Wilson fermions on a randomly triangulated manifold

Z. Burda
Laboratoire de Physique Théorique, Bâtiment 210, Université Paris-Sud, 91405 Orsay, France and Institute of Physics, ul. Reymonta 4, Jagellonian University, 30-059 Kraków, Poland

J. Jurkiewicz
Institute of Physics, ul. Reymonta 4, Jagellonian University, 30-059 Kraków, Poland

A. Krzywicki
Laboratoire de Physique Théorique, Bâtiment 210, Université Paris-Sud, 91405 Orsay, France
(Received 18 May 1999; published 26 October 1999)

A general method of constructing the Dirac operator for a randomly triangulated manifold is proposed. The fermion field and the spin connection live, respectively, on the nodes and on the links of the corresponding dual graph. The construction is carried out explicitly in 2D, on an arbitrary orientable manifold without boundary. It can be easily converted into a computer code. The equivalence, on a sphere, of Majorana fermions and Ising spins in 2D is rederived. The method can, in principle, be extended to higher dimensions.
[S0556-2821(99)04320-9]

PACS number(s): 11.15.Ha, 04.60.Nc, 11.25.Pm

I. INTRODUCTION

A. Preamble

The statistical mechanics of random manifolds has been intensely studied for a dozen years (see the reviews [1]). The interest of endowing these manifolds with fermionic degrees of freedom seems rather evident. It turns out that little has been done in this direction. A notable exception is the paper by Bershadsky and Migdal [2], where it is demonstrated that the Ising model on a fixed planar 2D graph is equivalent to a certain Majorana fermion theory on the dual graph. However, we do not find the discussion of Ref. [2] fully satisfactory. The authors ignore the covariance aspects of the problem and, therefore, several features of the model, which have a natural explanation, come out as ''miracles.'' Furthermore, it does not seem possible to extend their arguments beyond 2D. Some useful ideas can be found in the pioneering papers of the Columbia group, in particular in Ref. [3], but they do not discuss at all the topological aspects of defining a spin structure on a piecewise linear manifold.

The aim of the present paper is to propose a rather general method of constructing the Wilson fermion action on a piecewise flat manifold, made up by gluing equilateral simplices at random. The metric is assumed to have the Euclidean signature. Our motivation for studying this problem originates from our involvement in the study of simplicial gravity. In this context, limiting oneself to such manifolds is a common practice and seems justified by the results obtained in the study of noncritical strings: for a class of exactly solvable models in 2D it has been shown that the continuum and the discrete version belong to the same universality class, when the ''dynamical triangulation'' recipe, is adopted [1]. According to this recipe, the sum over metrics in the Feynman integral is indeed replaced by the sum over all triangulations of the above mentioned type. In order to make the discussion as clear as possible we shall focus on 2D. The results of Ref. [2] will, of course, be recovered. In the last section we shall briefly discuss the extension of the idea to arbitrary triangulations and to higher dimensions.

The reader will notice that the concept of a continuous manifold appears in our discussion as a scaffolding that helps to achieve the goal of putting a spinor field on an abstract graph, with specific intrinsic symmetries.

B. General strategy

How to put a spinor field on a curved manifold is explained in textbooks of general relativity (see, e.g., [4]). The key concepts are local frames, parallel transport and spin connection. The application of the general recipe to the case of a manifold discretized in the manner of Regge is, however, not quite trivial, since one cannot limit the discussion to infinitesimal displacements and rotations.

For pedagogical reasons, let us recall some differential geometry. With each point x of the manifold is associated an arbitrary orthonormal local frame defined by a set of d vectors $e^j_\mu(x)$ called vielbeins. Here, the latin index j refers to the axes of the local frame transforming under local rotations belonging to SO(d). Using vielbeins one can eliminate, by contraction, the space (here Greek) indices, in order to deal only with objects transforming under SO(d), as in flat space. One can then introduce spinor fields, because the group SO(d) has spinor representations, contrary to the group GL(d) induced by general coordinate transformations. The program is incomplete, however, until one tells how to compare local frames at distinct points of space. When $e^j_\mu(x)$ is parallel transported from x to $x+\delta x$, one obtains a frame rotated with respect to the one that has been chosen at $x+\delta x$. This observation is formally expressed by the equation

$$e^j_\mu(x+\delta x)=e^j_\mu(x)+\Gamma^\lambda_{\mu\nu}(x)e^j_\lambda(x)\,\delta x^\nu-\omega^j_{k\nu}(x)e^k_\mu(x)\,\delta x^\nu. \tag{1}$$

Here, $\Gamma^\lambda_{\mu\nu}$ is the Christoffel symbol and $\omega^j_{k\nu}$ is the spin con-

nection, which takes care of the relative rotation of the neighboring local frames. Equation (1) can be used to define a covariant derivative, which gives zero acting on the vielbein field (and therefore also on the metric $g_{\mu\nu}=e^j_\mu e_{j\nu}$).

Since we are interested in piecewise flat manifolds, we can assume that there exists an orthonormal reference frame in a region comprising the points x and $x+\delta x$. Choosing this frame one gets, from Eq. (1),

$$e^j_\mu(x+\delta x)=[\,\delta^j_k+\omega^j_{k\nu}(x)\,\delta x^\nu]e^k_\mu(x). \tag{2}$$

The operation on the right-hand side is an infinitesimal rotation. The space index μ is inert since, by assumption, the parallel transport from x to $x+\delta x$ is trivial.

Consider now a closed curve C in our piecewise flat manifold. It is assumed that C can be covered with a set of flat patches $P_a, a=1,\ldots,n$: $P_a\cap P_{a+1}$ is nonempty, and there exists an orthonormal reference frame common to the whole of $P_a\cup P_{a+1}$. The conventions are such that $P_{n+1}=P_1$. We associate a common $e^j_\mu(a)$ with all points of P_a. The analogue of Eq. (2) is

$$e^j_\mu(b)=U^j_k(ba)e^k_\mu(a), \tag{3}$$

where P_b is the patch next to P_a and $U(ba)$ is a finite rotation from a to b. In the following, the space indices, like μ above, which are irrelevant to our discussion, will not be exhibited. It will be convenient to eliminate the latin indices too, using the matrix notation to write, for example

$$e(b)=U(ba)e(a), \tag{4}$$

instead of Eq. (3). As already mentioned, the choice of the local frames $e(a)$ and $e(b)$ is arbitrary. Thus $U(ba)$ is defined up to a local gauge transformation

$$U(ba)\to G(b)U(ba)G^{-1}(a). \tag{5}$$

A vector $v(a)$ is uniquely defined in P_a by its components along the d vielbeins $e(a)$. Equation (4) tells how these components change as one goes from a given patch to its neighbor.

For any two neighboring patches the rotation depends on the gauge choice at these patches. It is easy to see that for the closed path $P_1\to\ldots\to P_a\to P_{a+1}\ldots\to P_1$ the global rotation matrix $U(C)$ has the gauge transformation

$$U(C)\to G(1)U(C)G^{-1}(1) \tag{6}$$

and, therefore, $(1/d)\mathrm{Tr}\,U(C)$ is a gauge independent geometrical object, whose deviation from unity is a measure of the curvature of the manifold (see Fig. 1). An orthonormal frame common to all patches does not, in general, exist and the chain of spin connections has inherited information about the curved metric.

The transformation matrix in Eq. (4) belongs to the vector representation of SO(d). A similar transformation law holds for a spinor. The only difference is that the corresponding transformation matrix belongs to the spinor representation of SO(d).

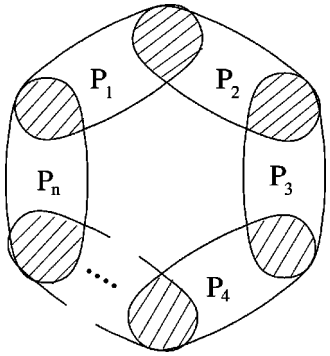

FIG. 1. The set of patches covering a closed curve (not shown explicitly). The region within two overlapping neighboring patches is flat.

In this paper, the simplices of the triangulation play the role of the flat patches above. Two neighboring simplices have a common $(d-1)$-dimensional face. The spinor field lives on simplices or, stated differently, on the vertices of the dual lattice. Only spinors belonging to neighboring simplices are directly coupled. There always exists an orthonormal frame common to two neighboring simplices. Therefore the coupling is given by the corresponding spinor connection, which can be constructed without too much effort for a pair of simplices. The problem is that one has to define the spinor connection consistently all over the lattice, which is nontrivial (there exist topologies where this is impossible). The whole construction will be done explicitly, in 2D, in the next section.

II. TWO-DIMENSIONAL MANIFOLDS

A. Spin connection in the vector representation

We consider a triangulation of an orientable 2D manifold. By convention, all triangles are oriented counterclockwise. In particular, given a triangle a, the angle between $e^1(a)$ and $e^2(a)$ is $\pi/2$ when measured counterclockwise. We focus on two triangles, say a and b, sharing a common link. It is convenient to introduce two auxiliary local frames, $f(ab)$ and $f(ba)$, attached to this link and rotated by the angle π with respect to each other: $f^1(ab)$ ($f^1(ba)$) is perpendicular to the link and points towards the exterior of a (b). The operation performed by the spin connection matrix $U(ba)$ can be defined by the following chain of rotations:

$$U(ba):e(a)\to f(ab)\to f(ba)\to e(b). \tag{7}$$

Denote by $R(\phi)$ the rotation by angle ϕ: $R(\phi)=\exp(\epsilon\phi)$, where ϵ is the rotation generator represented by the antisymmetric matrix $\epsilon_{12}=-\epsilon_{21}=1$. Hence

$$U(ba)=R^{-1}(\phi_{b\to a})R(\pi)R(\phi_{a\to b}). \tag{8}$$

Here $\phi_{a\to b}$ is the angle between $e^1(a)$ and $f^1(ab)$, while $\phi_{b\to a}$ is the angle between $e^1(b)$ and $f^1(ba)$ (see Fig. 2). It should be kept in mind that these angles are oriented. Notice that

$$U(ba)U(ab)=1. \tag{9}$$

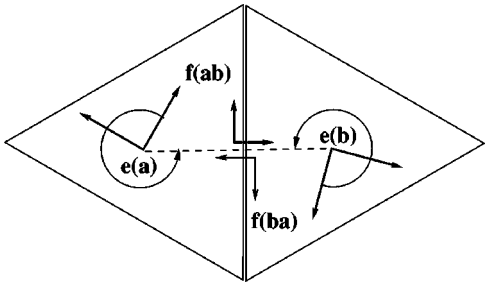

FIG. 2. The local frames $e(a)$ and $e(b)$ in two neighboring triangles a and b are shown. The two auxiliary frames $f(ab)$ and $f(ba)$ are also exhibited. They all have the same orientation. The oriented arc in the triangle a represents the angle $\phi_{a\to b}$. Rotation by this angle brings the frame $e(a)$ into $f(ab)$. Similarly, the arc in b represents the angle $\phi_{b\to a}$. The rotation by this angle brings $e(b)$ into $f(ba)$. Notice, that the angle is always measured from the local to the auxiliary frame. This is why an inverse rotation appears in Eq. (8).

Repeating the above argument, one eventually associates a connection matrix with every oriented link of the *dual* lattice. Let L_n be an elementary loop of the dual lattice going through n triangles labeled $1,2,\ldots,n$ and let $U(L_n)$ be the parallel transporter around L_n. One has

$$\mathrm{Tr}\, U(L_n)=\mathrm{Tr}\prod_{k=1}^{n} R(\pi)R(\phi_{k\to k-1})R^{-1}(\phi_{k\to k+1}), \tag{10}$$

using the cyclic labeling convention ($0\equiv n$ and $n+1\equiv 1$). Each factor above corresponds to a rotation by the angle $\pi+\phi_{k\to k-1}-\phi_{k\to k+1}$ equal, modulo 2π, to $\pi/3$ ($-\pi/3$) for L_n oriented clockwise (counterclockwise). Hence

$$\tfrac{1}{2}\,\mathrm{Tr}\, U(L_n)=\cos(n\pi/3) \tag{11}$$

independently of the orientation of the loop. For $n=6$, when the angular defect is zero and therefore the lattice is locally flat, the right-hand side equals unity, as expected.

B. The spinor case

The logic underlying the construction of the spinor connection, transforming a spinor field from one local frame to another, is close to that of the preceding subsection. There are, however, extra complications due to the fact that spinor rotations by ϕ and $\phi+2\pi$ are not equivalent. The resulting sign ambiguities require care.

The analogue of $U(ba)$, to be denoted $V(ba)$, is a matrix belonging to the spin 1/2 representation of the rotation group. In 2D there are two Hermitian Dirac matrices γ^1 and γ^2 satisfying the usual anticommutation relations $\{\gamma^j,\gamma^k\}=2\delta^{jk}$. Another standard matrix is $\gamma_5=-i\gamma^1\gamma^2$, which in 2D is equal to the generator of rotations $[\gamma^1,\gamma^2]/2i$.

The matrix representing a rotation by an angle ϕ is

$$S(\phi)=\pm\exp(i\gamma_5\phi/2). \tag{12}$$

The sign ambiguity cannot be resolved unless one is in a position to control the angle ϕ in the range 0 to 4π. The connection matrix $V(ba)$ has the same sign ambiguity. In analogy to Eq. (8) write

$$V(ba)=g_{ba}S^{-1}(\phi_{b\to a})S(\pi)S(\phi_{a\to b}), \tag{13}$$

where the rotation matrices are all taken with the positive sign [cf. Eq. (12)] and g_{ba} is a sign factor. The parallel transport of the spinor from a to b and back does not introduce any change, and therefore

$$V(ab)V(ba)=1, \tag{14}$$

which implies that

$$g_{ab}=-g_{ba}. \tag{15}$$

Replacing R by S on the right-hand side of Eq. (10) and including the sign factors, one finds the spinor analogue of Eq. (11) to be

$$\mathrm{Tr}\, V(L_n)=\mathrm{Tr}\prod_{k=1}^{n} g_{k\,k+1}S(\pi)S(\phi_{k\to k-1})S^{-1}(\phi_{k\to k+1}). \tag{16}$$

As in the vector case, the three rotations following the Π symbol correspond to a single rotation by $\pm\pi/3$ modulo 2π. But since we are now working in the spinor representation, the 2π is not innocent since it yields an extra negative sign. Hence, in general, the rotation is $\pm\exp(\pm i\gamma_5\pi/6)$. The sign in front of the exponential has to be determined carefully. It does not only depend on the way the dual lattice loop goes through the triangle k but also on the choice of the gauge, i.e., on the direction of the vielbein $e^1(k)$ (in 2D specifying the direction of a single vielbein suffices to fix the local frame). Since $\mathrm{Tr}\, V(L_n)$ is gauge invariant, we can fix the gauge at our convenience. We shall assume that in each triangle e^1 points from the center of the triangle towards one of the vertices.[1] Figure 3 illustrates the six possible paths the loop can take through the triangle k. The direction of $e^1(k)$ is also indicated, and appears as a flag associated with the vertex of the dual lattice. The result of the calculation of the rotation matrix is given in Table I.

In the first three cases the path in the figure turns left. The corresponding elementary loop goes counterclockwise. In the remaining cases it goes clockwise. The exponential factors are known as the Kac-Ward factors. Drawing a dashed line parallel to the loop, on the right of it, as shown in Fig. 3, one can see from Table I that the sign factor is negative when the dashed line crosses the flag. Otherwise it is positive. Denote the sign factors by z_{kjl}, where the letters k,j,l refer to the three successive triangles on the loop. Collecting all the sign factors we finally get

$$\tfrac{1}{2}\,\mathrm{Tr}\, V(L_n)=F(L_n)\cos(n\pi/6), \tag{17}$$

[1]Actually, the result of the calculation would be the same if the vielbein were rotated forth or back by an angle less than $\pi/3$.

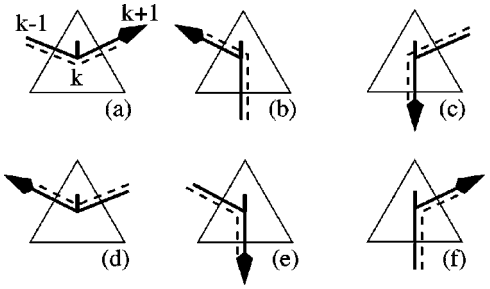

FIG. 3. Six possible ways a dual lattice loop can go through a triangle. The line segment pointing up from the center of the triangle is the vertex flag indicating the gauge choice, i.e., the direction of the vielbein $e^1(k)$. The dashed line is the loop slightly displaced to the right. The sign factor $z_{k-1\ k\ k+1}$ is negative when the dashed line crosses the flag.

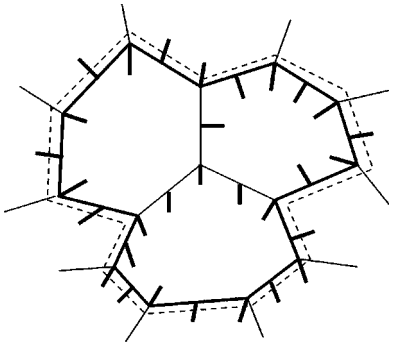

FIG. 4. A portion of a dual lattice with flags put on. The number of flags inside each elementary loop is odd. Notice that the dashed line following a nonelementary loop also crosses an odd number of flags.

where

$$F(L_n)=g_{12}z_{123}\ g_{23}\ z_{234}\ \ldots\ g_{n1}\ z_{n12}. \tag{18}$$

Notice that the loop sign factor can be defined for any closed loop, not only for an elementary one.

For $n=6$, the sign factors on the right-hand side of Eq. (18) should combine to give $F(L_6)=-1$ because the parallel transport is trivial in flat space. Equation (17) is a relation between an invariant measure of the local curvature on the left-hand side, and the angle deficit $2\pi-n\pi/3$, which determines the argument of the cosine function on the right-hand side. It is clear that if one could change the deficit angle continuously, like, e.g., at the top of a cone, the factor in front of the cosine would not change discontinuously. On a dynamically triangulated surface the quantization of the deficit angle is a lattice artifact, devoid of any deep physical significance. Therefore we set

$$F(L_n)=-1 \tag{19}$$

for all elementary loops L_n, whatever n is. We shall prove now that these constraints can be satisfied on an orientable 2D manifold.

C. Satisfying the constraints on an orientable 2D manifold

We have already introduced the vertex flags. Now we also attach flags to links. We put a flag on the right-hand side of the dual lattice link going from a to b if $g_{ab}=-1$. There is nothing fundamental in these flags. They are merely a convenient tool for helping to set the spin structure on the simplicial manifold.

TABLE I. The sign and the Kac-Ward factors.

Fig. 3	$\phi_{k\to k-1}$	$\phi_{k\to k+1}$	$S(\pi)S(\phi_{k\to k-1})S^{-1}(\phi_{k\to k+1})$
a	$\pi/3$	$5\pi/3$	$+\exp(-i\gamma_5\pi/6)$
b	π	$\pi/3$	$-\exp(-i\gamma_5\pi/6)$
c	$5\pi/3$	π	$-\exp(-i\gamma_5\pi/6)$
d	$5\pi/3$	$\pi/3$	$-\exp(+i\gamma_5\pi/6)$
e	$\pi/3$	π	$+\exp(+i\gamma_5\pi/6)$
f	π	$5\pi/3$	$+\exp(+i\gamma_5\pi/6)$

Let us follow a counterclockwise oriented elementary loop staying always slightly *outside* of it. Since each crossed flag corresponds to a negative sign factor, it follows from the rule established in the preceding section that Eq. (19) is satisfied provided we cross an odd number of flags. Following a clockwise oriented elementary loop, staying always slightly *inside* of it, one also requires that the number of crossed flags is odd. The two requirements are equivalent, since the total number of flags attached to a loop is even. In short, there should be an odd number of flags both outside and inside every elementary loop independent of its orientation. A portion of a dual lattice with flags put on vertices and links is shown in Fig. 4. It should be clear at this point of the discussion that the exact direction of a flag is irrelevant. One can rotate them as long as one does not cross a link of the dual lattice.

Let us briefly outline the strategy adopted to prove that the constraint (19) can be satisfied on every orientable manifold. First, by direct inspection, we check that it can be satisfied on a minimal sphere. Then we show that it can be preserved when one locally deforms the geometry with an ergodic move, whose repeated application enables one to construct an arbitrary sphere. Since the constraint is satisfied for the initial configuration and it is preserved by the moves, it can be satisfied on an arbitrary spherical lattice. Higher genus surfaces can be produced by gluing spheres. We show that this gluing can also be done in preserving the constraint.

It is easy to convince oneself that one can satisfy the constraint (19) on a minimal sphere made up of four triangles [see Fig. 5(a)]. A sphere of arbitrary size can be constructed by using the moves introduced, for example, in Ref. [5]. One move consists of splitting an elementary loop of the dual lattice by inserting a new link. The inverse move consists in removing a link. We show below that the construction of a sphere of arbitrary size can be done, thereby preserving (19) for every elementary loop.

When a link is added, the flags preexisting *inside* the loop are partitioned among the two new loops. Since their total number is odd, one loop gets an even number of old flags

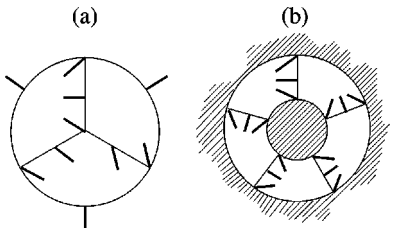

FIG. 5. (a) A possible assignment of flags on a minimal sphere. (b) The shaded areas represent two identical spheres with one loop cut out. We show a way of gluing them together with a bridge made up of new links. Only the new flags on the bridge are exhibited. In each new loop there is an even number of flags, old and new, which are not shown explicitly. Repeating the operation enables one to create a sphere with handles.

and the other an even number. One has to put five new flags, three on the new links and two on the new vertices. This can always be done so as to have an odd number of flags at the end inside the two new loops, without modifying the outside. A similar argument holds for the inverse move. One checks first that the number of flags to be removed inside (or outside) the new loop to be created is odd (it cannot be 0). If this is not the case, one flips one of them outside (inside), simultaneously flipping inside (outside) a neighboring link flag, the one not to be removed. The new flag assignment satisfies the constraint if the old one did. The total number of flags inside two adjacent elementary loops is even. By removing a link, one erases an odd number of flags. The final number of flags inside the new, larger loop is therefore odd, as it should be.

One can extend this result to a sphere with handles. One starts by creating the required topology and then one uses the split-join moves and the ergodicity argument, as above. First, duplicate a sphere and choose a pair of identical loops, one on each sphere. Join two such twin loops with a minimal number of new links as in Fig. 5(b). By symmetry, there is always an even number of old flags in the newly created loops. It is easy to convince oneself that new flags, three for each new link, can always be placed so as to have an odd number of flags in the new loops. A surface with an arbitrary number of handles can be obtained by repeating this operation. Of course, spheres have to be glued in more than one place.

Finally, let us observe that the satisfaction of the constraint (19) by all elementary loops implies that it is also satisfied by an arbitrary contractable and self-avoiding loop C. This results from the fact that such a loop can be obtained by fusing elementary loops with the same orientation. The total number of flags inside two loops that fuse is even and one drops an odd number of flags (see Fig. 6). Hence

$$F(C)=-1. \tag{20}$$

D. Dirac-Wilson operator, fermion loops and the Ising model

The Dirac operator is defined by contracting the connection matrix $V(ab)$ with the Dirac matrix

$$\gamma_{ab}=f^1(ab)\cdot\gamma=S^{-1}(\phi_{a\to b})\gamma^1 S(\phi_{a\to b}). \tag{21}$$

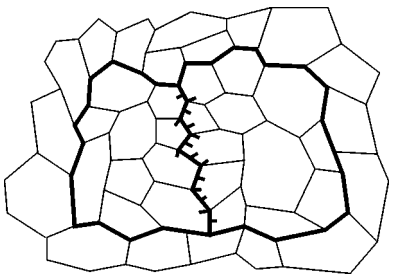

FIG. 6. Fusion of two loops. The number of flags on the common line is always odd.

Using Eq. (13) one finds that it has the following form:

$$D(ab)=\gamma_{ab}V(ab)=g_{ab}S^{-1}(\phi_{a\to b})\gamma^1 S(\pi)S(\phi_{b\to a}). \tag{22}$$

From here on we shall use the Dirac-Wilson operator, for which we keep the symbol D:

$$D(ab)=g_{ab}S^{-1}(\phi_{a\to b})PS(\pi)S(\phi_{b\to a}), \tag{23}$$

where $P=(1+\gamma^1)/2$ is a projector. We recall that g_{ab} has been set following the rules discussed in Sec. II B.

The Dirac-Wilson operator satisfies the following two conditions:

$$D(ab)D(ba)=0 \tag{24}$$

and

$$D^T(ab)=CD(ba)C^{-1}, \tag{25}$$

where C is the charge conjugation matrix.[2] The former means that the Wilson fermions do not propagate back and forth on the same link. The latter one results from Eq. (15). For Majorana fermions one obtains, from Eq. (25),

$$\overline{\psi}(a)D(ab)\psi(b)=\overline{\psi}(b)D(ba)\psi(a), \tag{26}$$

which means that fermion lines are not oriented.

Let us calculate the contribution to the partition function of a closed loop $C=\{1,2,\ldots,n,1\}$;

$$\begin{aligned}\langle\overline{\psi}(1)D(1,2)\psi(2)\overline{\psi}(2)D(2,3)\psi(3)\ \ldots\\ \times\overline{\psi}(n)D(n,1)\psi(1)\rangle\\ =-\mathrm{Tr}\,D(1,2)D(2,3)\ldots\ D(n,1)\\ =-F(C)\mathrm{Tr}\prod_{j=1}^{n}PK_j\\ =-F(C)(\sqrt{3}/2)^n,\end{aligned} \tag{27}$$

where $K_j=\exp(\pm i\gamma_5\pi/6)$ is the Kac-Ward factor at the jth turn of the loop. The presence of the projector P makes the

[2] $C^{-1}\gamma^j C=-\gamma^{jT}$, $C^\dagger C=1$ and $C^T=-C$. For Majorana fermions $\psi=C\overline{\psi}^T$.

final result independent of the $\pm$ sign in the exponent of K_j. We see that the right-hand side of Eq. (27) is positive when $F(C)=-1$. We have already proved that this is the case for contractable, self-avoiding loops. Now the loops are necessarily self-avoiding for Majorana fermions. From Eq. (27) easily follows the isomorphism, on a triangulated spherical surface, of the Majorana fermion model and of the Ising spin model in 2D. It rests on the identity of the pattern of phase boundaries, in the Ising model, and that of closed fermion loops. We refer the reader to Ref. [2] for more details.

In two dimensions and for Majorana fermions, Eq. (23) can be rewritten in another, particularly elegant and suggestive form. Choose the representation $\gamma^1=\sigma_3$ and $\gamma^2=\sigma_1$, where σ_i denotes the Pauli matrices. Then $\gamma_5=\sigma_2$ and $C=i\sigma_2=\epsilon$. Thus for Majorana fermions[3] $\psi=\epsilon\overline{\psi}^T$ and $\overline{\psi}=\psi^T\epsilon$. With these conventions it is easy to check that Eq. (23) can be written

$$D(ab)=g_{ab}s(\phi_{a\to b})\otimes\overline{s}(\phi_{b\to a}), \tag{28}$$

where

$$s(\phi)=\begin{pmatrix}\cos\frac{\phi}{2}\\ \sin\frac{\phi}{2}\end{pmatrix},\quad \overline{s}=s^T\epsilon=\begin{pmatrix}-\sin\frac{\phi}{2} & \cos\frac{\phi}{2}\end{pmatrix}. \tag{29}$$

With our choice of the gauge, the angles $\phi_{a(n)}, n=1,2,3$ are $\pi/3,\pi$ and $5\pi/3$ (the links emerging from a are ordered counterclockwise, starting with the flag). Hence, with each dual link n emerging from a one can associate a ''spinbein'' s^n:

$$s^1=\begin{pmatrix}\sqrt{3}/2\\ 1/2\end{pmatrix},\quad s^2=\begin{pmatrix}0\\ 1\end{pmatrix},\quad s^3=\begin{pmatrix}-\sqrt{3}/2\\ 1/2\end{pmatrix}. \tag{30}$$

One has

$$\begin{aligned}&\overline{s^1}s^1=\overline{s^2}s^2=\overline{s^3}s^3=0,\\ &\overline{s^1}s^2=\overline{s^1}s^3=\overline{s^2}s^3=\sqrt{3}/2.\end{aligned} \tag{31}$$

Equation (28) can be rewritten as

$$D(ab)=g_{ab}s^{n(a)}\otimes\overline{s}^{m(b)}, \tag{32}$$

where $n(a)$ and $m(b)$ refer to the link ab, but are labeled according to the gauge chosen at a and b, respectively. Finally, we write

$$\overline{\psi}(a)D(ab)\psi(b)=g_{ab}[\overline{\psi}(a)s^{n(a)}][\overline{s}^{m(b)}\psi(b)]. \tag{33}$$

It should be clear that the particular choice (30) is not important—only the invariant relations (31) matter. Notice that in the above formulation one only works with the dual lattice, decorated with flags, with spinbeins living on the links of the graph. One is no longer referring explicitly to the underlying continuous manifold.

E. A sketch of the computer implementation

The fermion action can be constructed explicitly for use in computer simulations. The crucial part of the construction is to choose the gauge and determine the dual link sign factors consistently. Hence, on each particular random lattice, one first puts the flags on links and vertices. In practice, this is most simply done recursively; one starts with an arbitrary dual loop, puts an odd number of flags in its interior, enlarges the domain with flags put on by considering a neighboring loop, and so on. Our general results insure that one has no problem arriving at the last loop.

With each dual link one associates one of the nine matrices defined by Eq. (23), or what amounts to the same, by Eq. (28). These matrices can be calculated beforehand and stored in the computer's memory. Which matrix is associated with a given link depends on the gauge choice at the ends of the link (i.e., the orientation of the vertex flags). For example, in the spinbein formalism, the flag at a given dual vertex determines the labeling of spinbeins at this vertex. Since each link connects two vertices, there are two spinbeins associated with each link and the right matrix is the one which is equal to their Cartesian product, see Eq. (32).

Let L denote the number of dual links. The Dirac-Wilson operator for the full lattice is an $L\times L$ matrix made up of 2×2 matrices. The off-diagonal ones have the form (32) and are found as explained above. The diagonal ones are unit 2×2 matrices multiplied by a mass coming from the mass term.

III. CONCLUSION

Since the conceptual difficulty in defining a spin structure on a random manifold is associated with the sign ambiguities, the extension of our discussion to an arbitrary triangulation is, in principle, straightforward. When the triangles making up the lattice are not equilateral, the exponents in the Kac-Ward factors are no longer equal to $\pm\pi/6$, but vary along the dual loop. On the other hand, the sign factor is a topological property of the loop and is independent of the triangulation details. The Dirac-Wilson operator can still be defined by Eq. (23) but its computer implementation becomes more tedious because of the additional freedom in the angles entering the spin connection. With the definition (23) the factor $(\sqrt{3}/2)^n$ in Eq. (27) is replaced by the product, along the loop, of the cosines of the Kac-Ward angles. This no longer corresponds to a simple Ising model with a constant coupling between neighbors. The isomorphism between the fermion theory and the simple Ising model is recovered, however, if one follows Ref. [2] and introduces, in the definition of the Dirac-Wilson operator in Eq. (23), an appropriate link-depending factor. In substance, this amounts to defining the Dirac operator by Eq. (32).

Another issue is the spectrum of the Dirac-Wilson operator. This very interesting problem, completely beyond the

[3]Using indices $\overline{\psi}\to\psi^\alpha$, $\psi\to\psi_\alpha$ the two equations can be rewritten in the standard form $\psi^\beta=\psi_\alpha\epsilon^{\alpha\beta}$ and $\psi_\alpha=\epsilon_{\alpha\beta}\psi^\beta$.

scope of this paper, deserves a separate study. As is known from the accumulated experience with the lattice fermions, spurious zero modes do occur. This pathology is also likely to be present for fermions interacting with the geometry. An extension of our discussion incorporating the recent progress in the lattice fermion theory would, of course, be welcome.

We have illustrated our method of putting Wilson fermions on a randomly triangulated manifold by carrying the program in detail in the 2D case. We do not see any major conceptual problem in extending this construction to higher dimensions, although we have not done this explicitly.

Let us briefly sketch how this could perhaps be done in 3D. The auxiliary frames f_{ab} and f_{ba} have their first axis perpendicular to the common face (triangle) of the tetrahedra a and b. One can assume, without the loss of generality, that the second axis is common to these two frames, which are then related by a rotation by π around the second axis. This does not specify these frames yet, because the orientation of the second axis can be arbitrary. The natural choice is to assume that the second axis points from the center towards one of the vertices of the triangle. There remains the freedom to make rotations by $2\pi/3$ within the triangle. A convenient choice of the gauge consists in associating the local frame e with one of the faces, in much the same way as for the auxiliary frames, in analogy to what was done in 2D.

Since the rotation group is now non-Abelian, manipulating rotations is less straightforward than in 2D. However, within a tetrahedron the rotations relating the various frames mentioned in the preceding paragraph satisfy a simple algebra.

The construction of the spin connection follows the rules formulated in the preceding section. The real problem, as in 2D, is the consistent determination of the sign factors. The sign associated with the particular way an elementary dual loop goes through a tetrahedron is related to the fact that in the spinor representation the rotation connecting two faces directly can differ by a sign from the compound rotation, where one goes from the face to the local frame and then from the local frame to the other face. The bookkeeping is more complicated than in the 2D case.

The bookkeeping starts being really complicated as one attempts to fix the signs all over the lattice. The right strategy is the one used in 2D, but the explicit implementation seems much more tedious. But these seem to be merely technical problems. We believe that the main conceptual issues are well illustrated by the present paper.

ACKNOWLEDGMENTS

We are indebted to John Madore and Bengt Petersson for interesting conversations. One of us (Z.B.) would like to thank the Laboratoire de Physique Théorique for hospitality. This work was partially supported by the KBN grant number 2P03B 044 12. Laboratoire de Physique Théorique is Unité Mixte de Recherche UMR 8627.

[1] F. David, in *Gravitation and Quantization*, edited by J. Zinn-Justin and B. Julia, Les Houches Summer School Proceedings Vol. LVII (North-Holland, Amsterdam, 1995); J. Ambjørn, B. Durhuus, and T. Jonsson, *Quantum Geometry* (Cambridge University Press, Cambridge, England, 1997).

[2] M.A. Bershadsky and A.A. Migdal, Phys. Lett. B **174**, 393 (1986).

[3] H. Ren, Nucl. Phys. **B301**, 661 (1988).

[4] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972); M.B. Green, J.H. Schwarz, and E. Witten, *Superstring Theory* (Cambridge University Press, Cambridge, England, 1987).

[5] J. Jurkiewicz, A. Krzywicki, and B. Petersson, Phys. Lett. B **177**, 89 (1986).