive in the final expression of a realistic amplitude, in particular, whether operators O_{12} and O_{34} will have a consistent CFT_d interpretation when we add up all the diagrams. Let us now consider what these operators could be if their contributions do survive in the final expression. A clue comes from the consideration of free theory, where the operator product expansion takes the form (see, e.g., [32]),

$$A(z)B(w)=\overbrace{A(z)B(w)}+:A(z)B(w):, \tag{6.1}$$

where the first term on the right-hand side denotes a contraction and $:A(z)B(w):$ stands for a normal-ordered operator whose explicit form can be obtained from a Taylor expansion,

$$:A(z)B(w):=\sum_{k=0}^{\infty}\frac{(z-w)^k}{k!}(\partial^k AB)(w).$$

In free theory the conformal dimension for $:A(z)B(w):$ is just $\lambda_A+\lambda_B$. Thus naturally we may expect that O_{12} and O_{34} should be the counterparts of $:O_{\lambda_1}O_{\lambda_2}:$ and $:O_{\lambda_3}O_{\lambda_4}:$ in interacting theory. In $\mathcal{N}=4$ super-Yang-Mills theory with gauge group SU(N), O_{12} and O_{34} may be interpreted as double-trace operators, i.e., operators of type $\mathrm{Tr}\,F^2\,\mathrm{Tr}\,F^2(x)$. Since we do not see a continuous spectrum of dimensions in Eqs. (5.6) and (5.4), we would expect O_{12} and O_{34} to correspond in AdS_{d+1} to two-particle bound states of supergravity/string theory. This is consistent with the expectation [33] that, to lowest order in $1/N$, there cannot be any two-particle cut in Yang-Mills four-point functions.

Finally, we note that since Eqs. (5.5) and (5.7) involve dimensions of other operators (e.g., λ and λ_{34}), it suggests a possibility that the mismatch in our OPE identification may be due to certain underlying mixing and interaction between different operators at the subprimary level. Moreover, when $\epsilon\to 0$, the pattern indicated in Eq. (5.12) for the degeneracy of subprimaries strongly reminds us of the behavior of a two-level system. Similarly, by examining the exchange amplitudes (5.6) and (5.5), we also find that the pattern near the degeneracy points is rather like a three-state system.

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APPENDIX A: MELLIN TRANSFORMATION AND ANALYTIC CONTINUATION

Here we give a brief introduction to the Mellin transformation and how to use it to evaluate integrals.[13]

The Mellin transform of a function $g(x)$ is

$$h(s)=\int_0^\infty dx\, g(x)x^{s-1}. \tag{A1}$$

If the set of convergence of the integral (A1) has a nonempty interior $\alpha<\mathrm{Re}(s)<\beta$ and $h(s)$ is analytic in this strip, we can have the inverse Mellin transformation,

$$g(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}h(s)x^{-s}ds \tag{A2}$$

for all c such that $\alpha<c<\beta$.

Some well-known integral representations of higher transcendental functions can be interpreted as (inverse) Mellin transforms, for example,

$$\Gamma(s)=\int_0^\infty dx\, x^{s-1}e^{-x},\quad \Gamma(s)\zeta(s)=\int_0^\infty dx\, x^{s-1}(e^x-1)^{-1},$$

and the Mellin-Barnes representation of a hypergeometric function,

$$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\times ds\,\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)}\Gamma(-s)(-z)^s. \tag{A3}$$

Taking $b=c$ in Eq. (A3) we get

$$F(a,b;b;z)=(1-z)^{-a}=\frac{1}{\Gamma(a)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\times ds\,\Gamma(a+s)\Gamma(-s)(-z)^s. \tag{A4}$$

Generally, to evaluate an integral

$$I=\int dx\, g(x)f(x), \tag{A5}$$

we can first plug into Eq. (A5) the inverse Mellin transform (A2) of $g(x)$, then do the x integral and finally inverse-Mellin transform back

$$I=\int_{c-i\infty}^{c+i\infty}ds\, h(s)J(s),\quad J(s)=\int dx\, x^s f(x). \tag{A6}$$

Normally, Mellin transform (A1), (A2) is not as convenient as Fourier or Laplace transform as it requires the functions to be transformed have reasonably ‘‘good behaviors’’ both at zero and infinity to ensure the existence of the strip where the transform can be defined. But for those functions g in Eq. (A1) which have a convenient power series expansion (such as hypergeometric functions) Mellin representation is more powerful since the indices and the coefficients of the expansion are represented by the poles and the corresponding residues of $h(s)$ in the complex s plane.

Let us look at a simple example,

$$I=\int_0^\infty dx x^{\nu-1}(1+x)^{-\mu}(x+t)^{-\rho}. \tag{A7}$$

The integral is defined when $\mathrm{Re}(\nu)>0$, $\mathrm{Re}(\mu+\rho-\nu)>0$, and $\arg(t)<\pi$ which we assume is the case. For convenience we will also take $|t|<1$. The result of Eq. (A7) is well known, given by a hypergeometric function,

$$I=t^{\nu-\rho}B(\nu,\mu-\nu+\rho)F(\mu,\nu;\mu+\rho;1-t), \tag{A8}$$

where $B(\nu,\mu-\nu+\rho)$ is a β function.

Here we would like to reproduce the result by using the Mellin transformation technique of Eq. (A6). For the moment we first assume $\mathrm{Re}(\rho)>0$. Note that from Eq. (A4),

$$(x+t)^{-\rho}=\frac{1}{\Gamma(\rho)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}ds\,\Gamma(\rho+s)\Gamma(-s)t^s x^{-\rho-s}, \tag{A9}$$

where $-\mathrm{Re}(\rho)<c<0$. Plugging the above expression into Eq. (A7), we get

[13] I would like to thank T. W. B. Kibble, F. G. Leppington, and A. B. Zamolodchikov for discussions related to the content of this appendix.

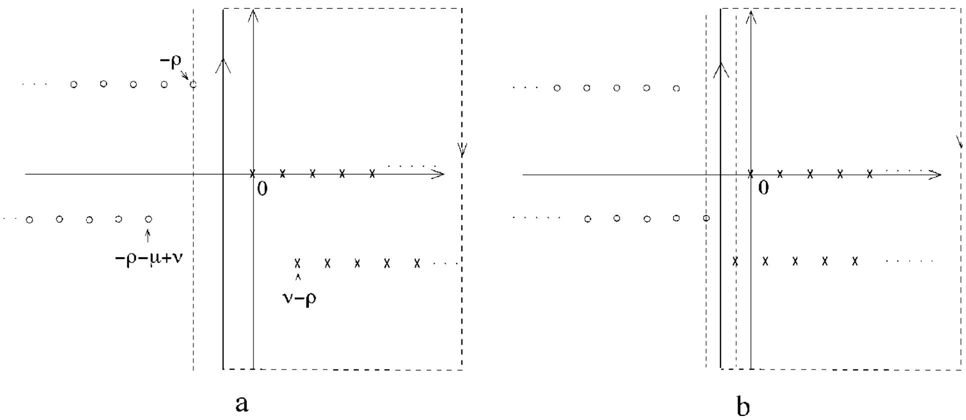

FIG. 7. Pole structures and contours. (a) Re(μ)>Re(ν)>Re(ρ)>0; (b) Re(ρ)>Re(ν)>Re(μ)>0.

$$I=\frac{1}{\Gamma(\rho)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}ds\,\Gamma(\rho+s)\Gamma(-s)t^s \times\int_0^\infty dx\,x^{\nu-\rho-s-1}(1+x)^{-\mu}$$
$$=\frac{1}{\Gamma(\rho)\Gamma(\mu)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}ds\,t^s\Gamma(\rho+s)\Gamma(-s) \times\Gamma(\nu-\rho-s)\Gamma(\mu+\rho-\nu+s), \tag{A10}$$

and the convergence of the x integral requires

$$-\mathrm{Re}(\mu+\rho-\nu)<\mathrm{Re}(s)<\mathrm{Re}(\nu-\rho). \tag{A11}$$

Since the x integral in Eq. (A10) has generated new pole sequences in the complex s plane, we have to check that they will not cause any ambiguity in carrying the inverse-Mellin integral in the second line of Eq. (A10).

1. Re(μ)>Re(ν)>Re(ρ)>0

In this case, it is easy to check that there is no overlap between descending and ascending pole sequences and the new pole sequences sit outside the strip $-\mathrm{Re}(\rho)<c<0$ [which means the convergence condition (A11) is trivially satisfied], where the inverse Mellin transform (A9) is defined. Thus there is no ambiguity in defining the integral in Eq. (A10) and we can take the integral around a contour C, which consists of the path in Eq. (A10) and encloses the right half plane. See Fig. 7(a) for the pole structure and the contour. Since $|t|<1$ the contribution from part of the contour other than Eq. (A10) vanishes and Eq. (A10) can be written by the calculus of residues as the sum of the residues of the integrand at the poles $s=0,1,\ldots$ and $s=\nu-\rho+n$, $n=0,1,\ldots$. It is easy to see the sum gives us Eq. (A8).

2. Re(μ), Re(ν), Re(ρ)>0

In this case, there is still no overlap between descending and ascending pole sequences, but there are poles sitting inside the strip $-\mathrm{Re}(\rho)<c<0$ if $\mu<\nu$ or $\nu<\rho$, which seems to cause ambiguity in the choice of c in Eq. (A9) as they may enclose different poles inside the strip. However, the convergent condition (A11) requires that we squeeze the integration path in Eq. (A10) into a smaller strip $\max[-\mathrm{Re}(\mu+\rho-\nu),-\mathrm{Re}(\rho)]<c<\min[0,\mathrm{Re}(\nu-\rho)]$. It is clear that in this refined strip there is indeed no ambiguity to define the integral and again we get the desired result. See Fig. 7(b) for the pole structure and the contour for the case: Re(ρ)>Re(ν)>Re(μ)>0.

3. Re(μ)>0, Re(ν), Re(ρ)>0 and others

Now the convergent condition (A11) can no longer be satisfied. There is an overlap between the ascending poles from $\Gamma(\nu-\rho-s)$ and descending poles from $\Gamma(\mu+\rho-\nu+s)$ and there does not exist a uniform strip that the inverse Mellin integral (A10) is well defined. In this case we can define the integral by analytic continuation from the convergent region of μ. It is clear that as we vary μ continuously from $\mu>0$ to $\mu<0$, the only way to avoid the sudden jump of the value of integral by crossing the poles from $\Gamma(\mu+\rho-\nu+s)$ is to deform the integration path so that it still separates the descending and ascending pole sequences [see Fig. 8(a)].

It is obvious that by repeating the above procedure of deforming the path of Eq. (A10) [see Fig. 8(b)] we can analytically continue the integral (A7) to arbitrary complex values of μ, ν, ρ except for some discrete surfaces in the space of μ, ν, ρ where one or more of ρ, μ and ν, $\mu+\rho-\nu$ become nonpositive integers. In these cases there are coincidences between the ascending poles and descending poles and it is no longer possible to separate them. The analytic continuation breaks down at these surfaces. The pathology at μ, $\rho=-k$, $k=0,1,2,\ldots$ may be attributed to the method we are using [see Eq. (A9)],[14] while at ν, $\mu+\rho-\nu=-k$, $kk=0,1,2,\ldots$ the analytic continuation truly breaks down (similar to the poles in Γ functions).

APPENDIX B: DETAILED EVALUATION OF *J*

Here we present the details of the calculation leading from Eq. (2.8) to Eq. (2.9). To avoid making formulas too

[14]When μ, $\rho=-k$, $k=0,1,2,\ldots$, $(1+x)^{-\mu}$ or $(x+t)^{-\rho}$ becomes a finite series and can be expanded directly to evaluate the integral. The result can be expressed in terms of the terminating series of hypergeometric functions.

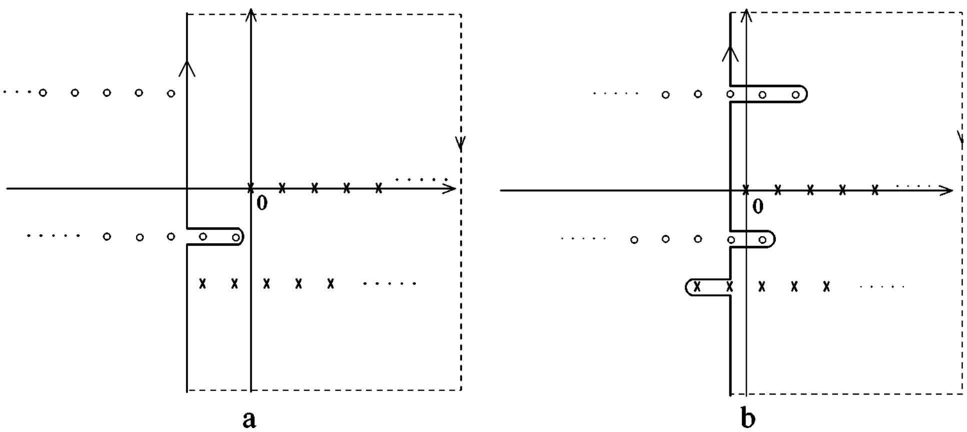

FIG. 8. Pole structure and analytic continuation. (a) Re(ρ) >Re(ν)>0, Re(μ)<0; (b) general values of parameters.

long, we will suppress the prefactors (numerical constants and powers of x_{ij}) of the integrals and give their final expression only at the end. We use the following definitions:

$$\tilde{\lambda}=d-\lambda,\;\; \nu=\lambda-\frac{d}{2},\;\; \nu_i=\lambda_i-\frac{d}{2},\;\; i=1,\ldots,4,$$

$$\lambda_{ij}=\lambda_i+\lambda_j,\;\; \Delta_{ij}=\lambda_i-\lambda_j,\;\; \epsilon_{ij}=\lambda_{ij}-\lambda,\;\; \tilde{\epsilon}_{ij}=\lambda_{ij}-\tilde{\lambda}, \tag{B1}$$

$$\delta_1=\lambda+\Delta_{12},\;\; \delta_2=\lambda+\Delta_{21},\;\; \delta_3=\lambda+\Delta_{34},\;\; \delta_4=\lambda+\Delta_{43},$$

$$\tilde{\delta}_1=\tilde{\lambda}+\Delta_{12},\;\; \tilde{\delta}_2=\tilde{\lambda}+\Delta_{21},\;\; \tilde{\delta}_3=\tilde{\lambda}+\Delta_{34},\;\; \tilde{\delta}_4=\tilde{\lambda}+\Delta_{43}.$$

$J(s)$ in Eq. (2.8) can be further simplified by applying the inversion trick [10]: set $x_4=0$, then use a simultaneous inversion of the external coordinates and the integration variables, $u_\mu\to u_\mu/|u|^2$, $v_\mu\to v_\mu/|v|^2$, $\vec{x}_i=\vec{x}_i'/|\vec{x}_i'|^2$, $i=1,2,3$, after which J becomes

$$J(s)=\frac{2^{\lambda+2s}}{|x_1|^{2\lambda_1}|x_2|^{2\lambda_2}|x_3|^{2\lambda_3}}\int\frac{du_0d^du}{u_0^{d+1}}\frac{dv_0d^dv}{v_0^{d+1}}\left(\frac{u_0}{|u-x_1'|^2}\right)^{\lambda_1}\left(\frac{u_0}{|u-x_2'|^2}\right)^{\lambda_2}\times\left(\frac{u_0v_0}{u_0^2+v_0^2+|\vec{u}-\vec{v}|^2}\right)^{\lambda+2s}\left(\frac{v_0}{|v-x_3'|^2}\right)^{\lambda_3}v_0^{\lambda_4}. \tag{B2}$$

Using

$$\frac{\Gamma(\lambda)}{z^\lambda}=\int_0^\infty d\rho\,\rho^{\lambda-1}e^{-\rho z}, \tag{B3}$$

we can rewrite Eq. (B2) as

$$J(s)=\int_0^\infty d\rho_1d\rho_2d\rho_3d\rho\,\rho_1^{\lambda_1-1}\rho_2^{\lambda_2-1}\rho_3^{\lambda_3-1}\rho^{\lambda+2s-1}\int du_0d\vec{u}\,dv_0d\vec{v}\,u_0^{\tilde{\epsilon}_{12}+2s-1}v_0^{\tilde{\epsilon}_{34}+2s-1}\times e^{-(\rho_1+\rho_2+\rho)u_0^2}e^{-(\rho+\rho_3)v_0^2}\exp\{-\rho_1|\vec{u}-\vec{x}_1'|^2-\rho_2|\vec{u}-\vec{x}_2'|^2-\rho|\vec{u}-\vec{v}_2|^2-\rho_3|\vec{v}-\vec{x}_3'|^2\}.$$

Integrating over u_0, v_0 we get[15]

$$J(s)=\int_0^\infty d\rho_1d\rho_2d\rho_3d\rho\,\rho_1^{\lambda_1-1}\rho_2^{\lambda_2-1}\rho_3^{\lambda_3-1}\rho^{\lambda+2s-1}\left(\frac{1}{\rho_1+\rho_2+\rho}\right)^{\tilde{\epsilon}_{12}/2+s}\left(\frac{1}{\rho_3+\rho}\right)^{\tilde{\epsilon}_{34}/2+s}\times\int d\vec{u}\,d\vec{v}\exp\{-\rho_1|\vec{u}-\vec{x}_1'|^2-\rho_2|\vec{u}-\vec{x}_2'|^2-\rho|\vec{u}-\vec{v}|^2-\rho_3|\vec{v}-\vec{x}_3'|^2\}.$$

Now we use the following expression:

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[15]The convergence of u_0, v_0 integrals requires $\mathrm{Re}(\tilde{\epsilon}_{12}+2s)>0$ and $\mathrm{Re}(\tilde{\epsilon}_{34}+2s)>0$. Since $\mathrm{Re}(s)\sim 0$, the convergence conditions are indeed satisfied with λ_i, $\lambda>d/2$.

$$\int d^d\vec{u}\,\exp\left\{-\sum_i \rho_i\left|\vec{u}-\vec{x}_i\right|^2\right\}=\left(\frac{\pi}{\Sigma_i\rho_i}\right)^{d/2}\exp\left\{-\frac{\Sigma_{i<j}\rho_i\rho_j x_{ij}^2}{\Sigma_i\rho_i}\right\} \tag{B4}$$

to integrate over $\vec{u},\vec{v}$, which leads to

$$J(s)=\int_0^\infty d\rho_1 d\rho_2 d\rho_3 d\rho\,\rho_1^{\lambda_1-1}\rho_2^{\lambda_2-1}\rho_3^{\lambda_3-1}\rho^{\lambda+2s-1}\left(\frac{1}{\rho_1+\rho_2+\rho}\right)^{\tilde{\epsilon}_{12}/2+s}\left(\frac{1}{\rho_3+\rho}\right)^{\tilde{\epsilon}_{34}/2+s}$$
$$\times\left(\frac{1}{\rho_3(\rho_1+\rho_2+\rho)+\rho(\rho_1+\rho_2)}\right)^{d/2}\exp\left\{-\frac{\rho\rho_2\rho_3|x'_{23}|^2+\rho\rho_1\rho_3|x'_{13}|^2+\rho_1\rho_2(\rho_3+\rho)|x'_{12}|^2}{\rho_3(\rho_1+\rho_2+\rho)+\rho(\rho_1+\rho_2)}\right\},$$

where $|x'_{ij}|=|\vec{x}'_i-\vec{x}'_j|$.

Let $\rho_i\to\rho\rho_i$, $i=1,2,3$ and integrate over ρ,

$$J(s)=\int_0^\infty d\rho_1 d\rho_2 d\rho_3\rho_1^{\lambda_1-1}\rho_2^{\lambda_2-1}\rho_3^{\lambda_3-1}\left(\frac{1}{1+\rho_1+\rho_2}\right)^{\tilde{\epsilon}_{12}/2+s}\left(\frac{1}{1+\rho_3}\right)^{\tilde{\epsilon}_{34}/2+s}$$
$$\times[(\rho_1+\rho_2)(1+\rho_3)+\rho_3]^{(\lambda_{12}\Delta_{34}-d)/2}\left[\frac{1}{\rho_2\rho_3|x'_{23}|^2+\rho_1\rho_3|x'_{13}|^2+\rho_1\rho_2(\rho_3+1)|x'_{12}|^2}\right]^{(\lambda_{12}+\Delta_{34})/2}.$$

Note that the convergence of the ρ integral requires $\lambda_{12}+\Delta_{34}>0$.

Now we define new variables, $\rho_1=\sigma u$, $\rho_2=\sigma(1-u)$, $\rho_3=\rho$, and

$$\eta=\frac{|x'_{13}|^2}{|x'_{12}|^2},\quad \xi=\frac{|x'_{23}|^2}{|x'_{12}|^2},\quad z=\frac{\eta}{1-u}+\frac{\xi}{u}, \tag{B5}$$

after which the integrals become

$$J(s)=\int_0^1 du\,u^{(\Delta_{12}-\Delta_{34})/2-1}(1-u)^{(\Delta_{21}-\Delta_{34})/2-1}\int d\sigma\,d\rho\,\sigma^{(\lambda_{12}-\Delta_{34})/2-1}(1+\sigma)^{-\tilde{\epsilon}_{12}/2-s}$$
$$\times\rho^{\lambda_3-1}(1+\rho)^{-\tilde{\epsilon}_{34}/2-s}(\sigma+\rho+\sigma\rho)^{(\lambda_{12}+\Delta_{34}-d)/2}[\sigma(1+\rho)+\rho z]-^{(\lambda_{12}+\Delta_{34})/2}.$$

Further define $t=\rho/(1+\rho)$, so that

$$J=\int_0^1 du\,u^{(\Delta_{12}-\Delta_{34})/2-1}(1-u)^{(\Delta_{21}-\Delta_{34})/2-1}\int_0^\infty d\sigma\int_0^1 dt\,\sigma^{(\lambda_{12}-\Delta_{34})/2-1}(1+\sigma)^{-\tilde{\epsilon}_{12}/2-s}$$
$$\times t^{\lambda_3-1}(1-t)^{\delta_4/2+s-1}(\sigma+t)^{(\lambda_{12}+\Delta_{34}-d)/2}(\sigma+tz)^{-(\lambda_{12}+\Delta_{34})/2}. \tag{B6}$$

Our next step is to use the inverse Mellin transformation Eq. (A4),

$$\left(\frac{1}{1+x}\right)^\alpha=\frac{1}{\Gamma(\alpha)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}ds\,\Gamma(-s)\Gamma(\alpha+s)x^s \tag{B7}$$

in $(\sigma+tz)^{-(\lambda_{12}+\Delta_{34})/2}$ in Eq. (B6). Then the σ-t part of the integrals in Eq. (B6) becomes

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}ds_1\Gamma(-s_1)\Gamma\left(\frac{\lambda_{12}+\Delta_{34}}{2}+s_1\right)z^{-s_1-(\lambda_{12}+\Delta_{34})/2}J_1 \tag{B8}$$

with

$$J_1=\int_0^\infty d\sigma\int_0^1 dt\,t^{(\lambda_{34}-\lambda_{12})/2-s_1-1}(1-t)^{\delta_4/2/+s-1}\sigma^{(\lambda_{12}-\Delta_{34})/2/+s_1-1}(1+\sigma)^{-\tilde{\epsilon}_{12}/2-s}(\sigma+t)^{(\lambda_{12}+\Delta_{34}-d)/2}.$$

After σ integration we get

$$J_1=B\left(s_1+\frac{\lambda_{12}-\Delta_{34}}{2},s-s_1-\frac{\epsilon_{12}}{2}\right)$$
$$\times\int_0^1 dt\, t^{\delta_4/2/+s-1}(1-t)^{(\lambda_{34}+\lambda_{12}-d)/2-1}F\left(\frac{\tilde{\epsilon}_{12}}{2}+s,s_1+\frac{\lambda_{12}-\Delta_{34}}{2};s+\frac{\delta_4}{2},t\right).$$

Notice that the power in t coincides with the third parameter of the hypergeometric function inside the integral. In this case the integration over t can be done very easily and we get

$$J_1=\frac{\Gamma[(\lambda_{34}+\lambda_{12}-d)/2]}{\Gamma(\lambda_4)}\frac{\Gamma[s_1+(\lambda_{12}-\Delta_{34})/2]\Gamma(s-s_1-\epsilon_{12})/2\Gamma[(\lambda_{34}-\lambda_{12})/2-s_1]}{\Gamma(s-s_1+\tilde{\epsilon}_{34}/2)}. \tag{B9}$$

Plugging Eq. (B9) back into Eqs. (B8), (B6) and rearranging the integrals

$$J(s)=C_2\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}ds_1\Gamma(-s_1)\Gamma\left(\frac{\lambda_{12}+\Delta_{34}}{2}+s_1\right)\Gamma\left(s_1+\frac{\lambda_{12}-\Delta_{34}}{2}\right)\Gamma\left(\frac{\lambda_{34}-\lambda_{12}}{2}-s_1\right)$$
$$\times\frac{\Gamma(s-s_1-\epsilon_{12}/2)}{\Gamma(s-s_1+\tilde{\epsilon}_{34}/2)}\int_0^1 du\, u^{(\Delta_{12}-\Delta_{34})/2-1}(1-u)^{(\Delta_{21}-\Delta_{34})/2-1}\left(\frac{\eta}{1-u}+\frac{\xi}{u}\right)^{-s_1-(\lambda_{12}+\Delta_{34})/2}. \tag{B10}$$

The integral in the second line gives us a hypergeometric function and the final expression for J is (we shifted s_1 by $s_1+\lambda_{12}\to s_1$),

$$J(s)=C_2\frac{1}{2\pi i}\int_C ds_1\xi^{-s_1-\Delta_{34}/2}\Gamma\left(\frac{\lambda_{12}}{2}-s_1\right)\Gamma\left(\frac{\lambda_{34}}{2}-s_1\right)F\left(\frac{\Delta_{34}}{2}+s_1,\frac{\Delta_{12}}{2}+s_1;2s_1;1-\frac{\eta}{\xi}\right)$$
$$\times\frac{\Gamma(\Delta_{34}/2+s_1)\Gamma(\Delta_{43}/2+s_1)\Gamma(\Delta_{12}/2+s_1)\Gamma(\Delta_{21}/2+s_1)}{\Gamma(2s_1)}\frac{\Gamma(\lambda/2+s-s_1)}{\Gamma[(\lambda_{12}+\tilde{\epsilon}_{34})/2+s-s_1]}. \tag{B11}$$

Now restore x_4 by taking $x_i\to x_i-x_4$, $i=1,2,3$. η and ξ defined in Eq. (B5) become the cross ratios

$$\eta=\frac{|\vec{x}_{14}'-\vec{x}_{34}'|^2}{|\vec{x}_{14}'-\vec{x}_{24}'|^2}=\frac{|x_{13}|^2|x_{24}|^2}{|x_{12}|^2|x_{34}|^2},\quad \xi=\frac{|\vec{x}_{24}'-\vec{x}_{34}'|^2}{|\vec{x}_{14}'-\vec{x}_{24}'|^2}=\frac{|x_{14}|^2|x_{23}|^2}{|x_{12}|^2|x_{34}|^2}. \tag{B12}$$

Finally, the prefactor C_2 is given by

$$C_2=\frac{\pi^d}{4}\frac{\Gamma[(\lambda_{12}+\lambda_{34}-d)/2]\Gamma(s+\tilde{\epsilon}_{12}/2)\Gamma(s+\tilde{\epsilon}_{34})/2}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma(\lambda_4)\Gamma(\lambda+2s)}\frac{2^{\lambda+2s}}{|x_{12}|^{\lambda_{12}+\Delta_{34}}|x_{14}|^{\Delta_{12}-\Delta_{34}}|x_{24}|^{\Delta_{21}-\Delta_{34}}|x_{34}|^{2\lambda_3}}.$$

We note that it can be checked that the integrals in intermediate steps from Eq. (B2) to (B11) are convergent only when the parameters satisfy the following conditions: [Re(s)$\sim$0],

$$\frac{\delta_1}{2},\frac{\delta_2}{2},\frac{\delta_3}{2},\frac{\delta_4}{2}>0,\frac{\lambda_{12}\pm\Delta_{34}}{2}>0,\frac{\lambda_{34}\pm\Delta_{12}}{2}>0. \tag{B13}$$

For the above range of parameters there is no overlap between ascending and descending poles in Eq. (B11) and the s_1 integral can be unambiguously defined by squeezing (also determined by the convergence of the intermediate integrals) the integration path $\mathcal{C}$ to lie inside the strip:

$$\max\left[\frac{\Delta_{12}}{2},\frac{\Delta_{21}}{2},\frac{\Delta_{34}}{2},\frac{\Delta_{34}}{2}\right]<\mathrm{Re}(s_1)<\min\left[\frac{\lambda}{2},\frac{\lambda_{12}}{2},\frac{\lambda_{34}}{2}\right].$$

When parameters are outside the range of Eq. (B13), some integrals in intermediate steps may not be convergent and can only be defined by analytic continuation. Further there are overlaps between the ascending and descending pole sequences in Eq. (B11) and there does not exist a uniform strip in which the Mellin integral can be well defined. In this case we can define the integral in Eq. (B11) by analytic continuation by deforming the integration path $\mathcal{C}$ so that it separates the ascending and descending poles. In Appendix A we have given a detailed discussion of this procedure in a simpler example. The whole discussion can be applied to this more complicated case without change. We note that when some poles from ascending and descending series coincide with one another, e.g., if one or more of $(\lambda_{12}\pm\Delta_{34})/2$, $(\lambda_{34}\pm\Delta_{12})/2$, $\delta_i/2$, $i=1,\ldots,4$ are nonpositive integers, there is no way to separate the ascending and descending poles. The analytic continuation breaks down at these points.

APPENDIX C: EVALUATION OF CONTACT CONTRIBUTION

Here we would like evaluate Eq. (3.9). Again we will suppress the prefactor of the integrals most of the time and follow the notations defined in Eq. (B1).

As in Appendix B, using Eq. (B3) and integrating over u_0, we get

$$S_c=\int \Pi_{i=1}^4 d\rho_i \rho_i^{\lambda_i-1}\frac{1}{(\rho_1+\rho_2+\rho_3+\rho_4)^{(\lambda_{12}+\lambda_{34}-d)/2}}\int d\vec{u}\exp\left[-\sum_i \rho_i|\vec{u}-\vec{x}_i|^2\right]$$
$$=\int \Pi_{i=1}^4 d\rho_i \rho_i^{\lambda_i-1}\frac{1}{(\rho_1+\rho_2+\rho_3+\rho_4)^{(\lambda_{12}+\lambda_{34})/2}}\exp\left\{-\frac{\sum_{i<j=1}^4 \rho_i\rho_j|x_{ij}|^2}{\rho_1+\rho_2+\rho_3+\rho_4}\right\}, \tag{C1}$$

where in the second line we have used Eq. (B4). Now let $\rho_i'=\rho_i(\rho_1+\rho_2+\rho_3+\rho_4)^{-1/2}$ and note that $\det(\partial\rho_i/\partial\rho_j')=2(\Sigma_{i=1}^4\rho_i')^4$,

$$S_c=\frac{1}{\pi^{3d/2}}\frac{\Gamma[(\lambda_{12}+\lambda_{34})-d/2]}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)}K(\lambda_1,\lambda_2,\lambda_3,\lambda_4), \tag{C2}$$

where we have included the numerical prefactor and K is defined by

$$K(\lambda_1,\lambda_2,\lambda_3,\lambda_4)=\int_0^\infty d\rho_1 d\rho_2 d\rho_3 d\rho_4 \Pi_i \rho_i^{\lambda_i-1}\exp\left[-\sum_{i<j=1}^4 \rho_i\rho_j|x_{ij}|^2\right]. \tag{C3}$$

Since K has translational invariance, we can take $x_4=0$. Defining $\rho_i'=\rho_i|x_i|^2$, $i=1,2,3$, we find that

$$\sum_{i<j=1}^4 \rho_i\rho_j|x_{ij}|^2=\rho_4(\rho_1'+\rho_2'+\rho_3')+\sum_{i<j=1}^4 \rho_i'\rho_j'|x_{ij}'|^2,$$

where $\vec{x}_{ij}'=\vec{x}_i'-\vec{x}_j'$ and $\vec{x}_i'=\vec{x}_j/|x_i|^2$. Then in terms of ρ_i' (we omit primes on ρ_i below) and x_i', J becomes

$$K=\frac{1}{|x_1|^{2\lambda_1}|x_2|^{2\lambda_2}|x_3|^{2\lambda_3}}\int_0^\infty \Pi_{i=1}^3 d\rho_i\rho_i^{\lambda_i-1}\exp\left[-\rho_4(\rho_1+\rho_2+\rho_3)-\sum_{i<j=1}^3 \rho_i\rho_j|x_{ij}'|^2\right]$$
$$=\frac{\Gamma(\lambda_4)}{|x_1|^{2\lambda_1}|x_2|^{2\lambda_2}|x_3|^{2\lambda_3}}\int_0^\infty \Pi_{i=1}^3 d\rho_i\rho_i^{\lambda_i-1}\frac{1}{(\rho_1+\rho_2+\rho_3)^{\lambda_4}}\exp\left[-\sum_{i<j=1}^3 \rho_i\rho_j|x_{ij}'|^2\right]. \tag{C4}$$

Now let $\rho_3=\alpha$, $\rho_1=\alpha\beta$, $\rho_2=\alpha\gamma$,

$$K=\int d\alpha\, d\beta\, d\gamma\frac{\beta^{\lambda_1-1}\gamma^{\lambda_2-1}\alpha^{\lambda_{12}+\Delta_{34}-1}}{(1+\beta+\gamma)^{\lambda_4}}\exp[-\alpha^2(\beta|x_{13}'|^2+\gamma|x_{23}'|^2+\beta\gamma|x_{12}'|^2)]$$
$$=\int d\beta\, d\gamma\frac{\beta^{\lambda_1-1}\gamma^{\lambda_2-1}}{(1+\beta+\gamma)^{\lambda_4}}\frac{1}{[\beta|x_{13}'|^2+\gamma|x_{23}'|^2+\beta\gamma|x_{12}'|^2]^{(\lambda_{12}+\Delta_{34})/2}}. \tag{C5}$$

Further let $\beta=\sigma u$, $\gamma=\sigma(1-u)$ and define,

$$\eta=\frac{|x_{13}'|^2}{|x_{12}'|^2},\quad \xi=\frac{|x_{23}'|^2}{|x_{12}'|^2},\quad z=\frac{\eta}{1-u}+\frac{\xi}{u},$$

after which K becomes

$$K=\int_0^1 du\, u^{(\Delta_{12}-\Delta_{34})/2-1}(1-u)^{(\Delta_{21}-\Delta_{34})/2-1}\int_0^\infty d\sigma\,\sigma^{(\lambda_{12}-\Delta_{34})/2-1}(1+\sigma)^{-\lambda_4}(\sigma+z)^{-(\lambda_{12}+\Delta_{34})/2}. \tag{C6}$$

The σ integral above is nothing but the familiar integral representation of a hypergeometric function.

We now restore x_4, after which ξ and η become the cross ratios defined in Eq. (B12). Including the prefactor of the integral, we get the final expression for K,

$$K=\frac{1}{2}\frac{\Gamma(\lambda_3)\Gamma(\lambda_4)\Gamma[(\lambda_{12}-\Delta_{34})/2]\Gamma[(\lambda_{12}+\Delta_{34})/2]}{\Gamma[(\lambda_{12}+\lambda_{34})/2]}\frac{1}{|x_{12}|^{\lambda_{12}+\Delta_{34}}|x_{14}|^{\Delta_{12}-\Delta_{34}}|x_{24}|^{\Delta_{21}-\Delta_{34}}|x_{34}|^{2\lambda_3}}$$
$$\times\int_0^1 du\, u^{(\Delta_{12}-\Delta_{34})/2-}(1-u)^{(\Delta_{21}-\Delta_{34})/2-1}z^{-\Delta_{34}}F\left(\frac{\lambda_{12}+\Delta_{34}}{2},\frac{\lambda_{12}-\Delta_{34}}{2};\frac{\lambda_{12}+\lambda_{34}}{2};1-\frac{1}{z}\right). \tag{C7}$$

Alternatively, we may use the Mellin-Barnes representation for a hypergeometric function

$$F(a,b;c;1-z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}ds\, z^s\Gamma(-s)\Gamma(c-a-b-s)\Gamma(a+s)\Gamma(b+s)$$

in Eq. (C7), and then integrate over u. In this form S_c can be written as

$$S_c=C_c\frac{1}{2\pi i}\int_{\lambda_{12}-i\infty}^{\lambda_{12}+i\infty}ds\,\xi^{-s}\Gamma\left(\frac{\lambda_{12}}{2}-s\right)\Gamma\left(\frac{\lambda_{34}}{2}-s\right)F\left(\frac{\Delta_{34}}{2}+s,\frac{\Delta_{12}}{2}+s;2s;1-\frac{\eta}{\xi}\right)$$
$$\times\frac{\Gamma(\Delta_{34}/2+s)\Gamma(\Delta_{43}/2+s)\Gamma(\Delta_{12}/2+s)\Gamma(\Delta_{21}/2+s)}{\Gamma(2s)} \tag{C8}$$

with

$$C_c=\frac{1}{2\pi^{3d/2}}\frac{\Gamma[(\lambda_{12}+\lambda_{34}-d)/2]}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_4)}\frac{1}{|x_{12}|^{\lambda_{12}}|x_{14}|^{\Delta_{12}}|x_{24}|^{\Delta_{21}-\Delta_{34}}|x_{23}|^{\Delta_{34}}|x_{34}|^{\lambda_{34}}}.$$

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