# Gauge invariance of the one-loop effective action of the Higgs field in the SU(2) Higgs model

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The one-loop effective action of the Abelian and non-Abelian Higgs models has been studied in various gauges, in the context of instanton and sphaleron transition, bubble nucleation, and most recently in nonequilibrium dynamics. Gauge invariance is expected on account of Nielsen's theorem if the classical background field is an extremum of the classical action, i.e., a solution of the classical equation of motion. We substantiate this general statement for the one-loop effective action, as computed using mode functions. We show that in the gauge-Higgs sector there are two types of modes that satisfy the same equation of motion as the Faddeev-Popov modes. We apply the general analysis to the computation of the fluctuation determinant for bubble nucleation in the SU(2) Higgs model in the 't Hooft gauge with general gauge parameter $\xi$. We show that due to the cancellation of the modes mentioned above, the fluctuation determinant is independent of $\xi$.
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## I. INTRODUCTION

The effective potential of gauge theories has been considered extensively as it is of interest for the phase structure of these theories, and in particular for the discussion of phase transitions and the associated bubble nucleation rates [1–6]. More generally the effective action appears when computing fluctuation corrections to the sphaleron [7–10] and instanton [11] transition rates in such theories.

It is well known that the effective potential is gauge dependent except for the region around the extrema of the effective action, where Nielsen's theorem states that the gauge dependence should disappear [12]. This has been verified in various cases [13–15]. In addition to the static extrema such as the minima and maxima of the effective *potential* Nielsen's theorem more generally applies to the extrema of the effective *action*, i.e., to the extremal, classical paths in configuration space, such as the bubble and sphaleron actions. It has been verified [16], using the gradient expansion, that for the leading orders in the coupling the quantum corrections to the bubble nucleation rate are gauge independent. Exact numerical computations are based on the analysis and numerical computation of mode functions in the background of the classical solution. Such computations are quite demanding numerically, as well as algebraically and analytically, so in general the authors just used one particular gauge, such as, e.g., the 't Hooft Feynman gauge and a concise discussion of gauge independence is lacking.

We have recently analyzed the evolution equations for a Higgs condensate and the gauge and Higgs field fluctuations in the SU(2) Higgs model, in one-loop approximation [17]. Here again the question of gauge dependence arises and has not yet been analyzed. For the Abelian Higgs model, a gauge invariant formalism has been developed, which, however, has not yet been implemented numerically [18,19].

We consider here the SU(2) Higgs model with an isoscalar Higgs background field. Such a configuration plays a central role in the discussion of the electroweak phase transition. Its finite temperature effective potential has been discussed extensively, and it has been used to predict the rate of bubble nucleation in a first order phase transition [1–6].

The plan of the paper is as follows. In Sec. II we present the basic equations, and we expand the Lagrangian into a classical and a second order fluctuation part. In Sec. III we present the equations of motion without gauge-fixing, and in the 't Hooft background gauge for arbitrary gauge parameter $\xi$. We show that there are two types of modes, the gauge modes and the gauge-fixing modes that satisfy the same equations of motion as the Faddeev-Popov ghosts, if the classical background field satisfies its equation of motion. This observation is the clue for a reduction of the mode equations into equations for the physical degrees of freedom and into equations whose functional determinant is cancelled by the Faddeev-Popov one. This reduction depends on the system under consideration. Here we demonstrate the cancellation of the unphysical modes for the fluctuation determinant which determines the fluctuation corrections to bubble nucleation, in the SU(2) Higgs model. We briefly introduce the model and its fluctuation operator in Sec. IV. The coupled gauge-Higgs system is analyzed in the partial-wave reduced equations in Sec. V. After some suitable transformations the system is reduced to a triangular form, with the consequence that the fluctuation determinant can be computed from the diagonal part. Thereby the cancellation against the Faddeev-Popov contributions to the fluctuation determinant is demonstrated explicitly. We present some conclusions in Sec. VI.

## II. FLUCTUATION LAGRANGIAN AND MODE EQUATIONS

The Lagrangian of the SU(2) Higgs model reads

$$\mathcal{L}=-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}+\frac{1}{2}(D_\mu\Phi)^\dagger(D^\mu\Phi)-V(\Phi^\dagger\Phi), \tag{2.1}$$

with the field strength tensor

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$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+g\epsilon^{abc}A^b_\mu A^c_\nu, \tag{2.2}$$

and the covariant derivative

$$D_\mu\equiv\partial_\mu-i\frac{g}{2}A^a_\mu\tau^a. \tag{2.3}$$

The potential has the form

$$V(\Phi^\dagger\Phi)=\frac{\lambda}{4}(\Phi^\dagger\Phi-v^2)^2. \tag{2.4}$$

We will assume in the following a classical field (condensate):

$$\Phi(x)=H(x)\begin{pmatrix}0\\1\end{pmatrix}, \tag{2.5}$$

its space-time dependence is not further specified here. A time-independent, metastable, radially symmetric configuration will be relevant for bubble nucleation, a spatially homogenous time dependent field describes a nonequilibrium situation, as considered in Ref. [17]. The fluctuations around this space-time dependent condensate are parametrized as

$$\Phi(x)=[H(x)+h(x)+i\tau_a\phi_a(x)]\begin{pmatrix}0\\1\end{pmatrix}, \tag{2.6}$$

with the isoscalar Higgs mode $h(x)$ and the would-be Goldstone fields $\phi_a(x)$, $a=1\cdots3$. As there is no classical gauge field, we have

$$A^\mu_a(x)=a^\mu_a(x). \tag{2.7}$$

The Lagrangian can then be split into a classical part

$$\mathcal{L}_{\rm cl}(x)=\frac{1}{2}\left[\partial_\mu H\partial^\mu H-\frac{\lambda}{4}(H^2-v^2)^2\right] \tag{2.8}$$

and a fluctuation Lagrangian. The part of first order in the fluctuating field vanishes, if the classical equation of motion

$$\Box H+\lambda(H^2-v^2)H=0 \tag{2.9}$$

is fulfilled. The part of second order in the fluctuations reads

$$\begin{aligned}\mathcal{L}^{(2)}=\frac{1}{2}\Big\{&-\partial_\mu a^a_\nu\partial^\mu a^\nu_a+\partial_\mu a^a_\nu\partial^\nu a^\mu_a+\frac{g^2}{4}H^2a^a_\mu a^\mu_a\\&+\partial_\mu\phi_a\partial^\mu\phi_a+g\partial_\mu Ha^\mu_a\phi_a-gHa_{a\mu}\partial^\mu\phi_a\\&-\lambda(H^2-v^2)\phi_a\phi_a+\partial_\mu h\partial^\mu h-\lambda(3H^2-v^2)h^2\Big\}.\end{aligned} \tag{2.10}$$

In the one-loop approximation we do not have to consider higher order terms. The gauge-fixing term, in the 't Hooft background gauge is given by

$$\mathcal{L}^{(2)}_{\rm gf}=-\frac{1}{2\xi}\mathcal{F}_a\mathcal{F}_a \tag{2.11}$$

with the gauge conditions

$$\mathcal{F}_a=\partial_\mu a^\mu_a+\xi eH\phi_a, \tag{2.12}$$

the Faddeev-Popov Lagrangian is

$$\mathcal{L}_{\rm FP}=\frac{1}{2}\left\{\partial_\mu\eta_a\partial^\mu\eta_a-\xi\frac{g^2}{4}H^2\eta_a\eta_a\right\}. \tag{2.13}$$

## III. GAUGE MODE AND GAUGE-FIXING MODE

Before discussing the fluctuation operator for a specific physical setting we specify here the unphysical degrees of freedom in the gauge field and would-be Goldstone sector whose cancellation against the Faddeev-Popov modes will lead to a gauge invariant fluctuation determinant. The fluctuation operator of the isoscalar Higgs mode $h(x)$ is gauge invariant from the outset.

We arrange the gauge field fluctuations $a^\mu_a$ and the would-be Goldstone fields $\varphi_a$ in a $(4+1)$ column vector

$$\psi_a=\begin{Bmatrix}a^\mu_a\\\phi_a\end{Bmatrix}. \tag{3.1}$$

We start with the equations of motion obtained *without* the gauge-fixing term. The differential operator (fluctuation operator) governing the mode evolution then takes the form

$$\mathcal{M}=\begin{Bmatrix}-\left(\Box+\frac{g^2}{4}H^2\right)\delta^\nu_\mu+\partial^\nu\partial_\mu & -\frac{g}{2}\partial^\nu H+\frac{g}{2}H\partial^\nu\\ -g\partial_\mu H-\frac{g}{2}H\partial_\mu & \Box+\lambda(H^2-v^2)\end{Bmatrix}. \tag{3.2}$$

The mode equations are the same for all $a=1,2,3$,

$$\mathcal{M}\psi_a=0. \tag{3.3}$$

An infinitesimal gauge transform is given by

$$\psi^g_a(x)=\begin{Bmatrix}\partial^\mu\\\frac{g}{2}H(x)\end{Bmatrix}f_a(x). \tag{3.4}$$

These modes satisfy the mode equation (3.3) if $H(x)$ satisfies the classical field equation (2.9). The latter condition is crucial. It arises from the mode equation for $\varphi_a$, the one for the vector potentials is fulfilled trivially.

If the gauge mode is substituted into the gauge condition one finds

$$(\mathcal{F}_a)_{\rm g}=\left[\Box+\xi\frac{g^2}{4}H^2(x)\right]f_a, \tag{3.5}$$

the differential operator on the right-hand side (RHS) is just the Faddeev-Popov operator. So, if the gauge mode is inserted into the Lagrangian, the gauge-fixing term contains the Faddeev-Popov operator *squared*. It is very suggestive that the contribution of this squared operator to the effective action, i.e., to the log det of the fluctuation operator, is cancelled by twice the log det of the Faddeev-Popov operator.

If the gauge-fixing term is included the fluctuation operator takes the form

$$\mathcal{M}_f=\begin{Bmatrix} -\left(\Box+\frac{g^2}{4}H^2\right)\delta^\nu_\mu+\left(1-\frac{1}{\xi}\right)\partial^\nu\partial_\mu & -g\partial^\nu H \\ -g\partial_\mu H & \Box+\lambda(H^2-v^2)+\frac{g^2}{4}\xi H^2 \end{Bmatrix}. \tag{3.6}$$

If we apply the fluctuation operator to the gauge mode, and use the classical equation of motion, we obtain

$$\mathcal{M}_{\rm f}\psi^g_a(x)=\begin{Bmatrix} -\frac{1}{\xi}\partial_\mu \\ \frac{g}{2}H(x)\end{Bmatrix}\left[\Box+\xi\frac{g^2}{4}H^2(x)\right]f_a(x)$$
$$=\mathcal{M}_{\rm FP}f_a(x). \tag{3.7}$$

The differential operator appearing on the right hand side is just the Faddeev-Popov operator

$$\mathcal{M}_{\rm FP}=\Box+\xi\frac{g^2}{4}H^2(x). \tag{3.8}$$

If $f_a$ is an eigenfunction of the Faddeev-Popov operator, $\mathcal{M}_{\rm FP}f_a=\omega^2_{\rm FP}f_a$, then the associated gauge mode satisfies

$$\begin{Bmatrix} -\xi & 0 \\ 0 & 1\end{Bmatrix}\mathcal{M}_{\rm f}\psi^g_a=\omega^2_{\rm FP}\psi^g_a. \tag{3.9}$$

The factor $\xi$ in the matrix multiplies the four gauge field components. So the fluctuation operator modified by multiplication with a constant matrix, has a class of eigenfunctions with the same eigenvalues as the Faddeev-Popov operator. In the effective action the modification by the constant matrix is irrelevant, as one computes the ratio between the fluctuation determinants in the background field and in a standard vacuum configuration, to which the same arguments apply.

Now consider the gauge condition $\mathcal{F}_a$. We introduce the covector

$$\mathbf{u}_\xi=\left[\partial_\mu,\xi\frac{g}{2}H(x)\right], \tag{3.10}$$

so that

$$\mathcal{F}_a=\mathbf{u}_\xi\phi_a. \tag{3.11}$$

Consider an arbitrary mode $\psi_a$. We then find, using again the classical equation of motion,

$$\mathbf{u}_\xi\begin{Bmatrix} -\xi & 0 \\ 0 & 1\end{Bmatrix}\mathcal{M}_{\rm f}\psi_a=\left[\Box+\xi\frac{g^2}{2}H^2(x)\right]\mathbf{u}_\xi\psi_a=\mathcal{M}_{\rm FP}\mathcal{F}_a. \tag{3.12}$$

Let $\psi^\alpha_a$ now be an eigenmode of the modified fluctuation operator with eigenvalue $\omega^2_\alpha$. Then this equation entails

$$\mathbf{u}_\xi\omega^2_\alpha\psi^\alpha_a=(\omega^\alpha_a)^2\mathcal{F}^\alpha_a=\mathcal{M}_{\rm FP}\mathcal{F}^\alpha_a. \tag{3.13}$$

So if the projection on the vector $\mathbf{u}_\xi$ is different from zero the eigenvalue is simultaneously an eigenvalue of $\mathcal{M}_{\rm FP}$. We thereby have a second class of modes on which the fluctuation operator of the gauge-Higgs system has the same spectrum as the Faddeev-Popov operator. We call them gauge-fixing modes. We have to make sure that this class of modes, obtained by a projection, is not empty, and not identical with the gauge modes.

Obviously the modes on which the projector $\mathbf{u}_\xi$ yields zero are those which satisfy the gauge condition, these are the ''physical modes.'' We know that out of the five components of the gauge-Higgs modes $\psi$ only three are physical, they represent the spatial components of the massive gauge field.

We next consider the action of the projector on the gauge eigenmodes. It is convenient to introduce a vector $\mathbf{v}$ that generates the gauge modes via

$$\psi^g_a=\mathbf{v}f_a=\begin{Bmatrix}\partial^\mu \\ \frac{g}{2}H(x)\end{Bmatrix}f_a. \tag{3.14}$$

We note that

$$\mathbf{u}_\xi\mathbf{v}=\Box+\xi\frac{g^2}{4}H^2. \tag{3.15}$$

This implies that the gauge-fixing mode obtained by projection of a gauge mode satisfies

$$\mathcal{F}_a=\mathbf{u}_\xi\psi^g_a=\mathbf{u}_\xi\mathbf{v}f_a=\left[\Box+\xi\frac{g^2}{4}H^2(x)\right]f_a. \tag{3.16}$$

So if $f_a$ is an eigenfunction of the Faddeev-Popov operator, then the gauge-fixing mode generated from it does not represent a new, independent mode. However, the gauge modes and the physical modes do not exhaust the Hilbert space that is based on five field degrees of freedom, and we are sure that the projector does not give zero on the remaining subspace.

We have shown up to now, that for a background field satisfying the classical equation of motion there are two classes of modes whose contribution to the effective action will be cancelled by the one of the Faddeev-Popov sector. We have not shown, thereby, that the remaining ''physical'' part of the gauge-Higgs sector becomes independent of $\xi$. Furthermore, the way in which the modes are eliminated is a technical matter, it depends on the structure of the background field, and on the problem under consideration. So if we want to illustrate the application of these general results we have to consider specific models. We will here analyze the modes introduced above, and the cancellation of their contribution to the fluctuation determinant, for the case of bubble nucleation in the SU(2) Higgs model.

## IV. BUBBLE NUCLEATION IN THE SU(2) HIGGS MODEL

Bubble nucleation occurs in the SU(2) Higgs model if the phase transition from the symmetric high temperature phase to the broken symmetry phase at low temperature is first order. It has been considered as providing a possible mechanism for baryogenesis, a possibility ruled out by the present lower limit for the Higgs mass. Still the model is of interest, in particular it can be studied in lattice simulations for sufficiently low Higgs masses. The phase transition is described (see, e.g., Ref. [20]), by the three-dimensional high-temperature action

$$S_{ht}=\frac{1}{g_3(T)^2}\int d^3x\left[\frac{1}{4}F_{ij}F_{ij}+\frac{1}{2}(D_i\Phi)^\dagger(D_i\Phi)+V_{ht}(\mathbf{\Phi}^\dagger\mathbf{\Phi})+\frac{1}{2}A_0\left(-D_iD_i+\frac{1}{4}\mathbf{\Phi}^\dagger\mathbf{\Phi}\right)A_0\right]. \tag{4.1}$$

Here the coordinates and fields have been rescaled as [8]

$$\vec{x}\to\frac{\vec{x}}{gv(T)},\quad \mathbf{\Phi}\to v(T)\mathbf{\Phi},\quad A\to v(T)A. \tag{4.2}$$

The vacuum expectation value $v(T)$ is defined as

$$v^2(T)=\frac{2D}{\lambda_T}(T_0^2-T^2). \tag{4.3}$$

$T_0$ is the temperature at which the high-temperature potential $V_{ht}$ changes its extremum at $\mathbf{\Phi}=0$ from a minimum at $T>T_0$ to a maximum at $T<T_0$. The temperature dependent coupling of the three-dimensional theory is defined as

$$g_3^2(T)=\frac{gT}{v(T)}. \tag{4.4}$$

We use the standard parameters

$$D=(3m_W^2+2m_t^2)/8v_0^2, \tag{4.5}$$

$$E=3g^3/32\pi, \tag{4.6}$$

$$B=3(3m_W^4-4m_t^4)/64\pi^2v_0^4, \tag{4.7}$$

$$T_0^2=(m_H^4-8v_0^2B)/4D, \tag{4.8}$$

$$\lambda_T=\lambda-3\left(3m_W^4\ln\frac{m_w^2}{a_BT^2}-4m_t^4\ln\frac{m_t^2}{a_FT^2}\right)\Big/16\pi^2v_0^4. \tag{4.9}$$

We use in the following a somewhat different rescaling, introduced in [21,22], based on the secondary minimum of the high-temperature potential which occurs at

$$\tilde{v}(T)=\frac{3ET}{2\lambda}+\sqrt{\left(\frac{3ET}{2\lambda}\right)^2+v^2(T)}. \tag{4.10}$$

The high-temperature potential then takes the form

$$V_{ht}(\mathbf{\Phi}^\dagger\mathbf{\Phi})=\frac{\lambda_T}{4g^2}\left\{(\mathbf{\Phi}^\dagger\mathbf{\Phi})^2-\epsilon(T)(\mathbf{\Phi}^\dagger\mathbf{\Phi})^{3/2}+\left[\frac{3}{2}\epsilon(T)-2\right]\mathbf{\Phi}^\dagger\mathbf{\Phi}\right\} \tag{4.11}$$

with

$$\epsilon(T)=\frac{4}{3}\left(1-\frac{v(T)^2}{\tilde{v}(T)^2}\right). \tag{4.12}$$

The standard formula [23–27] for the bubble nucleation rate is given by

$$\Gamma/V=\frac{\omega_-}{2\pi}\left(\frac{\tilde{S}}{2\pi}\right)^{3/2}\exp(-\tilde{S})\quad\mathcal{J}^{-1/2}. \tag{4.13}$$

Here $\tilde{S}$ is the high-temperature action, Eq. (4.1), minimized by a classical minimal bubble configuration (see below), $\mathcal{J}$ is the fluctuation determinant which describes the next-to-leading part of the semiclassical approach and which will be defined below; its logarithm is related to the one-loop effective action by

$$S_{\rm eff}^{1-l}=\frac{1}{2}\ln\mathcal{J}. \tag{4.14}$$

Finally $\omega_-$ is the absolute value of the unstable mode frequency.

The classical bubble configuration is described by a vanishing gauge field and a real spherically symmetric Higgs field $H(r)=|\mathbf{\Phi}|(r)$ which is a solution of the Euler-Lagrange equation

$$-H''(r)-\frac{2}{r}H'(r)+\frac{dV_{ht}}{dH(r)}=0 \tag{4.15}$$

with the boundary conditions

$$\lim_{r\to\infty} H(r)=0 \quad \text{and} \quad H'(0)=0. \tag{4.16}$$

We expand the gauge and Higgs fields around this classical configuration via

$$W^a_\mu(\mathbf{x})=a^a_\mu(\mathbf{x}),$$

$$\mathbf{\Phi}(\mathbf{x})=[H(r)+h(\mathbf{x})+\tau^a\phi_a(\mathbf{x})]\begin{pmatrix}0\\1\end{pmatrix}, \tag{4.17}$$

where $a^a_\mu,h$ and $\phi_a$ are the fluctuating fields, denoted collectively by $\varphi_i$.

If the action is expanded with respect to the fluctuating fields, the first order term vanishes if $H(r)$ satisfies the classical equation of motion (4.15). The second order part defines the fluctuation operator via

$$S^{(2)}=\frac{1}{\tilde{g}_3^2(T)}\int d^3x\frac{1}{2}\varphi_m\mathcal{M}_{mn}\varphi_n. \tag{4.18}$$

The fluctuation determinant $\mathcal{J}$ appearing in the rate formula is defined by[1]

$$\mathcal{J}=\frac{\det\mathcal{M}}{\det\mathcal{M}^0}, \tag{4.19}$$

where $\mathcal{M}_0$ is the fluctuation operator obtained by expanding around a spatially homogenous classical field that is a minimum of effective potential. The gauge conditions for the three-dimensional theory read

$$\mathcal{F}_a=\partial_\mu a^\mu_a+\frac{\xi}{2}H\phi_a=0. \tag{4.20}$$

The total gauge-fixed action $S_t$ is obtained from the high-temperature action by adding to it the gauge-fixing action

$$S_{\rm GF}=\frac{1}{\tilde{g}_3^2(T)}\int d^3x\frac{1}{2\xi}\mathcal{F}_a\mathcal{F}_a, \tag{4.21}$$

the corresponding Faddeev-Popov action reads

$$S_{\rm FP}=\frac{1}{\tilde{g}_3^2(T)}\int d^3x\,\eta^\dagger\left(-\Delta+\xi\frac{H^2(r)}{4}\right)\eta. \tag{4.22}$$

The fluctuation operator is obtained from the total action $S_t=S_{ht}+S_{\rm GF}+S_{\rm FP}$. The fluctuation operator, and along with it the fluctuation determinant, decomposes under partial wave expansion into fluctuation operators for fixed angular momentum. It is these that we will consider in the following.

[1]We omit some sophistications related to zero and unstable modes.

The background field is isoscalar, so the isospin index $a$ just results in multiplicity factors, we will omit it in the following. The scalar fields $h(\mathbf{x}),\phi_a(\mathbf{x}),\eta(\mathbf{x})$, and $a_0(\mathbf{x})$ are expanded with respect to the spherical harmonics $Y^m_l(\hat{\mathbf{x}})$, the partial wave mode functions are denoted by $f^l_h(r),f^l_\phi(r),f^l_\eta(r)$, and $f^l_0(r)$. The vector spherical harmonics $\hat{\mathbf{x}}Y^m_l,r\nabla Y^m_l$, and $\vec{L}Y^m_l$ are used for expanding the space components of the gauge fields via

$$\mathbf{a}(\mathbf{x})=\sum_{lm}\left(\frac{f^l_a(r)}{\sqrt{l(l+1)}}r\nabla Y^m_l+f^l_b(r)\hat{\mathbf{x}}Y^m_l+\frac{f^l_c(r)}{\sqrt{l(l+1)}}\mathbf{x}\times\nabla Y^m_l\right). \tag{4.23}$$

The fluctuation operator is block diagonal. In the following we consider just one partial wave and omit the superscript $l$. We denote the partial wave reduction of the fluctuation operator $\mathcal{M}$ by $\mathbf{M}^l$, we omit the superscript, however. The components $f_h(r),f_\eta(r),f_c(r)$, and $f_0(r)$ are decoupled, the operator has the form

$$\mathbf{M}_{nn}=-\frac{d^2}{dr^2}-\frac{2}{r}\frac{d}{dr}+\frac{l(l+1)}{r^2}+m_n^2+V_m(r). \tag{4.24}$$

The masses are $m_\eta=m_0=m_c=0$ and $m_h=m_H$ with the Higgs mass

$$m_H^2=\frac{\lambda_T}{4g^2}(3\epsilon-4). \tag{4.25}$$

The potentials are $V_0(r)=V_c(r)=H^2(r)/4$, $V_\eta(r)=\xi H^2(r)/4$ and

$$V_h(r)=\frac{\lambda_T}{4g^2}[12H^2(r)-6\epsilon H(r)]. \tag{4.26}$$

The Faddeev-Popov fluctuations are fermionic and two-fold degenerate, as usual.

The modes $f_a,f_b$, and $f_\phi$ are coupled. The nonvanishing components of the fluctuaion operator are

$$M_{aa}(r)=-\frac{d^2}{dr^2}-\frac{2}{r}\frac{d}{dr}+\frac{l(l+1)}{\xi r^2}+\frac{H^2(r)}{4}, \tag{4.27}$$

$$M_{bb}(r)=-\frac{1}{\xi}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+\frac{l(l+1)+2/\xi}{r^2}+\frac{H^2(r)}{4}, \tag{4.28}$$

$$M_{\phi\phi}(r)=-\frac{d^2}{dr^2}-\frac{2}{r}\frac{d}{dr}+\frac{l(l+1)}{r^2}+\xi\frac{H^2(r)}{4}+m_H^2+\frac{\lambda}{g^2}\left[H^2(r)-\frac{3}{4}\epsilon H(r)\right], \tag{4.29}$$

$$M_{ab}(r)=-\frac{\sqrt{l(l+1)}}{\xi r^2}\left[2+(1-\xi)r\frac{d}{dr}\right], \tag{4.30}$$

$$M_{ba}(r)=-\frac{\sqrt{l(l+1)}}{\xi r^2}\left[1+\xi-(1-\xi)r\frac{d}{dr}\right], \tag{4.31}$$

$$M_{b\phi}(r)=V_{\phi b}(r)=-H'(r). \tag{4.32}$$

The fluctuation operator of this coupled system is hermitean, as it should, because it arises from the variation of a Lagrangian. The asymmetry suggested by the explicit form arises from integrations by parts.

The gauge parameter $\xi$ only occurs in the coupled system and for the Faddeev-Popov modes. The cancellation of the $\xi$ dependence will have to occur between these two sectors. They will be analyzed in the next section.

## V. ANALYSIS OF THE FLUCTUATION OPERATOR

In analyzing the gauge dependence we will have to consider the coupled system of the modes $f_a, f_b$, and $f_\phi$, i.e., the radial mode functions for angular momentum $l$. In analogy to Sec. III we consider the fluctuation operator multiplied from the left by a constant matrix diag$(\xi,\xi,1)$. The eigenvalue problem for the fluctuation operator then takes the form of the three differential equations for the radial mode functions for angular momentum $l$:

$$-f_a''-\frac{2}{r}f_a'+\frac{l(l+1)}{\xi r^2}f_a+\frac{H^2(r)}{4}f_a-\frac{\sqrt{l(l+1)}}{\xi r^2}\times[2f_b+(1-\xi)rf_b']=\frac{\omega^2}{\xi}f_a, \tag{5.1}$$

$$-\frac{1}{\xi}\left(f_b''+\frac{2}{r}f_b'\right)+\frac{l(l+1)+2/\xi}{r^2}f_b+\frac{H^2(r)}{4}f_b-\frac{\sqrt{l(l+1)}}{\xi r^2}\times[(1+\xi)f_a-(1-\xi)rf_a']-H'(r)f_\phi=\frac{\omega^2}{\xi}f_b, \tag{5.2}$$

$$-f_\phi''-\frac{2}{r}f_\phi'+\frac{l(l+1)}{r^2}f_\phi+m_H^2f_\phi+\xi\frac{H^2(r)}{4}f_\phi+\frac{\lambda}{g^2}\left[H^2(r)-\frac{3}{4}\epsilon H(r)\right]f_\phi-H'(r)f_b=\omega^2f_\phi. \tag{5.3}$$

In view of the general arguments of Sec. III we now should identify the gauge and the gauge-fixing modes. A general gauge transformation is parametrized by a function $\chi(\mathbf{x})$. It can be expanded into partial waves with respect to spherical harmonics, the radial mode function is denoted by $f_\chi(r)$. The gauge mode then takes the form

$$f_a^g(r)=\sqrt{l(l+1)}f_\chi(r),$$
$$f_b^g(r)=f_\chi'(r), \tag{5.4}$$
$$f_\phi^g(r)=-\frac{H(r)}{2}f_\chi(r).$$

The partial wave amplitude of the gauge-fixing mode $\mathcal{F}$ is obtained from the general definition

$$\mathcal{F}(\mathbf{x})=\nabla\mathbf{a}(\mathbf{x})+\xi\frac{H(r)}{2}\phi(\mathbf{x}). \tag{5.5}$$

This equation is expanded into partial waves. The radial mode function of the mode $\mathcal{F}$ then reads

$$f_{\mathcal{F}}(r)=f_b'(r)+\frac{2}{r}f_b(r)-\frac{\sqrt{l(l+1)}}{r}f_a(r)+\xi\frac{H(r)}{2}f_\phi(r). \tag{5.6}$$

It can be checked, using the basic differential equations (5.1)–(5.3) and the differential equation for the background field (4.15), that the mode $f_{\mathcal{F}}$ satisfies the differential equation for the Faddeev-Popov modes

$$-f_{\rm FP}''-\frac{2}{r}f_{\rm FP}'+\frac{l(l+1)}{r^2}f_{\rm FP}+\xi\frac{H^2(r)}{4}f_{\rm FP}=\omega^2f_{\rm FP}. \tag{5.7}$$

Likewise, if the gauge function $f_\chi(r)$ satisfies this differential equation then the mode functions $f_n^g$ generated from it via Eq. (5.4), satisfy the basic differential equations (5.1)–(5.3). This is as to be expected from the general arguments.

We now try to separate the system of differential equations by introducing a suitable set of new mode functions. We first eliminate the mode $f_a(r)$ in favor of $f_{\mathcal{F}}(r)$,

$$f_a(r)=-r\frac{f_{\mathcal{F}}(r)+f_b'(r)+2f_b(r)-(\xi/2)H(r)f_\phi(r)}{\sqrt{l(l+1)}}. \tag{5.8}$$

As mentioned above $f_{\mathcal{F}}(r)$ satisfies

$$-f_{\mathcal{F}}''-\frac{2}{r}f_{\mathcal{F}}'+\frac{l(l+1)}{r^2}f_{\mathcal{F}}+\xi\frac{H^2(r)}{4}f_{\mathcal{F}}=\omega^2f_{\mathcal{F}}. \tag{5.9}$$

Having eliminated $f_a(r)$ in this way it cannot be used anymore as gauge mode, for which now $f_b(r)$ is a possible candidate, however, one cannot use a simple algebraic substitution. We introduce the new mode function $f_g(r)$, analogous to $\chi(r)$, and eliminate $f_b(r)$ with the substitution

$$f_b(r)=\frac{d}{dr}f_g(r). \tag{5.10}$$

We make the two other amplitudes gauge invariant by defining

$$\tilde{f}_\phi(r)=f_\phi(r)+\frac{H(r)}{2}f_g(r), \tag{5.11}$$

$$\tilde{f}_{\mathcal{F}}(r)=f_{\mathcal{F}}(r)+\omega^2 f_g(r). \tag{5.12}$$

The latter equation follows from the general relation (3.16). We now have to find the equation of motion for the amplitude $f_g(r)$. In view of its close relation to the gauge function $\chi(r)$ we make the ansatz

$$-f_g''-\frac{2}{r}f_g'+\frac{l(l+1)}{r^2}f_g+\xi\frac{H^2(r)}{4}f_g=\omega^2 f_g+\mathcal{R}(r). \tag{5.13}$$

We insert the substitutions into the differential equations (5.2), (5.3), and (5.9) for the amplitudes $f_b(r), f_\phi(r)$, and $f_{\mathcal{F}}(r)$, respectively. We find, after some algebra, the equation

$$\begin{aligned}\frac{1}{r}\frac{d}{dr}r^2\mathcal{R}(r)=&\frac{1}{2r}\frac{d}{dr}r^2[\xi H(r)\tilde{f}_\phi(r)-\tilde{f}_{\mathcal{F}}(r)]\\&+\frac{1}{2}\left[\frac{d}{dr}H(r)\tilde{f}_\phi(r)-H(r)\frac{d}{dr}\tilde{f}_\phi(r)\right]\\&+\frac{1}{\xi}\frac{d}{dr}\tilde{f}_{\mathcal{F}}(r)\end{aligned} \tag{5.14}$$

as a consistency condition for $\mathcal{R}$. It can be solved readily

$$\begin{aligned}\mathcal{R}(r)=&\frac{\xi}{2}H(r)\tilde{f}_\phi(r)-\tilde{f}_{\mathcal{F}}(r)+\frac{1}{2r^2}\int_0^r dr'r'^2\Big[H'(r')\tilde{f}_\phi(r')\\&-H(r')\tilde{f}_\phi{}'(r')+\frac{2}{\xi}\tilde{f}_{\mathcal{F}}'(r')\Big].\end{aligned} \tag{5.15}$$

This fixes the right hand side of Eq. (5.13) for $f_g(r)$ which is one of the basic ones for the new amplitudes. The equations for the other amplitudes become

$$\begin{aligned}&-\tilde{f}_\phi''-\frac{2}{r}\tilde{f}_\phi'+\frac{l(l+1)}{r^2}\tilde{f}_\phi+\left\{m_H^2+\frac{\lambda}{g^2}\left[H^2(r)-\frac{3}{4}\epsilon H(r)\right]\right\}\tilde{f}_\phi\\&=\omega^2\tilde{f}_\phi-\frac{1}{2}H(r)\tilde{f}_{\mathcal{F}}-\frac{H(r)}{4r^2}\int_0^r dr'r'^2\Big[H'(r')\tilde{f}_\phi(r')\\&\quad-H(r')\tilde{f}_\phi'(r')+\frac{2}{\xi}\tilde{f}_{\mathcal{F}}'(r')\Big]\end{aligned} \tag{5.16}$$

$$\begin{aligned}&-\tilde{f}_{\mathcal{F}}''-\frac{2}{r}\tilde{f}_{\mathcal{F}}'+\frac{l(l+1)}{r^2}\tilde{f}_{\mathcal{F}}+\xi\frac{H^2(r)}{4}\tilde{f}_{\mathcal{F}}\\&=\omega^2\left\{-\xi\frac{H(r)}{2}\tilde{f}_\phi-\frac{1}{2r^2}\int_0^r dr'r'^2\Big[H'(r')\tilde{f}_\phi(r')\right.\\&\quad\left.-H(r')\tilde{f}_\phi'(r')+\frac{2}{\xi}\tilde{f}_{\mathcal{F}}'(r')\Big]\right\}.\end{aligned} \tag{5.17}$$

Obviously, we have not succeeded in separating the system. However, in this form the gauge and gauge-fixing modes are easy to identify. We see that with the choice $\tilde{f}_\phi=0$ and $\tilde{f}_{\mathcal{F}}=0$ the function $\mathcal{R}(r)$ vanishes and the differential equation for $f_g$ becomes the Faddeev-Popov equation again, with a corresponding energy spectrum. Likewise, the combination $f_{\mathcal{F}}=\tilde{f}_{\mathcal{F}}+\omega^2 f_g$ still satisfies Eq. (5.9) and has a Faddeev-Popov eigenvalue spectrum as well. However, we do not find another linearly independent combination of amplitudes involving the amplitude $\tilde{f}_\phi$ that would satisfy a differential equation independent of $\xi$. So that part of the energy spectrum that is not compensated by the Faddeev-Popov contributions apparently still depends on the choice of $\xi$.

Matters are different, however, if we evaluate the effective action. This can be done using the fluctuation modes at $\omega=0$, using a general theorem on fluctuation determinants [28], generalized to coupled systems, that has been used, e.g., for computing the fluctuation corrections to bubble nucleation [3]. It is based on the equation[2]

$$\mathcal{J}(\nu)\equiv\frac{\det(\mathbf{M}+\nu^2)}{\det(\mathbf{M}_0+\nu^2)}=\lim_{r\to\infty}\frac{\det\mathbf{f}(\nu,r)}{\det\mathbf{f}_0(\nu,r)}. \tag{5.18}$$

Here $\mathbf{M}$ is the partial wave fluctuation operator as defined previously, and the matrix $\mathbf{f}(\nu,r)$ is an $(n\times n)$ matrix formed by a fundamental system of $n$ linearly independent $n$-tuples of solutions for a given $\nu$, regular at $r=0$. The operator $\mathbf{M}_0$ and the solutions $\mathbf{f}_0$ refer to a trivial background field configuration, in the present case to the symmetric vacuum state characterized by $H(r)\equiv 0$. It is understood, that both systems $\mathbf{f}$ and $\mathbf{f}_0$ are started, at $r=0$ with identical initial conditions. Finally, the desired fluctuation determinant is given by $\mathcal{J}\equiv\mathcal{J}(0)$.

If we apply the theorem we only need the coupled system of differential equations for $\omega=i\nu=0$, and then it decouples in a triangular way. The right hand side of the equation for $\tilde{f}_{\mathcal{F}}$ vanishes entirely, the RHS of the differential equation for $\tilde{f}_\phi$ only depends on $\tilde{f}_{\mathcal{F}}$, while both $\tilde{f}_\phi$ and $\tilde{f}_{\mathcal{F}}$ appear on the RHS of the equation for $f_g$. Furthermore, for $\tilde{f}_{\mathcal{F}}=0$, the differential equation for $\tilde{f}_\phi$ becomes independent of $\xi$. We can choose the following set of linearly independent solutions: (i) a gauge mode solution $f_n^g$ with $\tilde{f}_{\mathcal{F}}^g\equiv 0$ and $\tilde{f}_\phi^g\equiv 0$, for which $f_g^g$ evolves in the same way as a pure Faddeev-

[2]For a short proof along the lines of Ref. [28] see Ref. [29].

Popov mode; (ii) a ''physical'' solution $f_n^\phi$ with $\tilde{f}_{\mathcal{F}}^\phi \equiv 0$; then $\tilde{f}_\phi^\phi$ evolves independently; it appears on the right hand side of the differential equation for $f_g^\phi$, which can be obtained by using the Green function of the homogenous equation, and finally (iii) a gauge-fixing mode solution $f_n^{\mathcal{F}}$, where $\tilde{f}_{\mathcal{F}}^{\mathcal{F}}$ is different from zero. For $\nu=0$ the RHS of Eq. (5.17) vanishes and $\tilde{f}_{\mathcal{F}}$ evolves is a Faddeev-Popov mode. Both other amplitudes are different from zero in this case. Note that the second type of solution is determined only modulo an arbitrary multiple of the first one, and the third one only modulo arbitrary multiples of both other ones. This does not affect the determinant $\det \mathbf{f}(0,r)$, however.

The structure of the matrix $\mathbf{f}(0,r)$ now is triangular and its determinant is obtained from the diagonal elements as

$$\det \mathbf{f}(0,r) = f_g^g(0,r)\tilde{f}_{\mathcal{F}}^{\mathcal{F}}(0,r)\tilde{f}_\phi^\phi(0,r) = f_{\mathrm{FP}}^2(0,r)\tilde{f}_\phi^\phi(0,r). \tag{5.19}$$

The same structure holds for the free solutions which have to be started at $r=0$ with identical initial conditions, i.e., with the same coefficients of the lowest powers of $r$, as determined by the centrifugal barriers. We have considered the behavior at $r=0$ in detail and have verified that an appropriate choice is possible.

The effective action is obtained by adding the logarithms of the various fluctuation determinants for all independent systems, and for all partial waves. The only $\xi$ dependence occurs in the gauge and gauge-fixing modes of the coupled system, and for the two Faddeev-Popov modes. As these compensate each other the total effective action becomes independent of $\xi$.

For the practical computation this means that for the coupled system we just have to solve the integrodifferential equation for $\tilde{f}_\phi$ with $\tilde{f}_{\mathcal{F}}=0$, i.e.,

$$-\tilde{f}_\phi'' - \frac{2}{r}\tilde{f}_\phi' + \frac{l(l+1)}{r^2}\tilde{f}_\phi + \left\{ m_H^2 + \frac{\lambda}{g^2}\left[ H^2(r) - \frac{3}{4}\epsilon H(r)\right]\right\}\tilde{f}_\phi = \frac{H(r)}{4r^2}\int_0^r dr' r'^2 [H'(r')\tilde{f}_\phi(r') - H(r')\tilde{f}_\phi'(r')]. \tag{5.20}$$

From this derivation and discussion it is clear that the gauge independence only holds for the effective action, and not for other physical quantities. The nondiagonal parts of the mode solutions still depend on $\xi$, so other expectation values are affected by the gauge parameter $\xi$.

## VI. CONCLUSION

We have given general arguments, based on the fluctuation operator and the mode expansion, for the gauge independence of the one-loop effective action, computed for a background field which solves the classical equation of motion. There are various cases for which the one-loop effective action, and its gauge independence, are of interest. It appears in particular in the corrections to quantum or thermal tunneling rates obtained in the semiclassical approximation. For the case of bubble nucleation the analysis of Sec. III fully applies. We have verified in the partial wave mode equations [3], that the system of these differential equations can be cast, at zero frequency, into a triangular form. A theorem on fluctuation determinants relates the fluctuation determinant to the asymptotic behavior as $r\to\infty$, of a linearly independent system of solutions regular at $r=0$. The matrix formed by the $n$ linearly independent $n$-tuples of solutions can be cast into a triangular form as well, two of the diagonal elements evolve similar to the Faddeev-Popov modes. Their contribution to the logarithm of the fluctuation determinant is cancelled by one of the Faddeev-Popov modes. The remaining diagonal elements are independent of $\xi$. The final conclusion is that the *exact* one-loop correction to the nucleation rate is gauge independent. This goes beyond the results of Ref. [16], where a similar statement was derived for the leading orders in the gauge coupling, using the gradient expansion. This latter publication is, in part, complementary to our work: we have not considered the divergent parts and renormalization. Within our method [2,3] the divergent parts can be separated analytically from the computation of the determinants and take the form of ordinary Feynman graphs. So the $\xi$ independence of the renormalized leading order contributions, as established in Ref. [16], closes our argument.

We should like to add a comment on the use of partial resummations. Indeed the high temperature effective potential (4.11), from which the classical solution is computed, already contains one-loop effects. This introduces some double counting that has to be compensated for [11]. If the resummation includes the coupled gauge-Higgs sector, the $\xi$ dependence is there from the outset and can disappear only if all higher-loop orders are summed up. One therefore has to make sure that the high temperature resummation, that in an essential way determines the structure of the phase transition, takes into account transverse gauge loops and the isoscalar Higgs loop only. Then this modification of the ''classical'' Higgs potential does not interfere with our analysis.

It is not clear how far the conclusions obtained for the special case considered here can be generalized to different gauge theories, and to different physical systems. For the sphaleron [9], and also for topologically nontrivial solutions in other models, as the instanton of the abelian Higgs model in $(1+1)$ dimensions [11], the application of the determinant theorem meets difficulties [30]: The contribution of the $s$ wave diverges and the compensation of this divergence by the sum over the higher partial waves requires a suitable regularization. It would be worthwhile to pursue this issue.

Another system which should be investigated are the quantum fluctuations for the SU(2) Higgs model in nonequilibrium dynamics [17]. Here the applicability is certainly limited by the inclusion of the quantum back reaction on the background field. This back reaction changes the classical equation of motion, while analogous changes of the quantum mode equations depend on the resummation.

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