Somewhere in the universe: Where is the information stored when histories decohere?

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In the context of the decoherent histories approach to quantum theory, we investigate the idea that decoherence is connected with the storage of information about the decohering system somewhere in the universe. The known connection between decoherence and the existence of records is extended from the case of pure initial states to mixed states, where it is shown that records may still exist but are necessarily imperfect. We formulate an information-theoretic conjecture about decoherence due to an environment; the number of bits required to describe a set of decoherent histories is approximately equal to the number of bits of information thrown away to the environment in the coarse-graining process. This idea is verified in a simple model consisting of a particle coupled to an environment that can store only one bit of information. We explore the decoherence and information storage in the quantum Brownian motion model, in which a particle trajectory is decohered as a result of coupling to an environment of harmonic oscillators in a thermal state. It is shown that the variables that the environment naturally measures and stores information about are nonlocal functions of time, which are essentially the Fourier components of the function $x(t)$ (describing the particle trajectory). In particular, the records storing the information about the Fourier modes are the positions and momenta of the environmental oscillators at the final time. We show that it is possible to achieve decoherence even if there is only one oscillator in the environment. The information count of the histories and records in the environment adds up according to our conjecture. These results give quantitative content to the idea that decoherence is related to ''information lost.'' Some implications of these ideas for quantum cosmology are discussed.
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I. INTRODUCTION

The notion of decoherence plays an important role in discussions of the foundations of quantum theory, particularly in investigations of the emergence of classical behavior [1–4]. Decoherence typically arises as a result of a coarse-graining scheme—dividing the system into subsystem and environment, for example, and then tracing out the environment. Decoherence is then often regarded as a kind of generalized measurement process: the environment producing the decoherence ''measures'' the decohering subsystem, and ''stores information'' about it. Indeed, it can be argued that the physical significance of decoherence is that it ensures the storage of information about the decohering system's properties somewhere in the universe [2,5].

These appealing ideas are frequently mentioned in the literature, and some general theorems supporting them exist [2,5]. However, it is probably fair to say that, despite the concrete mathematical grip we now have on the notion of information, there is still considerable scope for their development and implementation in physically interesting models. This paper will focus on precisely these issues, through two particular questions. First, when a system decoheres as a result of coupling to an environment, how, in practice, can the system's history be reconstructed by examining the environment? That is, which properties of the environment carry the information about the decohered system? Second, how is the *amount* of information stored by the environment related to the nature or degree of decoherence of the system?

We will address these issues in the context of the decoherent histories approach to quantum theory [6,2,7–10]. (Other approaches to decoherence, such as Zurek's ''einselection'' approach [11,4], related density matrix approaches [3], or quantum state diffusion [12,13], may be equally useful for analyzing these issues, but will not be explored here.) In the decoherent histories approach, probabilities are assigned to histories via the formula

$$p(\alpha_1,\alpha_2,\dots)=\mathrm{Tr}(C_{\underline{\alpha}}\rho C_{\underline{\alpha}}^{\dagger}) \tag{1.1}$$

where $C_{\underline{\alpha}}$ denotes a time-ordered string of projectors interspersed with unitary evolution,

$$C_{\underline{\alpha}}=P_{\alpha_n}e^{-(i/\hbar)H(t_n-t_{n-1})}P_{\alpha_{n-1}}\cdots e^{-(i/\hbar)H(t_2-t_1)}P_{\alpha_1} \tag{1.2}$$

and $\underline{\alpha}$ denotes the string $\alpha_1,\alpha_2,\dots,\alpha_n$. Of particular interest are sets of histories which satisfy the condition of decoherence, which is that the decoherence functional

$$D(\underline{\alpha},\underline{\alpha}')=\mathrm{Tr}(C_{\underline{\alpha}}\rho C_{\underline{\alpha}'}^{\dagger}) \tag{1.3}$$

is zero when $\underline{\alpha}\neq\underline{\alpha}'$. Decoherence implies the weaker condition that $\mathrm{Re}\,D(\underline{\alpha},\underline{\alpha}')=0$ for $\underline{\alpha}\neq\underline{\alpha}'$, and this is equivalent to the requirement that the above probabilities satisfy the probability sum rules.

But for us the stronger condition of decoherence is the more interesting one since it is related to the existence of records. In particular, if the initial state is pure, there exist a set of records at the final time t_n which are perfectly correlated with the alternatives $\alpha_1\dots\alpha_n$ at times $t_1\dots t_n$ [2]. This follows because, with a pure initial state $|\Psi\rangle$, the decoherence condition implies that the states $C_{\underline{\alpha}}|\Psi\rangle$ are an orthogonal set. It is therefore possible to introduce a projection operator $R_{\underline{\beta}}$ (which is generally not unique) such that

$$R_{\underline{\beta}} C_{\underline{\alpha}} |\Psi\rangle = \delta_{\underline{\alpha}\underline{\beta}} C_{\underline{\alpha}} |\Psi\rangle \tag{1.4}$$

It follows that the extended histories characterized by the chain $R_{\underline{\beta}} C_{\underline{\alpha}} |\Psi\rangle$ are decoherent, and one can assign a probability to the histories $\underline{\alpha}$ and the records $\underline{\beta}$, given by

$$p(\alpha_1,\alpha_2,\dots,\alpha_n;\beta_1,\beta_2\dots\beta_n) = \mathrm{Tr}(R_{\beta_1\beta_2\dots\beta_n} C_{\underline{\alpha}} \rho C_{\underline{\alpha}}^\dagger). \tag{1.5}$$

This probability is then zero unless $\alpha_k = \beta_k$ for all k, in which case it is equal to the original probability $p(\alpha_1,\dots,\alpha_n)$. Hence either the α's or the β's can be completely summed out of Eq. (1.5) without changing the probability, so the probability for the histories can be entirely replaced by the probability for the records at a fixed moment of time at the end of the history:

$$p(\underline{\alpha}) = \mathrm{Tr}(R_{\underline{\alpha}} \rho(t_n)) = \mathrm{Tr}(C_{\underline{\alpha}} \rho C_{\underline{\alpha}}^\dagger). \tag{1.6}$$

Conversely, the existence of records $\beta_1,\dots,\beta_n$ at some final time perfectly correlated with earlier alternatives $\alpha_1,\dots,\alpha_n$ at $t_1,\dots,t_n$ implies decoherence of the histories. This may be seen from the relation

$$D(\underline{\alpha},\underline{\alpha}') = \sum_{\beta_1\dots\beta_n} \mathrm{Tr}(R_{\beta_1\dots\beta_n} C_{\underline{\alpha}} \rho C_{\underline{\alpha}'}^\dagger). \tag{1.7}$$

Since each β_k is perfectly correlated with a unique alternative α_k at time t_k, the summand on the right-hand side is zero unless $\alpha_k = \alpha_k'$ (although note that, as we shall see later, a perfect correlation of this type is generally possible only for a pure initial state).

There is, therefore, a very general connection between decoherence and the existence of records. From this point of view, the decoherent histories approach is very much concerned with reconstructing possible past histories of the universe from records at the present time, and then using these reconstructed pasts to understand the correlations among the present records [14].

The above results on the existence of records are very general, but they do not give any idea as to how one can actually identify the records in a given physical situation. How, for example, can one identify the records in the much-studied quantum Brownian motion model, in which a large bath of oscillators in a thermal state decohere a sequence of particle positions? In that model, the environment in some sense ''measures'' the particle, so we expect the records to be stored in the environment. Is it in practice possible to examine the environment at the final time and explicitly reconstruct the past history of the particle? Little clue as to how one should do this is provided by the formal results above. One main aim of this paper, as indicated at the beginning of this section, is therefore to show how to actually find the records in the quantum Brownian motion model.

The second issue we will address, again as indicated, concerns the *amount* of information stored in the records. Since the environment is thought of as measuring and storing information about the system, we expect there to be a quantitative connection between the amount of information stored and the degree or nature of the decoherence of the decohered system. What is the relevant measure of the degree or ''amount'' of decoherence and how is it related to the amount of information stored about the histories?

Thinking of decoherence via an environment as a generalized measurement process, it is not difficult to see that the relevant measure of the amount of decoherence is, loosely speaking, the precision or width to within which the decoherent histories are defined, or equivalently, the number of histories in the decoherent set. (This issue, is, incidently, distinct from the question of the degree of *approximate* decoherence, discussed below and elsewhere [15,16].) To be more precise, a given set of histories requires a certain number of bits of information to describe it. In the general account of histories and records given above, suppose that the alternatives α_k run over A values. These could, for example, be projections onto ranges of position that partition the x axis into A different bins. Since $k=1,\dots,n$, there are therefore A^n different histories, requiring $\log_2 A^n$ bits of information to describe them. Clearly if these histories are decoherent, the records they are correlated with must be able to store at least $\log_2 A^n$ bits. For many practical instances of decoherence, most of this information is stored in the decohering environment, hence the environment Hilbert space must have an information storage capacity large enough to accommodate the information.

However, not all of this information needs to be stored in the environment. This is because there can be a certain amount of decoherence of histories even without coupling to an environment. For example, the decoherence functional equation (1.3) is automatically diagonal in the final alternatives α_n (because of the cyclic property of the trace and the exclusive property of the projectors). These alternatives do not require records since they exist at the final time. More generally, for a system Hilbert space of dimension D_s, since decoherence requires that the states $C_{\underline{\alpha}}|\Psi\rangle$ must be orthogonal, there can in principle be a decoherent set of as many as D_s histories, without having to appeal to an environment. (To reach this upper limit, however, requires that the operators projected onto at each time are carefully chosen and possibly not physically interesting.) Hence, most generally, the records consist of final projections onto both the distinguished system and the environment. Furthermore, it is then clear that what the environment stores information about is the *enhancement* in the number of histories in a decoherent set when the system is coupled to an environment.

To be precise, return to the set of A^n histories described above. Since, as stated there is automatic decoherence of the A final alternatives, it is the A^{n-1} alternatives at the $n-1$ earlier times that typically do not decohere without an environment, and thus it is the records of these A^{n-1} alternatives that is stored in the environment. If the labels of the records living in the environment $\underline{\beta}$ run over a total of B values, we expect that a necessary condition for decoherence is

$$B \geqslant A^{n-1}. \tag{1.8}$$

This effectively mean that there must be at least one register for each distinct history. If $B<A^{n-1}$, each history cannot be uniquely correlated with a record label β_k, since there are not enough records, hence there will be no decoherence (in the pure state case). Therefore, the amount of information stored in the environment places an upper limit on the number of histories in the decoherent set. Differently put, the environmental information storage capacity limits the permissible amount of fine graining of the system histories consistent with decoherence.

The notion of the information of histories used here is clearly the simplest one imaginable, but is actually sufficient for present purposes. The general question of the assignment of information (or entropy) to histories, and its relation to information storage in the environment, is a very interesting one [17,18,19], but we will not go into it here. The possible difficulty is that a Shannon-like information measure requires probabilities for histories, but here we would like to discuss the logically prior issue of decoherence, hence the existence of probabilities for histories cannot be assumed. In any reasonable assignment of information to histories, however, the value $\log_2 A^n$ will typically arise as the maximum information, when the probabilities for the histories are all equal, so here we are covering the worst possible case. This is actually appropriate to many of the system-environment models studied in the literature, such as the quantum Brownian motion model, where decoherence typically arises for a fixed environment initial state with a wide class of system initial states. Decoherence is due in these models to the joint system-environment dynamics and to the environmental initial state. It does not depend very much on the system initial state, hence it is appropriate to consider decoherence and information storage for a variety of initial states.

Some comments on the nomenclature ''information lost'' and ''records'' are in order. If the environment starts out in a pure state, and its Hilbert space has dimension D_e, then its maximum information storage capacity is $\log_2 D_e$ bits. Hence we would say that the ''information lost'' to the environment is $\log_2 D_e$ bits, and we would also say that the records have D_e different possible states (i.e., $B=D_e$, in the notation used above). If, on the other hand, the environment is in a mixed state, the ''information lost'' to the environment can be greater than $\log_2 D_e$, since it also includes pre-existing uncertainty (or ''information loss'') in the environment state. But the records accessible by projections onto the environment still have D_e different possible states, and in fact the number of *distinguishable* environment states is often diminished in the presence of a mixed state. This will be discussed in more detail later in the paper, but to be clear, the conjecture we will explore is that in the case of both pure and mixed states, the amount of decoherence is related to the ''information lost'' to the environment, whether or not that information is accessible through projections onto the environment.

Note also that the above observation about the connection between the information of histories and the size of the environment also ignores the usual requirement of effective irreversibility of practical information storage. To store one bit in an effectively irreversible fashion typically requires far more than one bit. Here we are not particularly concerned with practical information storage (although that is ultimately an interesting issue to pursue), rather the more fundamental question of the connection between decoherence and maximum information storage.

The above arguments imply that in a system-environment situation, if we throw away N bits of information by tracing out an environment of dimension 2^N, we could, in principle, find an enhancement in the number of histories in a decoherent set by up to 2^N. This means, for example, that if we throw away just one bit, by coupling to a two-state system and then tracing it out, we could increase the number of decoherent histories by a factor of 2. We will indeed produce such an example. Crudely speaking, tracing out anything ought to decohere something.

Another striking example is in the quantum Brownian motion model [20]. Conventional wisdom dictates that an environment of a large number of oscillators is required to decohere histories of position of a single point particle [3]. We will show, however, that even with an environment of just *one* oscillator, decoherence of certain variables describing the particle may be obtained. The variables in question are defined nonlocally in time, and are essentially the Fourier modes of the particle's trajectory. This result then points the way towards showing how the system's history may be recovered from the oscillator states of a many oscillator environment. This simple example also sheds some light on the question of recurrences and how it affects decoherence.

In addition to the issues of explicitly identifying the records, and of finding a concrete connection between decoherence and information storage, a third issue of relevance is the question of approximate versus exact decoherence. In most realistic situations, decoherence is only approximate. A reasonable conjecture is that an approximately decoherent set of histories is in some sense close to an exactly decoherent set, although it is generally difficult to find such exactly decoherent sets explicitly [21]. Since decoherence is related to the existence of records, one can imagine that the nature of exact versus approximate decoherence could be better understood by examining the nature of the records. To be more precise, since records exist at a fixed moment of time at the end of the histories, they are described by projections at just one time and they are therefore trivially decoherent. If these records are exactly correlated with a set of alternatives in the past, those alternatives would then be exactly decoherent. The extent to which these alternatives are then ''close'' to a certain approximately decoherent set of interest could then be assessed. Approximate decoherence, may, for example, be approximate correlation of past alternatives with an exactly decoherent set of records. We will have somewhat less to say about this issue than the other two, but some comments can be made on the basis of the models examined, and it will be taken up in more detail elsewhere.

In assessing the extent to which an environment ''measures'' or stores information about a system it is interacting with, two different approaches suggest themselves.

The first, and simpler, approach is to examine explicit models of the measurement process, in which the system of interest is coupled to a measuring device specifically designed to become correlated with the system in a particular way. In this way one can explicitly see the information transfer from system to apparatus. However, one can then also regard the apparatus as an environment for the system. The apparatus states can then be traced out to produce decoherence of certain system alternatives. One can then investigate the connection between the decoherence of the system, and the extent to which information about it is stored elsewhere.

The second approach is to do things the other way round. That is, to start with a system coupled to an environment in a more general way, which produces decoherence but less obviously corresponds to a particular type of measurement. We can then ask whether, when decoherence occurs, information about the system is in some sense stored in the environment. In this paper, we will address these issues in two models.

We begin in Sec. II with a general discussion of records in the case, not previously covered, in which the initial state is mixed. It is argued that recordlike projectors still exist, but their correlation with past alternatives is necessarily imperfect. Records in the case of decoherence by conservation are also discussed.

Section III concerns a model corresponding to the measurement process which can also be used as an environment. It is a model for position measurements which determine whether a particle has passed through a series of spatial regions $R_1, R_2, \ldots$ at a series of times $t_1, t_2, \ldots$. The measuring device consists of a series of two-state systems localized to the regions $R_1, R_2, \ldots,$ with delta functions in time, so the detectors are only on momentarily. The coupling causes the two-state system to flip from one state to the other. Hence at the end of the history, one can discover whether the particle was in R_1 at t_1, in R_2 at t_2, etc., by examining the state of the two-state systems. We thus obtain a very simple model of the measurement process. We then trace out the measuring devices and look for decoherence of the system alone. Histories in which the position is specified to be in or not in R_1, R_2, etc., at times t_1, t_2 are found to be exactly decoherent. We thus find verification of our conjecture: the number of bits required to describe a decoherent set of histories is equal to the number of bits of information about the system stored in the environment.

In Sec. IV we consider the quantum Brownian motion model in detail. It is first observed that classically, the response of each environmental oscillator in interacting with the particle trajectory is to shift its final position and momentum by an amount proportional to the Fourier modes of the trajectory. Essentially the same story is shown to persist in the quantum case—the shifted position and momentum of the oscillators are the records storing information about the Fourier modes. The information storage is essentially perfect for a pure initial state for the environment, but imperfect in the case of a mixed state. It is also seen that the set of Fourier modes, in contrast to the particle trajectories, are in some sense the natural variables in which to discuss decoherence. An elementary way of counting the number of histories in a decoherent set is introduced, and this number is shown to approximately coincide with the number of different possible record states in the environment, in agreement with the conjecture.

Section V contains a discussion, including the implications of some of these ideas for quantum cosmology.

This paper builds very much on the connection between decoherence and records in the decoherent histories approach, especially as put forward by Gell-Mann and Hartle [2,5], although as stated above, it is likely that other approaches to decoherence may be amenable to a similar analysis. It is also partly inspired by some of the ''it from bit'' ideas initiated by Wheeler [22] and explored in detail by Caves [23], Wooters [24], Zurek [4,25,26], and others [27]. One particular motivation is the recent remark by Zurek [27], that information-theoretic ideas have not been exploited to the degree that they might. Indeed, before the advent of the decoherent histories approach, it was Zurek who first spelled out the connection between decoherence and information storage in the environment [25,28]. Some recent papers on the assignment of information to histories by Hartle and Brun [19], Gell-Mann and Hartle [5], and Isham and Linden [17] have also been influential. Finally, it should be noted that there has recently been a surge of interest in the subject of *quantum* information but these interesting developments are not very closely related to the present work, since we are interested in the case in which the information stored by the environment is essentially classical.

II. RECORDS IN THE CASE OF MIXED INITIAL STATES

The connection between decoherence and records has been demonstrated only in the case of a pure initial state [2]. Yet many situations in which decoherence is studied, such as the quantum Brownian motion model, involve a thermal state for the environment [20,15], hence the overall initial state is mixed. In this situation, the connection between decoherence and records needs to be examined more carefully. There are two issues. First of all, to determine whether records still exist in this case, that is, whether it is still possible to add a record projector at the end of the chain and preserve decoherence. Secondly, to work out the degree of correlation between the records and the histories.

A. Mixed initial states

We start from the observation that a mixed state can always be regarded as the reduced density operator of a pure state defined on an enlarged Hilbert space. Take, for example, a mixed density operator of the form

$$\rho = \sum_n p_n |n\rangle\langle n|. \tag{2.1}$$

Suppose we enlarge the original Hilbert space $\mathcal{H}$ to $\mathcal{H}\otimes\tilde{\mathcal{H}}$, where $\tilde{\mathcal{H}}$ is an exact copy of $\mathcal{H}$. Now on $\mathcal{H}\otimes\tilde{\mathcal{H}}$, we may define the pure state

$$|\Psi\rangle = \sum_n p_n^{1/2} |n\rangle \otimes |\tilde{n}\rangle \tag{2.2}$$

and it is readily seen that

$$\rho = \mathrm{Tr}_{\tilde{\mathcal{H}}}(|\Psi\rangle\langle\Psi|). \tag{2.3}$$

Of course, there are many different ways of regarding finding a pure state which reduces to a given mixed state in this way, but this way is sufficient for illustrative purposes.

On the enlarged Hilbert space, the decoherence functional may be written

$$D(\underline{\alpha},\underline{\alpha}') = \mathrm{Tr}(C_{\underline{\alpha}} \otimes \mathbb{1}|\Psi\rangle\langle\Psi|C^{\dagger}_{\underline{\alpha}'} \otimes \mathbb{1}) \tag{2.4}$$

where the trace is over $\mathcal{H}\otimes\tilde{\mathcal{H}}$. When there is decoherence, the argument showing the existence of records may now be repeated: there exist records at the final time perfectly correlated with the alternatives $\underline{\alpha}$. The records exist, however, on the enlarged Hilbert space. The probability of both the records and the histories is

$$p(\underline{\alpha},\underline{\beta}) = \mathrm{Tr}(\tilde{R}_{\underline{\beta}} C_{\underline{\alpha}} \otimes \mathbb{1}|\Psi\rangle\langle\Psi|C^{\dagger}_{\underline{\alpha}} \otimes \mathbb{1}) \tag{2.5}$$

where $\tilde{R}_{\underline{\beta}}$ is defined on $\mathcal{H}\otimes\tilde{\mathcal{H}}$. It will generally not be possible to express this joint probability in terms of states and projections on $\mathcal{H}$ alone. The projector on the enlarged Hilbert space will generally be a sum of projectors of the form $R\otimes Q$, where R acts on $\mathcal{H}$ and Q acts on $\tilde{\mathcal{H}}$, and part of the records will be contained in the projector Q on $\tilde{H}$.

Nevertheless, the existence of this joint probability distribution, in which the addition of the records projector $\tilde{R}_{\underline{\beta}}$ does not disturb the decoherence of the histories, permits us to deduce the existence of an analogous formula on $\mathcal{H}$. For suppose we coarse grain the record projector in such a way that all components Q acting on $\tilde{\mathcal{H}}$ are replaced by the identity. The decoherence of the histories is preserved since coarse graining preserves decoherence. This implies that we may write down a joint probability distribution of the form

$$p(\underline{\alpha},\underline{\beta}) = \mathrm{Tr}(R_{\underline{\beta}} C_{\underline{\alpha}} \rho C^{\dagger}_{\underline{\alpha}}) \tag{2.6}$$

where everything is now defined on the original Hilbert space $\mathcal{H}$, and $R_{\underline{\beta}}$ is a projection operator. Hence, given decoherence in the case of a mixed initial state, we can always add an extra projector $R_{\underline{\beta}}$ at the end of the chain without affecting decoherence.

A less general, but perhaps more explicit discussion can be given by an appeal to the particular situations in which decoherence occurs. In the most physically interesting situations, the type of variables that decohere, and that are correlated with records, is primarily determined by the underlying Hamiltonian, and not by the initial state. The initial state affects only the degree of decoherence and correlation. Suppose we first take as an initial state one of the pure states in which the mixed initial state of interest is diagonal [i.e., one of the states $|n\rangle\langle n|$ in the notation (2.1)]. Because the variables that decohere depend only on the Hamiltonian we expect there to be *some* degree of decoherence for this state, and record projectors may therefore be added at the end of the histories, without affecting decoherence. Now suppose we go from this pure initial state to the mixed state. Since this is a coarse graining, the extended histories including the records *continue to be decoherent*. This implies that a formula of type (2.6) will exist and satisfy the probability sum rules.

The interpretation of the extra projector in Eq. (2.6) as an exact record makes sense only on the enlarged Hilbert space. On coarse graining to the original Hilbert space, the correlation between $\underline{\alpha}$ and the reduced set of records $\underline{\beta}$ will generally be imperfect because we have thrown away some of the information. This can be explicitly shown as follows. Consider the conditional probability of the records $\underline{\beta}$ given the past alternatives $\underline{\alpha}$. This is given by

$$p(\underline{\beta}|\underline{\alpha}) = \frac{p(\underline{\alpha},\underline{\beta})}{p(\underline{\alpha})} = \mathrm{Tr}(R_{\underline{\beta}}\rho_{\mathrm{eff}}(\underline{\alpha})) \tag{2.7}$$

where

$$\rho_{\mathrm{eff}}(\underline{\alpha}) = \frac{C_{\underline{\alpha}}\rho C^{\dagger}_{\underline{\alpha}}}{\mathrm{Tr}(C_{\underline{\alpha}}\rho C^{\dagger}_{\underline{\alpha}})}. \tag{2.8}$$

A perfect correlation between the records and the past alternatives is assured only if $p(\underline{\beta}|\underline{\alpha})=1$, which is possible only if $\rho_{\mathrm{eff}}(\underline{\alpha})$ is pure. If $\rho_{\mathrm{eff}}(\underline{\alpha})$ is mixed, $p(\underline{\beta}|\underline{\alpha})<1$, and the correlation is imperfect.

To see when $\rho_{\mathrm{eff}}(\underline{\alpha})$ is pure, insert the diagonal form for ρ in Eq. (2.8):

$$C_{\underline{\alpha}}\rho C^{\dagger}_{\underline{\alpha}} = \sum_n p_n C_{\underline{\alpha}}|n\rangle\langle n|C^{\dagger}_{\underline{\alpha}}. \tag{2.9}$$

$\rho_{\mathrm{eff}}(\underline{\alpha})$ is therefore pure if and only if just one of the terms in the sum on the right-hand side is nonzero. This can only come about if each $\underline{\alpha}$ picks out a single value of n, so that, for fixed $\underline{\alpha}$,

$$C_{\underline{\alpha}}|n\rangle = 0 \tag{2.10}$$

except for just one value of n corresponding to $\underline{\alpha}$. (The converse need not be true, i.e., the value of n for which $C_{\underline{\alpha}}|n\rangle$ is nonzero may correspond to many values of $\underline{\alpha}$.) The interesting case, however, is that in which the fact that the initial state is mixed is essential for decoherence; i.e., there is no decoherence for the constituent pure initial states. In this case the states $C_{\underline{\alpha}}|n\rangle$ are generally not orthogonal:

$$\langle n|C^{\dagger}_{\underline{\alpha}'}C_{\underline{\alpha}}|n\rangle \neq 0 \tag{2.11}$$

for $\underline{\alpha}'\neq\underline{\alpha}$. This is incompatabile with Eq. (2.10). Hence, $\rho_{\mathrm{eff}}(\underline{\alpha})$ can only be pure when there is decoherence for every constituent pure component of the mixed initial state, and in addition, the (rather special) condition (2.10) is satisfied.

We therefore conclude the following: *when decoherence relies on the impurity of the initial state, there are no records that are perfectly correlated with the past alternatives*. When there is decoherence for the constituent pure components of the initial state, it will generally still be true that there are no perfect records, except for the special types of histories for which the condition (2.10) is satisfied.

Physically, the decay in quality of the records is no surprise. We assign a mixed state when the system is subject to fluctuations which are genuinely beyond our control so we average over them. For example, all systems are subject to scattering by microwave background radiation, and the scattered photons may subsequently disappear beyond the horizon and so are truly lost. This means that the records themselves are in a mixed state, so suffer inescapable fluctuations, and can therefore not be perfectly correlated with anything. However, although exact records are impossible, we might reasonably expect to find final alternatives which are correlated with a past alternative to a good approximation. Indeed, we will find this to be the case in the particular models we investigate.

The above arguments also illustrate why mixed initial states tend to give better decoherence than pure ones (e.g., in the quantum Brownian motion model the decoherence improves with increasing temperature of the thermal state of the environment). By better decoherence, we mean that more histories decohere, or equivalently, that the histories may be described more finely without encountering interference effects. Earlier we put forward the idea that the amount of decoherence is related to the amount of information about the histories stored somewhere in the universe. The more information stored the better the decoherence. Since a mixed state may be regarded as the reduced density operator of a pure state on an enlarged Hilbert space, it clearly represents, compared to a pure state, an enhanced ability to store information, since there is quite simply more Hilbert space available. Some of that information is inaccessible from the original Hilbert space, but that does not matter for the purposes of decoherence, which depends only on the storage of information *somewhere*.

B. Records in the case of decoherence by conservation

So far we have discussed decoherence arising from interaction with an environment, and the associated information storage. However, decoherence of histories seemingly of a rather different nature comes about when the alternatives characterizing the histories are exactly conserved [29]. This is an elementary property of the decoherence functional—the projectors commute with the unitary evolution operators, so may all be moved up to the final time, where the P_{α_k}'s act on the $P_{\alpha_k'}$'s, and thus give diagonality of the decoherence functional. A more general notion which also gives decoherence is *determinism* in the quantum theory. An example is histories of projections onto large cells of phase space. These projections have the property that under unitary evolution they evolve (approximately) into another projection of identical type, except that the center of the phase space cell is shifted according to the classical equations of motion [8]. This approximate determinism also guarantees (approximate) decoherence, for similar reasons to the case of exact conservation. These mechanisms are important in showing the emergent classicality of hydrodynamic variables [30–33].

In these cases it is natural to again ask for the connection with the existence of records, but the answer is almost trivial. Records do not need to exist in a separate environment. The existence of records is essentially the question of whether there exist alternatives at the final moment of time which are perfectly correlated with the alternatives describing the histories at earlier times. Clearly the answer is yes in this case: histories of exactly conserved quantities may always be expressed as projections at the final moment of time, since the projectors may quite simply be moved to the final time without changing anything. Each alternative at each time is, in a sense, its own record. A similar story applies in the case of approximate determinism.

III. A TWO-STATE ENVIRONMENT

In the Introduction, it was argued that decoherence is related to information storage in the environment, and that the number of histories in the decoherent set is related to the amount of information about the histories stored in the environment. Taken to the extreme, this means that even an environment consisting of a two-state system could potentially lead to decoherence of certain system alternatives. In this section, we will consider exactly such an environment, and show that it provides an instructive model of decoherence and information storage with exactly the expected properties.

The system in question is taken to be a point particle coupled to a two-state system environment via a coupling localized to a region of space and which, for simplicity, acts only at a single moment of time, $t=t_1$. The two-state system has states $|0\rangle$, $|1\rangle$, with associated raising and lowering operators, a, $a^\dagger$, where

$$a|0\rangle=0, \quad a|1\rangle=|0\rangle, \quad a^\dagger|0\rangle=|1\rangle, \quad a^\dagger|1\rangle=0. \tag{3.1}$$

The Hamiltonian is

$$H(t)=H_0+\lambda\,\delta(t-t_1)(a+a^\dagger)\Upsilon(x) \tag{3.2}$$

where $H_0=p^2/2m$. Here, $\Upsilon(x)$ is a window function equal to 1 in the interval $[a,b]$ and zero outside it. Therefore, although we regard the two-state system as an environment, it is also a very simple model of the measurement of position. If the two-state system is started out in the state $|0\rangle$, it will flip to $|1\rangle$ if the particle is in $[a,b]$ at time t_1, and remain in $|0\rangle$ otherwise. Hence by examining the state of the environment at any time after t_1, we may recover one bit of information about the particle at time t_1.

We assume that the initial state of the composite system is

$$|\Psi_0\rangle=|\psi\rangle\otimes|0\rangle. \tag{3.3}$$

It is convenient to introduce the eigenstates of $a+a^\dagger$, which are

$$|\pm\rangle=\frac{1}{\sqrt{2}}(|0\rangle\pm|1\rangle). \tag{3.4}$$

These we write as $|s\rangle$, where $s=\pm1$, and we also have

$$|0\rangle=\frac{1}{\sqrt{2}}\sum_s |s\rangle, \quad |1\rangle=\frac{1}{\sqrt{2}}\sum_s s|s\rangle. \tag{3.5}$$

The initial state may now be written

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}\sum_s |\psi\rangle\otimes|s\rangle. \tag{3.6}$$

Now consider unitary evolution from 0 to t, where $0<t_1<t$. Since the product form (3.3) is preserved up to t_1, there is no loss of generality in letting $t_1\to 0$, and

$$\begin{aligned}|\Psi\rangle &= T\exp\left(-\frac{i}{\hbar}\int_0^t dt' H(t')\right)|\Psi_0\rangle\\ &= \frac{1}{\sqrt{2}}\sum_s \exp\left(-\frac{i}{\hbar}Ht\right)\\ &\quad\times\exp\left(-\frac{i}{\hbar}s\lambda\Upsilon(\hat{x})\right)|\psi\rangle\otimes|s\rangle\end{aligned} \tag{3.7}$$

(where T denotes time ordering). The probability that the environment is then found in the state $|1\rangle$ is given by

$$\langle\Psi|(\mathbb{1}_{\mathcal{S}}\otimes|1\rangle\langle 1|)|\Psi\rangle = \int_a^b dx\,\sin^2\left(\frac{\lambda}{\hbar}\right)|\psi(x)|^2. \tag{3.8}$$

This is the correct result of standard quantum measurement theory if we choose the coupling λ to be $\lambda=\pi\hbar/2$, so we now adopt this value. With this value of λ, the second exponential in Eq. (3.7) may be written

$$\exp\left(-\frac{i}{2}\pi s\Upsilon(\hat{x})\right) = [1-\Upsilon(\hat{x})] - is\Upsilon(\hat{x}) \tag{3.9}$$

since Υ is a window function, and therefore $\Upsilon^2=\Upsilon$. It follows that $\Upsilon(\hat{x})$ is also a projection operator onto the region $[a,b]$, which we will denote by P_y, and we will denote its negation $1-\Upsilon(\hat{x})$ by P_n.

Now consider a history in which the system is hit by a projector P_{α_1} at the initial time, and then a second projector P_{α_2}, at time t, both of these projectors acting only on the particle, not the environment. Both will be projections onto ranges of position, described more below. The decoherence functional may be written

$$D(\alpha_1,\alpha_2|\alpha_1',\alpha_2) = \langle\Psi_{\alpha_1'\alpha_2}|\Psi_{\alpha_1\alpha_2}\rangle \tag{3.10}$$

where

$$\begin{aligned}|\Psi_{\alpha_1\alpha_2}\rangle &= P_{\alpha_2}\otimes\mathbb{1}_{\mathcal{E}}T\exp\left(-\frac{i}{\hbar}\int_0^t dt' H(t')\right)P_{\alpha_1}\otimes\mathbb{1}_{\mathcal{E}}|\psi\rangle\otimes|0\rangle\\ &= \frac{1}{\sqrt{2}}\sum_s P_{\alpha_2}\otimes\mathbb{1}_{\mathcal{E}}e^{-(i/\hbar)Ht}(P_n - isP_y)P_{\alpha_1}|\psi\rangle\otimes|s\rangle,\end{aligned} \tag{3.11}$$

where the second line follows from Eq. (3.7) and Eq. (3.9). Summing over s and using Eq. (3.5), we obtain

$$\begin{aligned}|\Psi_{\alpha_1\alpha_2}\rangle &= (P_{\alpha_2}e^{-(i/\hbar)Ht}P_nP_{\alpha_1}|\psi\rangle)\otimes|0\rangle\\ &\quad - i(P_{\alpha_2}e^{-(i/\hbar)Ht}P_yP_{\alpha_1}|\psi\rangle)\otimes|1\rangle.\end{aligned} \tag{3.12}$$

In this expression the projectors P_y and P_n have come entirely from the dynamics of the environment. It is therefore reasonably clear that exact decoherence and a perfect system-environment correlation is obtained if we choose the system projectors P_{α_1} to coincide with P_y and P_n. We have $P_nP_{\alpha_1}=0$, unless $\alpha_1=n$, and $P_yP_{\alpha_1}=0$ unless $\alpha_1=y$. Therefore,

$$|\Psi_{\alpha_1\alpha_2}\rangle = \begin{cases}(P_{\alpha_2}e^{-(i/\hbar)Ht}P_n|\psi\rangle)\otimes|0\rangle, & \text{if } \alpha_1=n,\\ -i(P_{\alpha_2}e^{-(i/\hbar)Ht}P_y|\psi\rangle)\otimes|1\rangle, & \text{if } \alpha_1=y,\end{cases} \tag{3.13}$$

from which the decoherence is easily seen. In these expressions the projector P_{α_2} can be onto anything, since decoherence in the final alternatives is always automatic.

An interesting alternative form of the decoherence functional is its path integral form, derived directly from Eq. (3.7), which is

$$\begin{aligned}D(\underset{\sim}{\alpha},\underset{\sim}{\alpha}') &= \sum_s \int_{\underset{\sim}{\alpha}}\mathcal{D}x\,\exp\left(\frac{i}{\hbar}S[x(t)] + \frac{\pi i}{2}s\Upsilon(x(t_1))\right)\psi(x_0)\int_{\underset{\sim}{\alpha}'}\mathcal{D}y\,\exp\left(-\frac{i}{\hbar}S[y(t)] - \frac{\pi i}{2}s\Upsilon(y(t_1))\right)\psi^*(y_0)\\ &= \int_{\underset{\sim}{\alpha}}\mathcal{D}x\int_{\underset{\sim}{\alpha}'}\mathcal{D}y\,\exp\left(\frac{i}{\hbar}S[x(t)] - \frac{i}{\hbar}S[y(t)]\right)\cos\left(\frac{\pi}{2}[\Upsilon(x(t_1)) - \Upsilon(y(t_1))]\right)\psi(x_0)\psi^*(y_0).\end{aligned} \tag{3.14}$$

The cosine term plays the role of an influence functional, in that it summarizes the effect of the environment. It may be seen that it destroys interference between histories partitioned according to whether they are in the region $[a,b]$ at time t_1, since it is equal to 1 if $x(t_1)$ and $y(t_1)$ are both either inside or outside the region $[a,b]$, and is zero if one is inside and the other outside.

Since the initial state of the whole system is pure, the existence of exact decoherence means that there must exist records at the final time. That is, we can add another projector R_β at the final time and construct the probability $p(\alpha_1,\alpha_2,\beta)$ where β is perfectly correlated with α_1. It is trivial to identify the records—they are clearly the states $|0\rangle$, $|1\rangle$ of the environment. The record projectors R_β are

$$R_0 = \mathbb{1}_{\mathcal{S}}\otimes|0\rangle\langle 0|, \quad R_1 = \mathbb{1}_{\mathcal{S}}\otimes|1\rangle\langle 1|. \tag{3.15}$$

From Eq. (3.13) it is clear that

$$R_\beta|\Psi_{\alpha_1\alpha_2}\rangle=|\Psi_{\alpha_1\alpha_2}\rangle$$

when $\alpha_1=y$ and $\beta=1$, or $\alpha_1=n$ and $\beta=0$, with $R_\beta|\Psi_{\alpha_1\alpha_2}\rangle=0$ otherwise. There is therefore a perfect correlation between the records and the past alternatives α_1. Essentially the same conclusions holds with different choices of pure initial state. The main difference is that the form of the record projectors change.

Turn now to the case in which the environment is in a mixed initial state. First, we introduce a convenient notation in which Eq. (3.12) is written

$$|\Psi_{\alpha_1\alpha_2}\rangle=|\overline{\psi}_{\alpha_1\alpha_2}\rangle\otimes|0\rangle+|\psi_{\alpha_1\alpha_2}\rangle\otimes|1\rangle. \tag{3.16}$$

The joint probability of the histories and the records may be written

$$p(\alpha_1,\alpha_2,\beta)=\mathrm{Tr}(R_\beta|\Psi_{\alpha_1\alpha_2}\rangle\langle\Psi_{\alpha_1\alpha_2}|) \tag{3.17}$$

where

$$|\Psi_{\alpha_1\alpha_2}\rangle\langle\Psi_{\alpha_1\alpha_2}|=|\overline{\psi}_{\alpha_1\alpha_2}\rangle\langle\overline{\psi}_{\alpha_1\alpha_2}|\otimes|0\rangle\langle 0|+|\psi_{\alpha_1\alpha_2}\rangle \times\langle\psi_{\alpha_1\alpha_2}|\otimes|1\rangle\langle 1|+\text{off-diagonal terms}. \tag{3.18}$$

The off-diagonal terms are irrelevant to both the discussion of correlations and decoherence, since they make no contribution. Equation (3.18) is the case in which the environment initial state is the pure state $|0\rangle$, and it shows very clearly the perfect correlation that exists between the system histories and the environment states. In particular, different system histories can be completely distinguished by projecting onto the two orthogonal environment states. If the initial state instead were $|1\rangle$, then the result would be similar to Eq. (3.18), but with the $|0\rangle$'s and $|1\rangle$'s interchanged. It follows that if we take the environment initial state to be the mixed state

$$\rho_1=a|0\rangle\langle 0|+b|1\rangle\langle 1| \tag{3.19}$$

then the joint probability of the histories and the records is

$$p(\alpha_1,\alpha_2,\beta)=\mathrm{Tr}[R_\beta(|\overline{\psi}_{\alpha_1\alpha_2}\rangle\langle\overline{\psi}_{\alpha_1\alpha_2}|\otimes\rho_1+|\psi_{\alpha_1\alpha_2}\rangle \times\langle\psi_{\alpha_1\alpha_2}|\otimes\rho_2)] \tag{3.20}$$

where

$$\rho_2=b|0\rangle\langle 0|+a|1\rangle\langle 1|. \tag{3.21}$$

As described in Sec. II, in the mixed state case the joint probability equation (3.20) must necessarily indicate less than perfect correlations between the records and the alternatives α_1 in the past. We can now see this in a different way. The point is that the record projector needs to be able to unambiguously distinguish between the different environment states the past alternatives are perfectly correlated with. This is possible in the pure state case, where the alternatives α_1 are perfectly correlated with the pair of orthogonal pure states $|0\rangle$, $|1\rangle$, and orthogonal pure states are completely distinguishable. In the mixed state case, the alternatives α_1 become correlated with the two mixed states ρ_1,ρ_2. These two states are *not* perfectly distinguishable. There is no projection operator that can unambiguously decide whether the environment is in state ρ_1 or ρ_2.

The model therefore illustrates the generally expected features. We can look at the environment and explicitly find the records. An environment consisting of a two-state system leads to a decoherent set of system histories enlarged by a factor of 2 compared to the set that decoherence without this environment. Clearly if we attempted to consider more than two alternatives α_1, we would not expect decoherence. Decoherence is preserved as we go to a mixed state (since there is decoherence for each consituent pure state), and we see that the reason the records are imperfectly correlated with past alternatives is due to the impossibility of completely distinguishing between the mixed environment states the system alternatives are correlated with.

This model can clearly be extended to more elaborate histories involving an arbitrary number of alternatives at each moment of time, and to an arbitrary number of times, but the essential ideas have been established in this simple model. It is also perhaps of interest to consider a slightly more realistic model of position samplings involving a genuinely irreversible detector model that does not involve a delta function in time. This has been considered in Ref. [34].

IV. DECOHERENCE AND INFORMATION STORAGE IN THE QUANTUM BROWNIAN MOTION MODEL

In this section we consider the question of how decoherence is related to storage of information by the environment in the quantum Brownian motion model. We begin with a brief review of the model. Although standard material [20,35,36,15,2], it is presented at some length in parts since it will be necessary to consider a modified version of the standard account later on in this section.

A. The quantum Brownian motion model

We are concerned with the class of quantum Brownian models consisting of a particle of large mass M moving in a potential $V(x)$ and linearly coupled to a bath of harmonic oscillators. The total system is therefore described by the action

$$S_T[x(t),q_n(t)]=\int dt\left[\frac{1}{2}M\dot{x}^2-V(x)\right]+\sum_n\int dt\left[\frac{1}{2}m_n\dot{q}_n^2-\frac{1}{2}m_n\omega_n^2q_n^2-c_nq_nx\right]. \tag{4.1}$$

The decoherence functional has the form

$$D(\underset{\sim}{\alpha},\underset{\sim}{\alpha}')=\int_{\underset{\sim}{\alpha}}\mathcal{D}x\int_{\underset{\sim}{\alpha}'}\mathcal{D}y\int\mathcal{D}q_n\mathcal{D}r_n \times\exp\left(\frac{i}{\hbar}S_T[x(t),q_n(t)]-\frac{i}{\hbar}S_T[y(t),r_n(t)]\right) \times\rho_0(x(0),y(0))\rho_0^{\mathrm{env}}(q_n(0),r_n(0)), \tag{4.2}$$

where we have assumed a factored initial state. We will make the standard assumption that the environment initial state is thermal

$$\rho_0^{\mathrm{env}}(q_n,r_n)=\prod_n \exp[-A(q_n^2+r_n^2)+Bq_nr_n] \tag{4.3}$$

where

$$A=\frac{m_n\omega_n}{2\hbar}\coth(\hbar\omega_n\beta),\quad B=\frac{m_n\omega_n}{\hbar\sinh(\hbar\omega_n\beta)}, \tag{4.4}$$

and $\beta=1/kT$. If the coarse graining $\underset{\sim}{\alpha}$, $\underset{\sim}{\alpha}'$ does not involve the environment, it may be integrated out, with the result

$$D(\underset{\sim}{\alpha},\underset{\sim}{\alpha}')=\int_{\underset{\sim}{\alpha}}\mathcal{D}x\int_{\underset{\sim}{\alpha}'}\mathcal{D}y \times\exp\left(\frac{i}{\hbar}S[x]-\frac{i}{\hbar}S[y]\right)\mathcal{F}[x(t),y(t)]\rho(x_0,y_0) \tag{4.5}$$

where

$$S[x]=\int dt\left[\frac{1}{2}M\dot{x}^2-V(x)\right] \tag{4.6}$$

and $\mathcal{F}[x(t),y(t)]$ is the Feynman-Vernon influence functional,

$$\mathcal{F}[x(t),y(t)]=\prod_n\int\mathcal{D}q_n\mathcal{D}r_n\rho_0^{\mathrm{env}}(q_n(0),r_n(0)) \times\exp\left(\frac{i}{\hbar}\int dt\left[\frac{1}{2}m_n\dot{q}_n^2-\frac{1}{2}m_n\omega_n^2q_n^2-c_nq_nx\right]\right) \times\exp\left(-\frac{i}{\hbar}\int dt\left[\frac{1}{2}m_n\dot{r}_n^2-\frac{1}{2}m_n\omega_n^2r_n^2-c_nr_ny\right]\right). \tag{4.7}$$

The sum is over all paths for which meet, $q_n=r_n$, at the final time and then there is an integral over q_n.

This expression may be evaluated by first using the standard path integral for the propagator of a harmonic oscillator in an external field,

$$g(q_n'',\tau|q_n',0)=\int\mathcal{D}q_n\exp\left(\frac{i}{\hbar}\int dt\left[\frac{1}{2}m_n\dot{q}_n^2-\frac{1}{2}m_n\omega_n^2q_n^2-c_nq_nx\right]\right), \tag{4.8}$$

where the sum is over all paths $q_n(t)$ from $q_n(0)=q_n'$ to $q_n(\tau)=q_n''$. The result is

$$g(q_n'',\tau|q_n',0)=\exp\left(\frac{i}{\hbar}(aq_n''^2+aq_n'^2+bq''q'-c[x]q'' -d[x]q_n'-f[x])\right) \tag{4.9}$$

where

$$a=\frac{m\omega_n\cos\omega_n\tau}{2\sin\omega_n\tau}, \tag{4.10}$$

$$b=-\frac{m\omega_n}{\sin\omega_n\tau}, \tag{4.11}$$

$$c[x(t)]=\frac{c_n}{\sin\omega_n\tau}\int_0^\tau dtx(t)\sin\omega_nt, \tag{4.12}$$

$$d[x(t)]=\frac{c_n}{\sin\omega_n\tau}\int_0^\tau dtx(t)\sin\omega_n(\tau-t), \tag{4.13}$$

$$f[x(t)]=\frac{c_n^2}{m_n\omega_n\sin\omega_n\tau}\int_0^\tau dt \times\int_0^t dsx(t)x(s)\sin(\omega_n(\tau-t)) \times\sin\omega_ns. \tag{4.14}$$

Using these expressions, the initial state is folded in, the final values of $q_n=r_n$ traced over, and the influence functional obtained is then normally written in the form

$$\mathcal{F}[x(t),y(t)]=\exp\left(\frac{i}{\hbar}W[x(t),y(t)]\right) \tag{4.15}$$

where, $W[x(t),y(t)]$ is influence functional phase, and has the form

$$W[x(t),y(t)]=-\int_0^t ds\int_0^s ds'[x(s)-y(s)]\eta(s-s')\times[x(s')+y(s')]+\frac{i}{2}\int_0^t ds\int_0^t ds'\times[x(s)-y(s)]\nu(s-s')[x(s')-y(s')]. \tag{4.16}$$

(In the imaginary part, the symmetry of $\nu(s-s')$ has been used to write the two integrals over the same range, $[0,t]$, and this will be exploited below.) The kernels $\eta(s)$ and $\nu(s)$ are defined by

$$\eta(s)=-\sum_n \frac{c_n^2}{2m_n\omega_n}\sin\omega_n s \tag{4.17}$$

and

$$\nu(s)=\sum_n \frac{c_n^2}{2m_n\omega_n}\coth(\tfrac{1}{2}\hbar\omega_n\beta)\cos\omega_n s. \tag{4.18}$$

They are commonly rewritten

$$\nu(s)=\int_0^\infty \frac{d\omega}{\pi}I(\omega)\coth\left(\frac{\hbar\omega}{2kT}\right)\cos\omega s, \tag{4.19}$$

$$\eta(s)=\frac{d}{ds}\gamma(s), \tag{4.20}$$

where

$$\gamma(s)=\int_0^\infty \frac{d\omega}{\pi}\frac{I(\omega)}{\omega}\cos\omega s \tag{4.21}$$

and $I(\omega)$ is the spectral density

$$I(\omega)=\sum_n \delta(\omega-\omega_n)\frac{\pi c_n^2}{2m_n\omega_n}. \tag{4.22}$$

Typically, the spectral density is chosen to have the Ohmic form

$$I(\omega)=M\gamma\omega\exp\left(-\frac{\omega^2}{\Lambda^2}\right). \tag{4.23}$$

Here, Λ is a cutoff, which will generally be taken to be very large. We then find that

$$\gamma(s)=M\gamma\frac{\Lambda}{2\pi^{1/2}}\exp\left(-\tfrac{1}{4}\Lambda^2 s^2\right) \tag{4.24}$$

and thus when Λ is very large,

$$\gamma(s)\approx M\gamma\delta(s). \tag{4.25}$$

The noise kernel $\eta(s)$ is nonlocal for large Λ, except in the so-called Fokker-Planck limit, $kT\gg\hbar\Lambda$, in which case one has

$$\nu(s)=\frac{2M\gamma kT}{\hbar}\delta(s). \tag{4.26}$$

Decoherence of histories of positions typically arises when there is essentially a continuum of oscillators at high temperatures. For in this case,

$$|\mathcal{F}[x(t),y(t)]|=\exp\left(-\frac{2M\gamma kT}{\hbar^2}\int dt(x-y)^2\right) \tag{4.27}$$

in the decoherence functional equation (4.5), hence the contribution from widely differing paths $x(t)$, $y(t)$ is strongly suppressed. It will be useful for what follows to spell out in more detail what this means. Suppose that the coarse graining of the position histories is chosen so that the histories are specified at each moment of time up to a width σ. This means that for pairs of histories to be ''distinct'' in Eq. (4.27), x and y must differ by at least σ. The decoherence condition, that Eq. (4.27) be very small, is then a lower limit on the value of σ. If the time scale of the entire history is τ, the condition is $\sigma^2\gg\hbar^2/(2M\gamma kT\tau)$ Hence, decoherence supplies a lower limit on the precision to within which the histories of positions may be used in an essentially classical way, without suffering interference effects. We can discuss the number of decoherent histories in the set by confining the particle's motion to a region of size L. Formally, this is of course achieved by putting the system in a box, with the accompanying complications. However, it is sufficient for our purposes to restrict the particle's motion in a more approximate way, by supposing that the potential $V(x)$ becomes very large outside the region, or by restricting to particle initial states that have negligible support outside the region during the time interval of interest. We can then say that for decoherence to order σ satisfying the above condition, the number of histories in the decoherent set is of order L/σ.

Under more general conditions, the oscillatory and nonlocal nature of the noise kernel $\eta(s)$ in W makes decoherence of positions at a series of times less obvious. This is not unrelated to the presence of recurrences in the master equation. Take, for example, the case of zero temperature and and a finite number of oscillators. An arbitrary initial density operator might initially tend towards diagonality in position, but over long periods of time, the correlations ''lost'' to the environment will eventually come back, and the density matrix will become off diagonal. In terms of the decoherence functional, a set of decoherent histories defined in terms of projections onto position at a sequence of times might lose decoherence if the projections are spread out over a timescale comparable to the recurrence time. This is why it is necessary, at least for decoherence of position, to take an essentially infinite environment. We will see below, however, how this conclusion may be modified.

B. Decoherence of the Fourier modes

According to the general discussion in the Introduction, the decoherence of histories of positions in the quantum Brownian motion model means that there ought to exist records about the trajectories $x(t)$ somewhere in the environment. We will now show how this comes about. Important clues can be found from studying the classical equations of motion of the environment of oscillators. These are

$$m_n \ddot{q}_n + m_n \omega_n^2 q_n = -c_n x(t). \tag{4.28}$$

The solution to this equation, with fixed $p_n(0)$, $q_n(0)$ is

$$q_n(\tau) = q_n(0)\cos\omega_n\tau + \frac{p_n(0)}{m_n\omega_n}\sin\omega_n\tau - \frac{c_n}{m_n\omega_n}\int_0^\tau dt\, x(t)\sin\omega_n(\tau - t), \tag{4.29}$$

$$p_n(\tau) = p_n(0)\cos\omega_n\tau - m_n\omega_n q_n(0)\sin\omega_n\tau - c_n\int_0^\tau dt\, x(t)\cos\omega_n(\tau - t), \tag{4.30}$$

where $p_n = m\dot{q}_n$. From this solution, one can see that at the final time τ, the positions and momenta of the environment of oscillators depend on the particle's trajectory $x(t)$ via the temporally nonlocal quantities

$$X_n^s = \int_0^\tau dt\, x(t)\sin\omega_n(\tau - t), \tag{4.31}$$

$$X_n^c = \int_0^\tau dt\, x(t)\cos\omega_n(\tau - t). \tag{4.32}$$

Hence, classically, the final values of p_n and q_n are correlated with the variables X_n^s and X_n^c—for given initial data for the environment, measurement of the final data permits the determination of X_n^s and X_n^c.

It now follows that, classically, the *entire trajectory* $x(t)$ for all t may be recovered by using an infinite number of oscillators, and by choosing the frequencies ω_n appropriately, since X_n^s and X_n^c are essentially the Fourier components of the function $x(t)$ in its expansion on the range $[0,\tau]$. This is the key observation about how the environment stores information about the system: each oscillator measures a Fourier component of the trajectory. We will demonstrate that essentially the same story persists in the quantum theory.

First, however, since we expect the nonlocal functions X_n^s, X_n^c to play a key role, let us explore their decoherence properties. This is readily done in the decoherent histories approach: the path integral form of the decoherence functional above comfortably accommodates coarse grainings involving variables defined nonlocally in time. We calculate the decoherence functional by summing over paths in which the functionals X_n^s, X_n^c of $x(t)$ are each constrained to lie in small widths, Δ_n. This can be achieved by inserting window functions Υ_Δ, which are 1 inside a region of width Δ_n and zero outside. Explicitly, the decoherence function has the form

$$D(\underline{\alpha},\underline{\alpha}') = \int \mathcal{D}x \int \mathcal{D}y \exp\left(\frac{i}{\hbar}S[x(t)] - \frac{i}{\hbar}S[y(t)] + \frac{i}{\hbar}W[x(t),y(t)]\right)\rho_0(x(0),y(0)) \times \prod_n \Upsilon_{\Delta_n}(X_n^s - \overline{X}_n^s)\Upsilon_{\Delta_n}(X_n^c - \overline{X}_n^c)\Upsilon_{\Delta_n}(Y_n^s - \overline{Y}_n^s)\Upsilon_{\Delta_n}(Y_n^c - \overline{Y}_n^c), \tag{4.33}$$

where Y_n^s and Y_n^c are defined in terms of $y(s)$ exactly as in Eqs. (4.31), (4.32), and $\underline{\alpha}$ now denotes the $\overline{X}_n^s$ and $\overline{X}_n^c$. To see how well the variables X_n^s, X_n^c decohere, we rewrite the influence functional in terms of them. Inserting the explicit form for $\nu(s)$, Eq. (4.18), and expanding the factor $\cos\omega_n(s-s')$, it is readily shown that

$$\text{Im}\, W = \sum_n \frac{c_n^2}{4m_n\omega_n}\coth\left(\frac{\hbar\omega_n}{2kT}\right)[(X_n^s - Y_n^s)^2 + (X_n^c - Y_n^c)^2]. \tag{4.34}$$

Since the part of decoherence the functional governing decoherence goes like $\exp(-\text{Im}\, W/\hbar)$ there is clearly decoherence of the Fourier variables, provided that the widths of their coarse graining are sufficiently large,

$$\Delta_n^2 \frac{c_n^2}{m_n\hbar\omega_n}\coth\left(\frac{\hbar\omega_n}{2kT}\right) \gg 1. \tag{4.35}$$

Again this may be regarded as a lower limit on the precision to within which the histories may be defined.

An interesting feature of these expressions is that the oscillatory functions of time are no longer present, since they have been absorbed into the new nonlocal variables. It is therefore not necessary to take an infinity of oscillators in the environment to obtain decoherence, nor to take high temperatures. In particular, there is a degree of decoherence, at any temperature, and *even if there is only one oscillator in the environment*.

This last result is perhaps surprising, but it is in keeping with the idea put forward in the Introduction, which loosely speaking is that tracing out *anything* coupled to the system ought to produce decoherence of *something*. The variables that decohere are nonlocal in time, and this is how they get round the old problem of recurrences. Furthermore, the uncomplicated nature of the decoherence of the Fourier modes, provides a useful alternative view on decoherence of histories of positions in the case of low temperatures, or finite environments, where the oscillatory and nonlocal character of the noise kernels makes it difficult to get a clear picture of the decoherence of position histories. That is, we regard the Fourier modes as in some sense more fundamental, and then approximately reconstruct histories of positions from them. From now on we will work entirely with particle trajectories characterized by fixed values of the Fourier modes.

C. System-environment correlations

We turn now to the question of establishing the correlations between the environment and system in the quantum case, and the consequent decoherence. We have shown that, classically, the final values of q_n and p_n are correlated with the Fourier components of the particle's trajectory. This can be established in the quantum theory by considering a probability in which, in addition to projecting onto the particle's trajectory at a series of times, we also consider projections onto the final state of the environment. In the quantum theory, one has to make a choice between projecting onto final values of q_n or p_n, or onto both using phase space quasiprojectors. We first consider final states of the environment characterized by fixed final values of q_n, denoted q_n''. The general question is, given the probability $p(\underline{\alpha})$ for a decoherent set of histories, under what conditions can one introduce a record projector $R_{\underline{\beta}}$ onto ranges of oscillator positions at the end of the history, so that the probabilities for histories are essentially undisturbed when the labels $\underline{\beta}$ are suitably chosen?

We have shown that histories of the Fourier modes decohere as long as they are coarse grained to a width Δ, defined above. The probability for a set of histories plus records consisting of a projection $R_{\underline{\beta}}$ onto ranges σ of value of q_n'' is

$$p(\underline{\alpha},\underline{\beta})=\prod_n \int dq_n''\Upsilon_\sigma(q_n''-\overline{q_n})\int_{\underline{\alpha}}\mathcal{D}x\int_{\underline{\alpha}}\mathcal{D}y \times\exp\left(\frac{i}{\hbar}S[x]-\frac{i}{\hbar}S[y]\right) \times\mathcal{F}[x(t),y(t);\{q_n''\}]\rho(x_0,y_0), \tag{4.36}$$

where $\{q_n\}$ denotes the set of all oscillator coordinates. Υ_σ is again a window function of width σ which implements the projection onto a range of q_n'', centered around $\overline{q_n}$ (which correspond to record labels $\underline{\beta}$). $\underline{\alpha}$ denotes the paths of the particle in configuration space specified by fixed values of the Fourier modes, as in Eq. (4.33). The object $\mathcal{F}[x(t),y(t);\{q_n''\}]$ is a generalized influence functional, given by the same path integral expression (4.7), but with the different boundary conditions that the final values of q_n and r_n are set to the value q_n'' (rather than summed over). Hence integrating $\mathcal{F}[x(t),y(t);\{q_n''\}]$ over all the q_n'''s, which is equivalent to letting $\sigma\to\infty$ in Υ_σ, yields the usual influence functional, and hence the original probability $p(\underline{\alpha})$. The question is therefore to determine the smallest value of σ for which the probability $p(\underline{\alpha},\underline{\beta})$ is the same as $p(\underline{\alpha})$, that is, the smallest value for which the integral of q_n'' over the range σ is essentially equivalent to integrating over an infinite range.

$\mathcal{F}[x(t),y(t);\{q_n''\}]$ may be written in terms of the propagator equation (4.9):

$$\mathcal{F}[x(t),y(t);\{q_n''\}]=\prod_n \int dq_n'dr_n'\rho_0^{\text{env}}(q_n,r_n)g(q_n'',\tau|q_n',0)g^*(q_n'',\tau|r_n',0). \tag{4.37}$$

The integrals are Gaussians, and at some length, one obtains the result

$$\mathcal{F}[x(t),y(t);\{q_n''\}]=\prod_n \exp\left[-A\left(q_n''-\frac{d[x]}{b}\right)^2-A\left(q_n''-\frac{d[y]}{b}\right)^2+B\left(q_n''-\frac{d[x]}{b}\right)\left(q_n''-\frac{d[y]}{b}\right)\right] \times\exp\left(-\frac{i}{\hbar}q_n''[c[x]-c[y]+\cos\omega_n\tau(d[x]-d[y])]\right) \times\exp\left(-\frac{i}{2\hbar m_n\omega_n}\sin\omega_n\tau\cos\omega_n\tau(d^2[x]-d^2[y])-\frac{i}{\hbar}(f[x]-f[y])\right), \tag{4.38}$$

where the coefficients A, B are given by Eq. (4.4), and b, $c[x]$, and $d[x]$ are given by Eqs. (4.10)–(4.14). From these, and comparing with Eq. (4.31), we see that

$$\frac{d[x]}{b}=-\frac{c_n}{m_n\omega_n}X_n^s\equiv-\tilde{X}_n^s. \tag{4.39}$$

'Similarly, we also see that

$$c[x]+\cos\omega_n\tau d[x]=c_nX_n^c. \tag{4.40}$$

Hence Eq. (4.38) may be rewritten

$$\mathcal{F}[x(t),y(t);\{q_n''\}]=\prod_n \exp[-A(q_n''+\tilde{X}_n^s)^2-A(q_n''+\tilde{Y}_n^s)^2+B(q_n''+\tilde{X}_n^s)(q_n''+\tilde{Y}_n^s)]\exp\left(-\frac{i}{\hbar}q_n''c_n(X_n^c-Y_n^c)\right)$$
$$\times\exp\left(-\frac{i}{2\hbar m_n\omega_n}\sin\omega_n\tau\cos\omega_n\tau(d^2[x]-d^2[y])-\frac{i}{\hbar}(f[x]-f[y])\right). \tag{4.41}$$

As expected from the classical analysis, the first exponential in Eq. (4.37) indicates that the oscillator coordinates q_n'' are approximately correlated with the Fourier modes $-\tilde{X}_n^s$.

To see more precisely the nature of the correlation, note that the Gaussian in Eq. (4.41) may be rewritten

$$\exp\{-(2A-B)[q_n''+\tfrac{1}{2}(\tilde{X}_n^s+\tilde{Y}_n^s)]^2-\tfrac{1}{4}(2A+B)(\tilde{X}_n^s-\tilde{Y}_n^s)^2\}. \tag{4.42}$$

Clearly the second term in this exponential gives the decoherence of the Fourier modes $\tilde{X}_n^s$ [since this corresponds exactly to the usual imaginary part of the influence functional phase (4.16) when the oscillator coordinates are integrated out]. The decoherence width $\tilde{\Delta}$ of these modes is

$$\tilde{\Delta}\equiv\frac{c_n}{m_n\omega_n}\Delta=(2A+B)^{-1/2}=\left(\tanh\left(\frac{\hbar\omega_n}{2kT}\right)\right)^{-1/2} \tag{4.43}$$

[in agreement with the earlier analysis, Eq. (4.35)]. Hence a projection onto a range of q_n'' of *any* width σ can be added at the end of the histories without affecting decoherence. In order to preserve the original probabilities for the histories as much as possible, however, the width σ of the record projection needs to satisfy

$$\sigma>(2A-B)^{-1/2}=\left(\coth\left(\frac{\hbar\omega_n}{2kT}\right)\right)^{-1/2} \tag{4.44}$$

for the integral to be equivalent to an integral over an infinite range.

Generally the width σ of the records q_n'' will be greater than the width $\tilde{\Delta}$ of the decoherent histories of Fourier modes, $\tilde{X}_n^s$. The correlation between them must necessarily be imperfect, therefore, since the records alternatives, being more coarsely defined, will not be able to completely distinguish between different past history alternatives. Differently, fixing a record alternative does not uniquely fix a past history alternative, hence the conditional probability of the histories given the records is less than one. Yet another way of putting it is to say that, in a suitably chosen counting technique [as we did after Eq. (4.27), for example], the number of records will be *less* than the number of decoherent histories. The imperfection of the records in the mixed state case can in fact be understood already at a classical level. For even classically, the amount of correlation between the phase space data of the environment and the Fourier modes will be reduced if the environment is subject to thermal fluctuations.

In the case of a pure initial state for the environment, $B=0$, and therefore $\sigma\sim\tilde{\Delta}$, and in this case we will have a near-perfect correlation between the records and the histories (as perfect as the degree of approximate decoherence, which is generally extremely good).

General expectations are therefore confirmed: records exist in the case of a pure initial state, with an almost perfect correlation between the history alternatives and the records. In the mixed state case, records continue to exist but with an imperfect correlation.

So far we have seen how projections onto ranges of the environmental coordinates q_n'' are correlated with the Fourier modes X_n^s describing the histories. This is, however, only a partial description of the histories, since the variables X_n^c, which are in some sense complementary to X_n^s, also decohere. We expect these to be correlated with the environmental momenta.

To investigate projections onto more general types of records, such as this, at the final time, we need to consider a more general type of influence functional in which the paths summed over to obtain the influence functional (4.37) are not constrained to meet at q_n'', but may take different values. This allows arbitrary states to be attached at the final time. It is straightforward to show that this more general influence functional is given by

$$\mathcal{F}[x(t),y(t);\{q_n''\},\{r_n''\}]=\prod_n \exp[-A(q_n''+\tilde{X}_n^s)^2-A(r_n''+\tilde{Y}_n^s)^2+B(q_n''+\tilde{X}_n^s)(r_n''+\tilde{Y}_n^s)]\exp\left(-\frac{i}{\hbar}c_n(q_n''X_n^c-r_n''Y_n^c)\right)$$
$$\times\exp\left(-\frac{i}{2\hbar m_n\omega_n}\sin\omega_n\tau\cos\omega_n\tau(d^2[x]-d^2[y])-\frac{i}{\hbar}(f[x]-f[y])\right). \tag{4.45}$$

This object is, in fact, essentially just the thermal initial state, unitarily shifted in positions and momenta by the classical equations of motion (4.29), (4.30) with vanishing initial positions and momenta:

$$\mathcal{F}[x(t),y(t);\{q_n''\},\{r_n''\}]=\langle q_n''|U(-\tilde{X}_n^s,-c_nX_n^c)\rho_0^{\rm env}\times U^\dagger(-\tilde{Y}_n^s,-c_nY_n^c)|r_n''\rangle \tag{4.46}$$

(up to a possible phase). Here, $U(q,p)$ represents the unitary displacement operator in positions and momenta. This result is not surprising for a linear system.

Projections onto final momenta may be considered by Fourier transforming with respect to both q_n'' and r_n''. In the zero temperature case, for which $B=0$, it is clear to see what is going on. $\mathcal{F}$ has the form of the pure state density operator for a coherent state of spatial width $A^{-1/2}$. Fourier transform therefore leads to a state which has exactly the same form; thus the discussion of decoherence and records is the same as the previous case. The mixed state case will be similar.

Perhaps more useful and general is to combined the above two cases and consider quasiprojectors onto the final values of the environmental phase space data. Using Eq. (4.46), the explicit decoherence functional for the situation involving any projection $R_{\underline{\beta}}$ onto environment states at the final time is

$$\begin{aligned}D(\underline{\alpha},\underline{\alpha}',\underline{\beta})=&\int_{\underline{\alpha}}\mathcal{D}x\int_{\underline{\alpha}'}\mathcal{D}y\\&\times\exp\left(\frac{i}{\hbar}S[x]-\frac{i}{\hbar}S[y]\right)\rho_0(x_0,y_0)\\&\times{\rm Tr}\big(R_{\underline{\beta}}U(-\tilde{X}_n^s,-c_nX_n^c)\rho_0^{\rm env}\\&\times U^\dagger(-\tilde{Y}_n^s,-c_nY_n^c)\big),\end{aligned} \tag{4.47}$$

where the trace is over the environment Hilbert space. It is clear that decoherence and the probabilities for histories are not disturbed if the records projectors $R_{\underline{\beta}}$ are taken to be phase space quasiprojectors onto suitable large regions of phase space, and the discussion is again very similar, so need not be spelled out in detail.

To summarize, the classical analysis shows that the Fourier modes of the particle trajectories are correlated with the final values of the phase space data of the environment at the final time. We have shown that essentially the same story persists in the quantum theory. For the zero temperature case, the record projectors need to be wide enough to beat quantum fluctuations. For finite temperature, they need in addition to beat the thermal fluctuations, and the correlation between the records and the history alternatives is then less than perfect, in accordance with general expectations.

It is also worth noting that the discovered correlation of the final phase space data with the Fourier modes of the environment means that the environment effectively performs a so-called spectral measurement of the particle's trajectory. Measurements of this type have previously been investigated by Mensky in the context of the path integral approach to continuous quantum measurement [37].

D. Information count

We may now check that, as asserted at the beginning of the paper, the amount of decoherence is related to the amount of information thrown away. That is, the number of histories in the decoherent set is approximately the same as the number of states thrown away to the environment. We will consider the most general case considered above, in which the system histories are characterized by X_n^s, X_n^c, and the records are phase space projectors onto the environmental oscillators.

Consider first the case of zero temperature. Since the variables we are dealing with are continuous and the Hilbert spaces infinite dimensional, we need to make some artificial restrictions in order to do any counting. Hence, as earlier in this section, lets us restrict the dynamics of the distinguished particle to a spatial region of size L. The Fourier variables (4.31), (4.32), are therefore restricted to a region of size of order $L\tau$.

For decoherence, the widths Δ_n of the Fourier variables must satisfy the inequality (4.35), which for $T=0$ reads $\Delta_n^2>m_n\hbar\omega_n/c_n^2$. The histories of the two types of Fourier variables, X_n^s and X_n^c are therefore each defined up to order Δ_n, satisfying this restriction, and there are of order $L\tau/\Delta_n$ decoherent histories of the variables X_n^s and the same number of the variables X_n^c. Hence, for each mode n, the total number of histories N_d in the decoherent set is

$$N_d=\left(\frac{L\tau}{\Delta_n}\right)^2<\frac{c_n^2L^2\tau^2}{m\hbar\omega_n}. \tag{4.48}$$

Now consider the environment states for each mode n. Each environment mode starts out centered around $q_n=0=p_n$, and as a result of interacting with the system, is displaced in q_n and p_n by the amounts (4.29), (4.30). (A partially classical analysis suffices since the system is linear.) Since $x(t)$ is assumed to be restricted to a region of size L, q_n will range over a region with size of order $c_nL\tau/m_n\omega_n$, and p_n will range over a region of size $c_nL\tau$. q_n and p_n therefore range over a phase space volume of size $c_n^2L^2\tau^2/m_n\omega_n$. The number of distinct environment states, for each mode n, corresponding to this phase space volume is therefore given by

$$N_\epsilon=\frac{c_n^2L^2\tau^2}{m_n\hbar\omega_n} \tag{4.49}$$

which coincides with Eq. (4.48). This is therefore the desired result: the number of distinct states of the environment thrown away in the coarse-graining process is equal to the upper limit on the number of histories in the decoherent set of histories. Differently put, the record of each individual history of the Fourier variables is stored in a single phase space cell of an environment oscillator.

In the case of a thermal environment with $T\neq 0$, decoherence is improved so that, from Eq. (4.35), the number of histories in the decoherent set has a larger upper limit:

$$N_d=\left(\frac{L\tau}{\Delta_n}\right)^2<\frac{c_n^2L^2\tau^2}{m\hbar\omega_n}\coth\left(\frac{\hbar\omega_n}{2kT}\right). \tag{4.50}$$

The effect of thermal fluctuations on the environment states is, from one point of view, to reduce the number of distinguishable states, since the elementary phase space cells are effectively increased in size from $\hbar$ to $\hbar\coth(\hbar\omega_n/2kT)$ in a thermal state. That is, the number of distinct *accessible* records in the environment is reduced. However, as discussed in Sec. II, a mixed environment state can be regarded as a pure state on an enlarged environment Hilbert space, much of which is inaccessible, and it is from the perspective of this enlarged environment Hilbert space that we expect to understand the connection between decoherence and information loss.

There are then a number of ways of understanding how much information is stored in the enlarged Hilbert space. For example, we can regard the smearing of the environment phase space cells from $\hbar$ to $\hbar\coth(\hbar\omega_m/2kT)$ as meaning that the environment is actually in one of a number $\coth(\hbar\omega_n/2kT)$ of $\hbar$-sized phase space cells, but the information as to exactly which of those cells it occupies is stored in the inaccessible part of the enlarged Hilbert space. This indicates that the mixed state case, regarded as a pure state on an enlarged Hilbert space, has its information storage capacity enhanced by a factor of $\coth(\hbar\omega_n/2kT)$ compared to the $T=0$ case. Hence Eq. (4.49), the information storage capacity of one mode of the environment in the $T=0$ case, is multiplied by the factor, $\coth(\hbar\omega_n/2kT)$, and we obtain agreement with Eq. (4.50). That is, in the mixed initial state case also, the number of histories in the decoherent set is approximately the same as the maximum number of states storing information about the histories.

Another way of understanding the increased information storage in the mixed state case is to consider the von Neumann entropy $S=-\mathrm{Tr}(\rho\ln\rho)$ of the environment. Loosely speaking, in going from a pure to a mixed state, the number of states available for information storage is increased by e^S. It is well known that the entropy of a harmonic oscillator in a thermal state is of order $\ln(kT/\hbar\omega)$, hence the information storage enhancement factor is of order $kT/\hbar\omega_n$, for large T. This agrees with the $\coth(\hbar\omega_n/2kT)$ factor deduced above in the limit of high temperatures. It does not generally agree at lower temperatures, although this is not surprising since measures of uncertainty or information loss in quantum theory are dependent on the particular dynamical variables of interest. (Since a thermal state is diagonal in energy, the von Neumann entropy may be regarded as a measure of uncertainty in energy, which will generally not be the same as the phase space uncertainty used above). Nevertheless, these two arguments are sufficient for it to be seen that the degree of decoherence (4.50) may be related to information lost to the environment in the mixed state case.

E. Exact and approximate decoherence

Finally, we may make some elementary remarks about approximate decoherence. Inserting Eq. (4.46) in the expression for the joint probability of the histories and the records, we obtain the particularly simple expression

$$\begin{aligned}p(\underline{\alpha},\underline{\beta})=&\int_{\underline{\alpha}}\mathcal{D}x\int_{\underline{\alpha}}\mathcal{D}y\\&\times\exp\left(\frac{i}{\hbar}S[x]-\frac{i}{\hbar}S[y]\right)\rho_0(x_0,y_0)\\&\times\mathrm{Tr}(R_{\underline{\beta}}U(-\tilde{X}_n^s,-c_nX_n^c)\rho_0^{\mathrm{env}}U^\dagger(-\tilde{Y}_n^s,-c_nY_n^c)),\end{aligned} \tag{4.51}$$

where the trace is over the environment Hilbert space. For simplicitly take the environment initial state to be pure, so it is the ground state of the harmonic oscillator $|0\rangle$. The unitary displacement operators then turn it into standard coherent states.

The issue of exact decoherence or exact correlation of the records with the histories, is then the question of finding a coarse graining of the Fourier modes X_n^s and X_n^c, which effectively brings the coherent states

$$U(-\tilde{X}_n^s,-c_nX_n^c)|0\rangle$$

into an orthogonal set of states. It is well known that given the coherent states $|p,q,\rangle$, which are overcomplete, a complete set of states may be found by restricting p,q to discrete values lying on a regular lattice, and this is clearly implementable by suitable coarse graining of the Fourier modes [38]. The resulting states, however, are not orthogonal. The orthogonalization process may not be straightforward to carry out. More significantly, it is by no means clear that a coarse graining of the Fourier modes is possible which puts this orthogonalization process into effect. The issue of finding an exactly decoherent set of histories which is close to the approximately decoherent sets discussed in this section therefore remains open.

V. SUMMARY AND DISCUSSION

We have obtained a number of results concerning the connection between decoherence, information loss and the existence of records. The existing basic result that we have very much built on is the fact that decoherence with a pure initial state implies the existence of records [2], i.e., alternatives that may be added to the end of the histories that are perfectly correlated with the past alternatives. Our main aim was to explore the connection between decoherence and *physical* information storage in the case of decoherence due to an environment. The main results may broadly be summarized as follows.

(1) The discussion of records in the decoherent histories approach has been extended to the case of mixed initial states, both in the general results of Sec. II, and the explicit models of Secs. III and IV.

(2) In the quantum Brownian motion model the records

carrying information about the distinguished particle's trajectory have been explicitly identified.

(3) We have formulated a concrete conjecture concerning the amount of decoherence and the information lost to the environment, and proved it in some important specific cases. This gives substance to the old idea that decoherence is related to information loss.

This last result also indicates how decoherence conditions in practical models can be interpreted. The commonly used decoherence condition

$$\exp\left(-\frac{\mathrm{Im}\,W[x(t),y(t)]}{\hbar}\right)\ll 1 \tag{5.1}$$

(where W is the influence functional) is normally physically interpreted as meaning that interference between trajectories $x(t)$, $y(t)$ is very small, and therefore that probabilities may be assigned to these histories. That is, the condition (5.1) puts a lower limit on the degree to which the histories may be fine grained without interference effects becoming significant.

To assign probabilities, one requires only the condition of consistency, $\mathrm{Re}\,D(\underset{\sim}{\alpha},\underset{\sim}{\alpha}')=0$ for $\underset{\sim}{\alpha}\neq\underset{\sim}{\alpha}'$, whereas the condition (5.1) corresponds to the stronger condition of (approximate) decoherence, $D(\underset{\sim}{\alpha},\underset{\sim}{\alpha}')\approx 0$ for $\underset{\sim}{\alpha}\neq\underset{\sim}{\alpha}'$, which surely merits a stronger interpretation. The physical meaning of decoherence is that it implies the existence of records, as discussed in Secs. I and II (and in Ref. [2]). Consistency alone does not guarantee this. The decoherence condition (5.1) should therefore be thought of in terms of the records, rather than just in terms of interference and the assignment of probabilities. In this paper we have effectively shown that *the decoherence condition is a reflection of the information storage capacity of the environment*. That is, it is a lower limit on the degree to which the histories may be fine grained without the information storage capacity of the environment being exceeded.

Some of the issues considered in this paper shed some light on an old problem with decoherence in the context of quantum cosmology, which is how to choose the division of the universe into ''system'' and ''environment.'' On perusing the literature on decoherence via tracing out an environment, one can find papers in which a matter field is traced out to produce decoherence of the gravitational field in quantum cosmology [39]. On the other hand, one can find other papers in which the gravitational field is regarded as a decohering environment for matter, since it is clearly the universal environment [40]. Which is correct? One man's system is another man's environment, at least, from the point of view of published papers on the subject,

The case of decohering the gravitational field is an interesting one, since the gravitational field is undeniably classical in all physical situations that can be checked observationally, so there is a strong incentive to discover the mechanism by which it becomes classical from the assumed underlying quantum gravity theory. On the other hand, in a certain sense we *never* actually measure the gravitational field itself. What we actually measure are the changes of motion of matter that we interpret as being due to an underlying gravitational field. From that point of view, nothing is really lost by tracing out the gravitational field since it is never really actually observed.

The ideas discussed in this paper perhaps offer some resolution to the dilemma over the choice of ''system'' and ''environment.'' As we have seen in a number of situations, decoherence is intimately connected with the existence of records at the final moment of time that are correlated with alternatives in the past. Furthermore, as we saw in the analysis of the quantum Brownian motion model, the records can be stored in the decohering environment, and by inspecting them at the final time we can recover the past history of the system. What the decoherence of the quantum Brownian particle by a thermal environment means, therefore, is that the history of the Brownian particle may be recovered by examining the thermal environment. Similarly, the decoherence of a gravitational field by a decohering matter field environment means that we can recover the history of the gravitational field by examining the matter field at late times, which is indeed exactly what is done in cosmology. From a practical point of view therefore, the significance of decoherence is that it ensures a correlation between present records and past events.

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