Thermal fermionic quantum field in a static background gauge potential

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We study at finite temperature the Green function and energy-momentum tensor $T_{\mu\nu}(x)$ of a spinor field in 1+1 dimensions, interacting with a static background electric field. $T_{\mu\nu}$ separates into a UV divergent part representing the virtual sea, and a UV finite part describing the thermal plasma of the spinor field. From $T_{\mu\nu}$ we find that the virtual sea remains uniform in the presence of a *uniform* electric field *E*, while the thermal plasma becomes position dependent. This remarkable property of the thermal plasma is found to be related to the topological properties of the manifold, and to the presence of zero modes. [S0556-2821(99)03622-X]

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I. INTRODUCTION

At finite temperature ($T>0$) a quantum field can be visualized as a *sea* of virtual particles and a thermal *gas* of real field excitations. The virtual particle sea is independent of the temperature *T*. It can, however, be deformed by coupling the quantum field to a static background involving external fields and/or boundary surfaces. This is generally known as the static vacuum Casimir effect.

Abelian (and non-Abelian) gauge theories present Casimir problems with particular features whose origin can be traced to the underlying gauge invariance. Indeed, the restrictions imposed by $T>0$ on the class of allowed gauge transformations are found to have remarkable consequences for the spatial energy distribution of a charged thermal matter field coupled to a static background electromagnetic field. The resulting local distortion of the virtual sea and thermal *gas* (or *plasma*, as the thermal gas consists of both particles and antiparticles) is revealed by a study of local quantities, such as the thermal stress energy momentum tensor. Though the problem of charged quantum fields coupled to a uniform electromagnetic background field is an old one, going back to famous papers of Euler and Heisenberg [1] and Schwinger [2], global aspects of this problem have received most of the attention [3–7], while local aspects seem to have been neglected.

In the present paper we try to gain some general insight into the local response of fermionic, thermal matter fields coupled to a background electromagnetic field. We restrict our considerations to 1+1 dimensions, where exact results can be obtained. Using the Matsubara formalism (see e.g., Refs. [8–10]), we work on a cylinder of circumference $\beta=1/T$ in the Euclidean time direction, choosing space to be flat and infinite. Our results reveal some interesting features:

(i) We find that a uniform background electric field causes the thermal plasma to become position dependent, while the sea remains spatially uniform. The position dependence of the thermal plasma can be traced to the underlying gauge invariance.

(ii) This position dependence is also directly related to the existence of zero modes of the Dirac operator. For a finite ''volume'' *L* we have a finite number κ of such zero modes; for each zero mode we have an infinite tower of equally spaced ''excited'' levels, each level being κ-fold degenerate. The situation is thus reminiscent of the Landau levels in the quantum Hall effect.

(iii) Our exact results for the heat-kernel are in disagreement with a theorem [11] concerning the factorization of the heat-kernel in a general external gauge field A_μ. In the conclusion we explain the limitations of this theorem, which turns out to be valid only for $A_0=0$ and arbitrary static magnetic field. We also show how it can be generalized to include static electric fields, as well.

All three of the above features are fully revealed for the case of a constant electric field. Even for a constant gauge field A_μ (vanishing electric field) we obtain observable effects, since on the ''cylinder,'' a constant time component of a gauge field cannot be gauged away [12,13].

The general solution for an arbitrary gauge field can then be constructed on ''top'' of the special solution for $E=\text{const}$ using standard methods familiar from the treatment of the Schwinger model. Since this has been amply discussed in the literature, we shall restrict our discussion to the case of a constant external electric field.

II. THERMAL FERMIONS IN AN EXTERNAL POTENTIAL

Electrodynamics of massless fermions in 1+1 dimensions is described by the Lagrangian density

$$\mathcal{L}(x)=-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)+\bar{\psi}(x)(i\partial\!\!\!/+eA\!\!\!/)\psi(x). \tag{2.1}$$

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For $T=0$ the fully quantized version of this model (Schwinger model) has been extensively discussed in the literature. (See for instance [14].) The case $T\neq 0$ has also been discussed by a number of authors [15–17]. We shall concentrate in this paper on the specific aspects mentioned in the Introduction. Aside from the usual complications of spin, we shall have to face in the infinite volume limit the existence of an infinite number of normalizable zero modes. We work in the imaginary time formalism [9,10], where the fermion field is required to satisfy anti-periodic boundary conditions[1]

$$\psi(x_1,x_2)=-\psi(x_1,x_2+\beta). \tag{2.2}$$

This implies the Euclidean space-time topology of a cylinder $\mathfrak{R}\times S^1$.

From the antiperiodicity property of the spinor field we can write the stationary modes of $\psi(x)$ in the Matsubara form

$$\psi_{mn}(x)=\frac{1}{\sqrt{\beta}}e^{i(2m+1)(\pi/\beta)x_2}\begin{pmatrix}\varphi_{mn}(x_1)\\ \overline{\varphi}_{mn}(x_1)\end{pmatrix} \tag{2.3}$$

with $m=0,\pm 1,\pm 2,\ldots$. The Dirac equation for the modes then takes the form

$$\left(-(2m+1)\frac{\pi}{\beta}+A_2(x_1)+\gamma^5 D_1\right)\varphi_{mn}(x_1)=\lambda_{mn}\gamma_2\varphi_{mn}(x_1), \tag{2.4}$$

with $D_1=\partial_1-ieA_1(x_1)$. In this form the Dirac equation displays the gauge equivalence of $A_2(x_1)$ and $A_2(x_1)\pm 2\pi/\beta$ configurations. $A_2(x_1)$ thus has the character of an angular variable, as will become also evident from our exact results.

In this section we discuss the case of a spinor field in a static background potential $A_2=\text{const}+Ex_1$, corresponding to a constant electric field. Our final results for local quantities such as the heat kernel display a smooth $E\to 0$ limit, although the mathematical details of the $E=0$ and $E\neq 0$ cases are quite different.

A. Constant electric field

Consider the case of a constant electric field, with the choice

$$A_1=0,\quad eA_2(x)=Ex_1+2\pi a/\beta \tag{2.5}$$

for the Euclidean gauge potential.[2] The eigenvalue equation for the corresponding Dirac operator then takes the form

$$\left[\left(i\partial_2+Ex_1+\frac{2\pi}{\beta}a\right)+\gamma^5\partial_1\right]\psi(x)=\lambda\gamma_2\psi(x). \tag{2.6}$$

Making the ansatz

$$\psi(x)=\frac{1}{\sqrt{\beta}}e^{i(2m+1)(\pi/\beta)x_2}\begin{pmatrix}\varphi(x_1)\\ \overline{\varphi}(x_1)\end{pmatrix} \tag{2.7}$$

and defining

$$x_m=x_1-(2m+1)\frac{\pi}{E\beta}+\frac{2\pi a}{E\beta} \tag{2.8}$$

we arrive at the coupled set of equations

$$\begin{aligned}\left(Ey+\frac{d}{dy}\right)\varphi&=-\lambda\overline{\varphi}\\ \left(Ey-\frac{d}{dy}\right)\overline{\varphi}&=-\lambda\varphi\end{aligned} \tag{2.9}$$

where we have set $y=x_m$. Define the operators

$$a=\frac{1}{\sqrt{2}}\left(\sqrt{|E|}y+\frac{1}{\sqrt{|E|}}\frac{d}{dy}\right) \tag{2.10}$$

$$a^\dagger=\frac{1}{\sqrt{2}}\left(\sqrt{|E|}y-\frac{1}{\sqrt{|E|}}\frac{d}{dy}\right). \tag{2.11}$$

These operators evidently satisfy the commutation relations of destruction and creation operators, respectively:

$$[a,a^\dagger]=1.$$

Substituting one equation into the other in (2.9) we have, depending on the sign of E,

$$E\ \ \text{positive:}\begin{Bmatrix}2|E|aa^\dagger\overline{\varphi}=\lambda^2\overline{\varphi}\\ 2|E|a^\dagger a\varphi=\lambda^2\varphi\end{Bmatrix} \tag{2.12}$$

and

$$E\ \ \text{negative:}\begin{Bmatrix}2|E|a^\dagger a\overline{\varphi}=\lambda^2\overline{\varphi}\\ 2|E|aa^\dagger\varphi=\lambda^2\varphi\end{Bmatrix}. \tag{2.13}$$

Now, $2|E|a^\dagger a$ is just the Hamiltonian of the harmonic oscillator with the zero-point energy omitted. Correspondingly φ and $\overline{\varphi}$ are given by the harmonic oscillator eigenfunctions. Defining the ground state $|0\rangle$ by $a|0\rangle=0$ we conclude that the eigenstates and corresponding eigenvalues are given by

$$E\ \ \text{positive:}|\Psi^{(\pm)}\rangle=\begin{pmatrix}|n\rangle\\ \mp|n-1\rangle\end{pmatrix},\quad \lambda_n=\pm\sqrt{2n|E|}$$

and

$$E\ \ \text{negative:}|\Psi^{(\pm)}\rangle=\begin{pmatrix}|n-1\rangle\\ \mp|n\rangle\end{pmatrix},\quad \lambda_n=\pm\sqrt{2n|E|}$$

[1]We use a Euclidean notation, where $x_2=ix^0$. Our Euclidean conventions for the Dirac γ-matrices are $\gamma_2=-\sigma_1$, $\gamma_1=\sigma_2$, $\gamma^5=\sigma_3$. Notice that since we restrict ourselves to static background field configurations, the gauge field trivially satisfies $A_\mu(x_1,x_2)=A_\mu(,x_1,x_2+\beta)$.

[2]As is well known (see e.g., [18]), the constant term $2\pi a/\beta$ in (2.5) corresponds to the introduction of a chemical potential in Minkowski space-time.

where $|n\rangle$ are the eigenstates of the harmonic oscillator and λ_n^2 are the corresponding energy-eigenvalues without the ''zero-point energy.'' Denoting by $\varphi_n(x_1)$ the eigenfunctions of the harmonic oscillator, normalized with respect to the interval $[-\infty,\infty]$ and setting $y=x_m$, we have for $n\geqslant 1$ the orthonormalized eigenfunctions of the Dirac operator (2.6),

$$\psi_{m,n}^{(\pm)}(x)=\frac{1}{\sqrt{2\beta}}e^{i(2m+1)(\pi/\beta)x_2}\begin{pmatrix}\varphi_n(x_m)\\ \mp\varphi_{n-1}(x_m)\end{pmatrix},\quad n\geqslant 1, \tag{2.14}$$

for positive E, and

$$\psi_{m,n}^{(\pm)}(x)=\frac{1}{\sqrt{2\beta}}e^{i(2m+1)(\pi/\beta)x_2}\begin{pmatrix}\varphi_{n-1}(x_m)\\ \mp\varphi_{n}(x_m)\end{pmatrix},\quad n\geqslant 1, \tag{2.15}$$

for negative E, each corresponding to the eigenvalues

$$\lambda_n=\pm\sqrt{2n|E|}, \tag{2.16}$$

respectively. Since the spectrum corresponds to the absence of the ''zero-point energy'' of the harmonic oscillator, we have an infinite set of orthonormalized zero modes labeled by m and chirality, of the form

$$\phi_m^{(+)}(x)=\frac{1}{\sqrt{\beta}}e^{i(2m+1)(\pi/\beta)x_2}\begin{pmatrix}\varphi_0(x_m)\\ 0\end{pmatrix} \tag{2.17}$$

for positive E, and

$$\phi_m^{(-)}(x)=\frac{1}{\sqrt{\beta}}e^{i(2m+1)(\pi/\beta)x_2}\begin{pmatrix}0\\ \varphi_0(x_m)\end{pmatrix} \tag{2.18}$$

for negative E, each corresponding to the eigenvalue $\lambda_0=0$. This is in line with the Atiyah-Singer index theorem in the infinite volume limit. Indeed, for a finite ''volume'' L we have (see [16])

$$n_+-n_-=\frac{e}{4\pi}\int_0^\beta dx_2\int_0^L dx_1\epsilon_{\mu\nu}F_{\mu\nu}(x)=\kappa, \tag{2.19}$$

where n_+ and n_- are the zero modes of positive and negative chirality, respectively. Equation (2.19) relates the index of the Dirac operator to the flux of the ''magnetic field''[3] through the torus. Notice that this flux is quantized in integer multiples of $2\pi/e$, and depends on the temperature $T=1/\beta$:[4]

$$E=\kappa\frac{2\pi}{\beta L}. \tag{2.20}$$

As we shall explicitly see in the following section, there exists again a ''vanishing theorem,'' just as in the $T=0$ case, stating that zero modes occur only for either positive or negative chirality. Notice that in the case of the zero-modes (2.17) and (2.18), the superscript denotes ''chirality.''

The wave functions (2.17) and (2.18) correspond to the ground state wave functions of the harmonic oscillator, localized at the positions

$$x_1=(2m+1)\pi/\beta E-2\pi a/E\beta$$

with $m\in Z$. This provides a physical interpretation of the degeneracy of the spectrum. In order to gain further insight into the problem, we examine next the effective Lagrangian giving rise to this degeneracy, as defined in terms of the ''local'' ζ-function.

In order to simplify the discussion, we shall restrict ourselves in the following to the case where E is positive.

B. Effective Lagrangian density

We begin by considering the local heat kernel. For the case in question it takes the form (we now take $E>0$; we also include the zero modes)

$$h_{\alpha\beta}^{(\beta)}(t;x,y)=\sum_{m=-\infty}^{\infty}\left(\sum_{n=1}^{\infty}\sum_{\sigma=\pm}e^{-2nEt}\psi_{n,m}^{(\sigma)}(x)_\alpha\psi_{n,m}^{(\sigma)}(y)_\beta^*+\phi_m^{(+)}(x)_\alpha\phi_m^{(+)}(y)_\beta^*\right)$$

or explicitly

$$h_{\alpha\beta}^{(\beta)}(t;x,y)=\sum_{m=-\infty}^{\infty}\frac{1}{\beta}e^{i(2m+1)(\pi/\beta)(x_2-y_2)}\times\begin{pmatrix}\sum_{n=0}^{\infty}e^{-2nEt}\varphi_n(x_m)\varphi_n^*(y_m) & 0\\ 0 & \sum_{n=1}^{\infty}e^{-2nEt}\varphi_{n-1}(x_m)\varphi_{n-1}^*(y_m)\end{pmatrix}. \tag{2.21}$$

[3]In Euclidean space-time we may regard F_{12} as the component of the magnetic field perpendicular to the (12)-plane.
[4]In the $T=0$ case such a quantization emerges only after stereographic projection [21,20] (see also chapter 4 of [14]).

The diagonal matrix structure is again a consequence of the existence of a pair of eigenfunctions $\psi_n^{(\pm)}$ corresponding to the eigenvalues $\pm\sqrt{2nE}$, if $n\neq 0$. We now observe that (note that the sum starts with $n=0$)

$$\sum_{n=0}^{\infty} e^{-2nEt}\varphi_n(x_m)\varphi_n^*(y_m)=e^{Et}\langle x_m|e^{-tH_{HO}}|y_m\rangle \tag{2.22}$$

where the matrix element on the right-hand side (RHS) is the propagation kernel of the harmonic oscillator known to be given by[5]

$$\langle x|e^{-tH_{HO}}|y\rangle = \sqrt{\frac{E}{\pi}}\frac{e^{-Et}}{\sqrt{1-e^{-4Et}}} \times\exp\left(-\frac{E}{2}\,\frac{(x^2+y^2)(1+e^{-4Et})-4xye^{-2Et}}{1-e^{-4Et}}\right). \tag{2.23}$$

Going to the limit of coincident points $x=y$, and taking the trace in matrix space, we arrive at

$$\mathrm{tr}h^{(\beta)}(t;x,x)=\sum_{m=-\infty}^{\infty}\frac{1}{\beta}[2\cosh Et\langle x_m|e^{-tH_{HO}}|x_m\rangle] =\frac{1}{\beta}\sqrt{\frac{E}{\pi}}\frac{1}{\sqrt{\tanh Et}}\sum_{m=-\infty}^{\infty} e^{-Ex_m^2\tanh Et}. \tag{2.24}$$

Making use of the identity

$$\frac{1}{\beta}\sum_{m=-\infty}^{\infty} e^{-Ex_m^2\tanh Et} =\left[\frac{E}{4\pi\tanh Et}\right]^{1/2}\sum_{n=-\infty}^{\infty} e^{-n^2\beta^2E/4\tanh Et} \times e^{-in(2\pi a+x_1\beta E)}. \tag{2.25}$$

We may thus write the Euclidean heat kernel (2.24) in the form

$$\mathrm{tr}\,h^{(\beta)}(t;x,x)=\frac{E}{2\pi}\left(\frac{1}{\tanh Et}\right)\left\{1+2\sum_{m=1}^{\infty}(-1)^m \times\cos[m(E\beta x_1+2\pi a)]e^{-m^2\beta^2E/4\tanh Et}\right\}. \tag{2.26}$$

In order to compute the effective Lagrangian density we first need to subtract the zero-mode contribution,

$$\mathrm{tr}\,h'^{(\beta)}(t;x,x)=\left[\mathrm{tr}\,h^{(\beta)}(t;x,x)-\frac{1}{\beta}\sum_{m=-\infty}^{\infty}\varphi_0(x_m)\varphi_0^*(x_m)\right],$$

where $\varphi_0(x)$ is the zero-energy harmonic oscillator wave function:

$$\varphi_0(x)=\left(\frac{E}{\pi}\right)^{1/4}e^{-(E/2)x^2}.$$

Using the Jacobi identity [19]

$$\sum_{m=-\infty}^{\infty} e^{-b(m-a)^2}=\sqrt{\frac{\pi}{b}}\sum_{l=-\infty}^{\infty} e^{-\pi^2l^2/b}e^{-i2\pi al} \tag{2.27}$$

we have

$$\mathrm{tr}\,h'^{(\beta)}(t;x,x)=\frac{E}{2\pi}\left[f_0(t)+2\sum_{m=1}^{\infty}(-1)^m \times\cos[m(E\beta x_1+2\pi a)]f_m(t)\right], \tag{2.28}$$

where

$$f_m(t)=\frac{1}{\tanh Et}e^{-m^2\beta^2E/4\tanh Et}-e^{-m^2\beta^2E/4}$$

and the ''prime'' indicates the exclusion of zero-modes. Finally we define the effective Lagrangian[6]

$$\mathcal{L}_{eff}(x_1)=\frac{1}{2}\left[\frac{d}{ds}\zeta^{(\beta)}(s;x,x)\right]_{s=0}+\frac{1}{2}\zeta(0;x,x)\ln\mu^2 =\mathcal{L}_{sea}+\mathcal{L}^{\beta}_{plasma}(x_1), \tag{2.29}$$

where

$$\zeta^{(\beta)}(s;x,x)=\frac{1}{\Gamma(s)}\int_0^{\infty}dtt^{s-1}h'^{(\beta)}(t;x,x)$$

and μ is an arbitrary scale paramenter reflecting the usual ambiguity associated with a change in scale of the dimensionful eigenvalues λ_n.

A simple calculation yields for $\mathcal{L}^{\beta}_{plasma}(x_1)$ the result

[5]The Hamiltonian in our case is of the form $H=p^2+E^2y^2$, and thus correponds to making the identifications $m=\frac{1}{2}$, $\omega=2E$ in the conventional Hamiltonian.

[6]In the ζ-function regularization the ambiguity in the calculation of the effective action is well known to be determined by $\zeta(0)$: $\ln\det A=-\zeta'(0)+\zeta(0)\ln\mu^2$, where μ is an arbitrary scale parameter.

$$\mathcal{L}^{\beta}_{plasma}(x_1)=\int_0^\infty dt t^{-1}\frac{E}{2\pi}\sum_{m=1}^{\infty}(-)^m f_m(t)\times\cos m(\beta E x_1+2\pi a). \quad (2.30)$$

The corresponding calculation of $\mathcal{L}_{sea}(x_1)$ is slightly more involved. The temperature independent term in (2.28) contributes to the thermal zeta function the term

$$\zeta_{sea}(s;x,x)=\frac{\epsilon}{\Gamma(s)}\int_0^\infty dt t^{s-1}f_0(t)=\frac{2\epsilon}{\Gamma(s)}\sum_{n=1}^{\infty}\int_0^\infty dt t^{s-1}e^{-2nEt}=2\epsilon\sum_{n=1}^{\infty}\frac{1}{(2nE)^s}=\frac{2\epsilon}{(2E)^s}\zeta_R(s) \quad (2.31)$$

where $\epsilon=E/2\pi$ and $\zeta_R(s)$ is the Riemann ζ-function

$$\zeta_R(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

Differentiating with respect to s, setting $s=0$ and using $\zeta_R(0)=-\frac{1}{2}$, $\zeta_R'(0)=-\frac{1}{2}\ln 2\pi$, we obtain

$$\mathcal{L}_{sea}=\frac{E}{4\pi}\ln\left(\frac{E}{\pi\mu^2}\right). \quad (2.32)$$

As expected, $\mathcal{L}_{sea}$ does not depend on x_1 or on temperature, and does depend on the gradient of A_2, but not on A_2 itself. On the other hand, as we see from Eq. (2.30), $\mathcal{L}^{\beta}_{plasma}(x_1)$ does depend directly on A_2, and consequently is not uniform, and in fact is periodic in A_2 (and hence in x_1). This periodicity can be understood as follows:

Let us compactify space to a circle of perimeter L, so that Euclidean space-time is now a torus $S^1\times S^1$. The bosonic (fermionic) observables should then be a periodic (antiperiodic) function of the spatial coordinate x_1, with period L. If we want a configuration with a non-trivial index (''winding number'') κ, then we must allow A_μ and ψ to change by a bonafide gauge transformation, as we go around a closed loop on the torus in the x_1 direction:

$$A_\mu(x_1+L,x_2)=A_\mu(x_1,x_2)+\frac{2\pi\kappa}{e\beta}\delta_{\mu,2} \quad (2.33)$$

$$\psi(x_1+L,x_2)=e^{i\frac{2\pi\kappa}{\beta}x_2}\psi(x_1,x_2). \quad (2.34)$$

The integer κ corresponds here to the index in the Atiya-Singer theorem (2.19), and labels also the constant gauge field configuration (2.5) via the flux quantization condition (2.20). In this way the observed periodicity in A_2 with period $2\pi/\beta$ (or x_1 with period $2\pi/E\beta$) gets intertwined with the allowed gauge transformations.

Periodicity in x_1 is however not a physical prediction. Indeed, in order to make contact with the physical problem in question, we must continue our results back to Minkowski space. This requires the substitution $A_2\to iA_0$ in the gauge potential. For the gauge potential (2.5) this corresponds to performing the analytic continuation $E\to iE$ and $2\pi a/\beta\to i\mu$, where $\mathcal{E}=E/e$ is now the (physical) electric field, and μ now plays the role of a chemical potential (see e.g., [18] and references therein). In the thermal heat kernel (2.28) and $\mathcal{L}_{plasma}$ this implies replacing $\cos m(\beta E x_1+2\pi a)$ under the sum by $\cosh m(\beta E x_1+\beta\mu)$. Spatial periodicity is therefore not a property of the thermal plasma in Minkowski space-time. Nonetheless, as already mentioned, the ''physical'' plasma turns out to be position dependent as a result of the above periodicity in Euclidean space.

The contribution of the plasma to $\mathcal{L}_{eff}$ is real, also in Minkowski space-time, and hence does not contribute to particle production. This is in line with the results of Refs. [5] and [7]. Physically it also seems reasonable that particle production should occur exclusively from the virtual sea, and therefore should be temperature independent. Indeed, the analytic continuation of (2.32) to Minkowski space-time is given by

$$\mathcal{L}_{sea}=-\frac{E}{8\pi}+i\frac{E}{4\pi}\ln\left(\frac{E}{\pi\mu^2}\right). \quad (2.35)$$

As expected, $\mathcal{L}_{sea}$ has an imaginary part, corresponding to particle production.

III. CONCLUSION

We have investigated the effect of a uniform background electric field on the distribution of thermal fermionic matter fields. In order to obtain exact results, we have restricted our discussion to fermions in 1+1 dimensions. By calculating the heat kernel and the Euclidean effective Lagrangian density we found that in the presence of a constant electric field the thermal plasma distribution of the fermion field becomes position dependent (in fact periodic) along the spatial direction x, while the virtual sea remains uniform. Compactifying space to a circle of perimeter L, this position dependence was traced to the quasi-periodicity property (2.33) under a bonafide gauge transformation. The periodic x-dependence of the plasma was also shown to reflect the existence of a degeneracy of the eigenvalue spectrum of the spectral operator, with degree equal to the number of zero modes as given by the Atiyah-Singer theorem. Attention was drawn to the formal similarity between this degneracy, and the degeneracy of the Landau levels in the quantum Hall effect. This results from the fact that, at finite temperature, one is working in two-dimensional Euclidean space, with the temporal direction compactified.

The diagonal heat kernel of a quantum field in a *static* background at finite temperature $T>0$ is in general expected to factorize in the following way:

$$h^{(\beta)}(t;x,x)=h(t;x,x)_{T=0}[1+f(t;x;T)]$$

where $h(t;x,x)_{T=0}$ is the temperature-zero heat kernel for the same background, and $f(t,x,T)$ is some function of the temperature T, the diffusion or ''proper'' time t and the spatial position. This function f vanishes exponentially as either $T\to 0$ or $t\to 0$. The factorization above is motivated by the expectation that $h^{(\beta)}(t;x,x)$ separates quite generally for a static background into an UV divergent sea part, and an UV finite gas part

$$h^{(\beta)}(t;x,x)=h(t;x,x)_{sea}+h(t;x)_{gas}$$

where

$$h(t;x,x)_{sea}=h(t;x,x)_{T=0}.$$

Defining $f(t;x;T)$ by

$$h(t;x)_{gas}=f(t;x;T)h(t;x,x)_{T=0}$$

we arrive at the factorization above.

We have shown in this paper for the case of thermal fermions in an external Euclidean electrostatic potential $A_2=Ex_1+2\pi a/\beta$, that

$$1+f=\sum_{n=-\infty}^{\infty}(-)^n e^{-n\beta^2 E/4\tanh Et}e^{-in\beta A_2}.$$

We thus see that $f(t,x;T)$ in general depends on A_2, when a gauge potential background is involved. This disagrees with the claim, made in Ref. [11], that f has the simple form

$$1+f=\sum_{n=-\infty}^{\infty}(-)^n e^{-n^2\beta^2/4t}$$

for a general (even time-dependent) background gauge potential A_μ. In fact, this claim applies only to the case where $A_1=A(x_1)$ is an arbitrary function of the spatial coordinate, and $A_2=0$. Although the general analysis in Ref. [11] is formally correct, the conclusion of the authors is incorrect, as they have taken terms involving powers of the covariant derivative $\mathcal{D}_0$ to give a vanishing contribution, when acting to the right on ''one.'' It is easy to see that, for the case in question, the contribution of the first two Seeley coefficients in the Seeley expansion of Ref. [11] combine to give the first two terms in the series expansion of the cosine term in (2.26).

Our results for the heat kernel and related functions are Fourier series in the Euclidean gauge field A_2 in (2.5). This is a consequence of the compactification of A_2 at finite temperature. This leads us to expect that such Fourier series also arise for an arbitrary static background time component A_2. The same is expected to be true in higher dimensions.

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