Universality and the magnetic catalysis of chiral symmetry breaking

G. W. Semenoff*
The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

I. A. Shovkovy[†] and L. C. R. Wijewardhana
Physics Department, University of Cincinnati, Cincinnati, Ohio 45221-0011
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The hypothesis that the magnetic catalysis of chiral symmetry breaking is due to interactions of massless fermions in their lowest Landau level is examined in the context of chirally symmetric models with short ranged interactions. It is argued that, when the magnetic field is sufficiently large, even an infinitesimal attractive interaction in the appropriate channel will break chiral symmetry. [S0556-2821(99)04220-4]

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I. INTRODUCTION

Field theoretical models in external electromagnetic fields are of great interest and have attracted a lot of attention [1–4]. They are relevant to the study of many physical systems whose properties depend on the effects produced by external fields, especially when such effects are non-perturbative in nature. A particular example of such a situation is the so-called magnetic catalysis of chiral symmetry breaking [5–10] (see also [11–13] for some earlier studies) which has many potential applications in condensed matter physics [14–16] and in studies of the early Universe [17].

It is well known that a sufficiently strong attractive interaction between massless fermions can result in a chiral symmetry breaking condensate and a dynamically generated fermion mass. In the Nambu–Jona-Lasinio model, for example, this occurs when the interaction strength is at a particular large critical value. It is also now known that the presence of an external magnetic field can be a strong catalyst for this effect, leading in some cases to the generation of a dynamical mass for fermions even by very weak attractive interactions. For example, in an external magnetic field, the critical value of the Nambu–Jona-Lasinio coupling is reduced to zero.

It was suggested in [5] that this magnetic catalysis of chiral symmetry breaking is a rather universal phenomenon and that its main features are model independent. The arguments were based on the observation that the effect is primarily due to the dynamics of the fermions from the lowest Landau level which are gapless and should therefore dominate the behavior of the system at long wavelengths.

In this paper we will develop this idea further. We shall investigate the low-energy dynamics of chirally symmetric models with a short range interaction in an external magnetic field. The idea is to construct an effective action for the low energy degrees of freedom. In strong external magnetic fields, these are the modes of charged particles in the lowest Landau level. In the absence of interactions, their effective dynamics is described by a field theory whose dimensionality is two less than the spacetime dimension. We shall take the hypothesis that, at least for strong magnetic fields and short ranged interactions, the dynamics of chiral symmetry breaking can be understood in the lower dimensional effective theory.

As an illustration, we shall argue that this assumption is very natural in 2+1 dimensions. There, the modes of the lowest Landau level are non-propagating, and in the absence of interactions, the ground state is highly degenerate and contains both chirally symmetric and chirally non-symmetric states. Interactions, even with infinitesimally small coupling constants, will split this degeneracy, and whether chiral symmetry breaking occurs or not depends on whether the interaction favors the chirally non-symmetric or chirally symmetric ground states. One by-product of our analysis will be the fact that the dynamical generation of parity violating masses seems to be suppressed by magnetic fields.

In the more complicated case of (3+1)-dimensional gauge theory, we shall argue that the infrared dynamics is governed by an effective (1+1)-dimensional Gross-Neveu like theory with an infinite number of flavors of fermions. We derive the most general low energy effective action which is compatible with the symmetries of the four dimensional theory. We then study the resulting model using the renormalization group. We will present a plausible argument for our main conclusion: that the presence of chiral symmetry breaking depends crucially on the sign (but not the magnitude) of one particular moment of the coupling constants which we denote by g_v. If it is of the appropriate sign, its infrared renormalization group flow is to strong coupling and chiral symmetry is broken. With the other sign it flows to zero coupling and chiral symmetry remains unbroken.

II. (2+1)-DIMENSIONAL PARADIGM

Consider a system of relativistic (2+1)-dimensional fermions with the Dirac Hamiltonian

$$H_0=\begin{pmatrix} i\vec{\sigma}\cdot\vec{D} & 0 \\ 0 & -i\vec{\sigma}\cdot\vec{D} \end{pmatrix}, \tag{1}$$

where $\vec{D}=\vec{\nabla}+ie\vec{A}(\vec{x})$ is the covariant derivative and $\vec{\sigma}=(\sigma^1,\sigma^2)$ are the first two Pauli matrices. This Hamiltonian

*On leave from Department of Physics and Astronomy, University of British Columbia, 6224 Agricultural Road, Vancouver, British Columbia V6T 1Z1, Canada.

[†]On leave of absence from Bogolyubov Institute for Theoretical Physics, Kiev 252143, Ukraine.

is obtained in the continuum limit of some tight-binding lattice models relevant to condensed matter physics — on a square lattice with half of a quantum of magnetic flux through each plaquette (see [18] and references therein) or on a honeycomb lattice without a magnetic field [19]. It has also been discussed in the context of the continuum limit of the d-wave state of high T_C superconductors [14,15].

A. Symmetries

H_0 commutes with any linear combination of the matrices

$$\mathcal{I}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad T^1=\begin{pmatrix}0 & \sigma^3\\ \sigma^3 & 0\end{pmatrix},$$

$$T^2=\begin{pmatrix}0 & -i\sigma^3\\ i\sigma^3 & 0\end{pmatrix},\quad T^3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\tag{2}$$

which generate the Lie algebra of a $U(2)$ flavor symmetry which we will call chiral symmetry. Possible mass terms which can be added to H_0 must be matrices which anti-commute with it. The basic matrix which anti-commutes with H_0 and which commutes with all of the flavor generators is

$$\beta=\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}\tag{3}$$

and a mass term which one could add to H_0 which would preserve the flavor symmetry is βm.

Generally, in 2+1 dimensions, fermion mass terms violate parity symmetry. A parity transformation which is a symmetry of the massless Dirac equation

$$H_0\psi_E(x)=E\psi_E(x)\tag{4}$$

and which commutes with the flavor symmetry generators T^a is

$$\psi_E(x^1,x^2)\rightarrow\begin{pmatrix}\sigma^2 & 0\\ 0 & -\sigma^2\end{pmatrix}\psi_E(-x^1,x^2).\tag{5}$$

The mass term βm is odd under this transformation and it would therefore violate parity symmetry. More generally, all mass terms of the form $\beta(m_0\mathcal{I}+\vec{m}\cdot\vec{T})$ are odd under the parity defined in Eq. (5). Moreover, there is no modification of the parity transformation which would make the mass βm parity symmetric.

It is possible to get a parity invariant mass by breaking the flavor symmetry. For this, we must define parity as a combination of the transformation (5) and a discrete flavor transformation, for example,

$$\psi_E(x^1,x^2)\rightarrow\begin{pmatrix}\sigma^2 & 0\\ 0 & -\sigma^2\end{pmatrix}T^1\psi_E(-x^1,x^2).\tag{6}$$

This is a symmetry of H_0 and it also commutes with the mass term $\beta(m_2T^2+m_3T^3)$. This mass term breaks the $U(2)$ flavor symmetry to $U(1)\times U(1)$.

B. Zero modes

This existence of the unitary matrix β which anti-commutes with H_0 implies that H_0 has a symmetric spectrum: if $H\psi_E=E\psi_E$, then $H\beta\psi_E=-E\beta\psi_E$ and, with a particular choice of phase, $\psi_{-E}=\beta\psi_E$. Thus, for each positive energy state, there is a negative energy state and the probability measure $\psi_E^\dagger(x)\psi_E(x)$ is a symmetric function of E.

The interesting feature of H_0 is the appearance of zero energy states in a background magnetic field. Consider the equation

$$\vec{\sigma}\cdot\vec{D}u_0=0,\tag{7}$$

where the background electromagnetic field is static and has the property $\vec{\nabla}\times\vec{A}=B(x)$. It has the solution

$$u_0(x)=\exp[ie\phi(x)-e\sigma^3\chi(x)]v(x),\tag{8}$$

where

$$-\vec{\nabla}^2\phi(x)=\vec{\nabla}\cdot\vec{A}(x),\quad -\vec{\nabla}^2\chi(x)=\vec{\nabla}\times\vec{A}(x)=B(x)\tag{9}$$

and

$$\vec{\sigma}\cdot\vec{\nabla}v(x)=0.\tag{10}$$

The second equation in Eqs. (9) has the solution

$$\chi(x)=-\frac{1}{2\pi}\int d^2y\,\ln|x-y|B(y),\tag{11}$$

with asymptotic behavior

$$\lim_{|x|\rightarrow\infty}\chi(x)=-\frac{\chi}{e}\ln|x|,\quad \chi=\frac{e}{2\pi}\int d^2yB(y).\tag{12}$$

(This is strictly correct if the magnetic field goes to zero sufficiently quickly at infinity for the total flux χ to be finite. If it is not finite, there is an infinite number of zero modes and their density is given by χ/V where V is the volume.) Asymptotically,

$$\lim_{|\vec{x}|\rightarrow\infty}u_0(x)=|\vec{x}|^{\chi\sigma^3}v(x).\tag{13}$$

In order to have a normalizable zero mode, the spinor $v(x)$ should be an eigenvector of σ^3 with eigenvalue $-\mathrm{sgn}(\chi)$ (-1 if $\chi>0$ and $+1$ if $\chi<0$). It then must be a normalizable solution of the equations

$$\sigma^3v(x)=-\mathrm{sgn}(\chi)v(x),\quad \left(\frac{\partial}{\partial x^1}+i\,\mathrm{sgn}(\chi)\frac{\partial}{\partial x^2}\right)v(x)=0.\tag{14}$$

Solutions of this equation are $v(x)\sim[x^1-i\,\mathrm{sgn}(\chi)x^2]^k$ for $k=0,1,\ldots,[|\chi|]-1$ where $[|\chi|]$ is the largest integer less than $|\chi|$ and the maximum value of k is the largest power allowed by normalizability of the wave function. Thus there are $[|\chi|]$ normalizable zero modes. This fact depends only on the total magnetic flux and is independent of the profile of the magnetic field. The result, which we have found by explicit computation, is also a result of the Atiyah-Singer index theorem generalized to open spaces [20]. Some other applications of this idea are discussed in [21].

There are thus $2[|\chi|]$ zero mode wave functions of H_0. All of them obey the equation

$$\beta\psi_0=-\mathrm{sgn}(\chi)\psi_0, \tag{15}$$

$[|\chi|]$ of them obey the equation

$$\beta T^3\psi_0=\mathrm{sgn}(\chi)\psi_0, \tag{16}$$

and $[|\chi|]$ of them obey

$$\beta T^3\psi_0=-\mathrm{sgn}(\chi)\psi_0. \tag{17}$$

Of course, any linear combination of them is also a zero mode and could be chosen to be eigenvalues of any of the flavor symmetry breaking mass operators $\beta\vec{m}\cdot\vec{T}$. All linear combinations would satisfy Eq. (15).

C. Interpretation

The existence of zero modes implies that the Dirac ground state of the second quantized system of fermions is degenerate. In the ground state, all negative energy states of the fermions are full and all positive energy states are empty. For overall charge neutrality, half of the fermion zero modes must be occupied — but the occupation of zero modes is otherwise unconstrained.[1] This means that the degeneracy of the ground state is $N_0!/[(N_0/2)!]^2$ where $N_0=[|\chi|]$ is the number of zero mode wave functions.

In second quantization, the expectation value of any fermion bilinear operator, using the Dirac commutator for normal ordering, is [20]

$$\left\langle \int d^2x\!:\!\Psi^\dagger(x)M\Psi(x)\!:\right\rangle =-\frac{1}{2}\left(\sum_{\text{occupied}}\int d^2x\,\psi_E^\dagger(x)M\psi_E(x) -\sum_{\text{unoccupied}}\int d^2x\,\psi_E^\dagger(x)M\psi_E(x)\right). \tag{18}$$

Since, in the Dirac ground state, all positive energy states are unoccupied and negative energy states are occupied, and since β maps positive energy states onto negative energy states, if the matrix M commutes with β, only the zero modes survive in this summation.

First of all, this means that in a magnetic field the expectation value of the parity violating mass operator vanishes,

$$\left\langle \int d^2x\!:\!\bar{\Psi}\Psi\!:\right\rangle=\left\langle \int d^2x\!:\!\Psi^\dagger\beta\Psi\!:\right\rangle=0, \tag{19}$$

regardless of the occupation of zero modes. This has the implication that external magnetic fields do not enhance the generation of parity violating mass terms. In fact, it indicates that if the external field is strong enough that the zero modes are well isolated, the expectation value of a parity breaking mass operator, $\langle\int\!:\!\Psi^\dagger\beta\Psi\!:\rangle$ should always vanish.

On the other hand, the parity invariant, chiral symmetry breaking mass (with $M=\beta T^3$) has the expectation value

$$\left\langle \int d^2x\!:\!\bar{\Psi}(x)T^3\Psi(x)\!:\right\rangle =-\frac{1}{2}\mathrm{sgn}(\chi)\left(\sum_{\text{occupied}}\int d^2x\,\psi_0^\dagger(x)T^3\psi_0(x) -\sum_{\text{unoccupied}}\int d^2x\,\psi_0^\dagger(x)T^3\psi_0(x)\right). \tag{20}$$

It is possible to choose ground states where this expectation value has any value between two bounds:

$$-[|\chi|]\le\left\langle \int d^2x\!:\!\bar{\Psi}(x)T^3\Psi(x)\!:\right\rangle\le[|\chi|]. \tag{21}$$

If we added an infinitesimal mass term $m\beta T^3$ to the Hamiltonian H_0, the system would choose the ground state with minimal βT^3, that is the one where[2]

$$\left\langle \int d^2x\!:\!\bar{\Psi}(x)T^3\Psi(x)\!:\right\rangle=-[|\chi|]\mathrm{sgn}(m\chi). \tag{22}$$

The observation that this is so for infinitesimal mass was made in [5]. However, this does not mean that chiral symmetry is necessarily broken. It only implies that the ground state is degenerate and there are chirally non-symmetric ground states which are degenerate with chirally symmetric ones. If there are additional weak interactions, of course, the correct ground state should be found by resolving the degen-

[1] Of course, occupying half of the zero modes is only possible when their number is even, which we shall assume. If it is odd, there are no neutral ground states.

[2] This is just the contribution of zero modes. If the total magnetic flux χ is finite, the Dirac Hamiltonian has a continuum spectrum whose support begins at zero energy. Furthermore, there is a threshold density of continuum states whose contribution to bilinear densities such as Eq. (22) makes up the difference between $[|\chi|]$ and $|\chi|$ which is proportional to the spectral asymmetry of the Hamiltonian and is the correct result. For details, see [20]. In the present paper, for simplicity we ignore the relatively small error which we make in neglecting the asymmetry of the continuum spectrum. When the magnetic flux is infinite, for example in the case of constant external field, $[|\chi|]$ should be replaced by $|eB|V/2\pi$ where V is the spatial volume.

eracy using degenerate perturbation theory. In this case, all that is required to break chiral symmetry is an interaction which favors a chirally nonsymmetric population of the zero modes.

If the interaction favors the breaking of chiral symmetry, symmetry breaking occurs for even an infinitesimal value of its coupling constant. This is the reason why the critical value of the interaction can be at zero coupling in the presence of a magnetic field. In fact, in the absence of external magnetic fields a critical coupling necessary to break chiral symmetry is typically large. In the presence of the magnetic field it is reduced to zero.

For example, it is well known that an interaction of the Nambu–Jona-Lasinio model,

$$L_{\text{int}}=\frac{\lambda}{2}(\bar{\Psi}T^3\Psi)^2, \tag{23}$$

when added to the Hamiltonian will break the chiral symmetry if the coupling constant is attractive and is greater than a particular critical value. This is seen by analyzing solutions of the Schwinger-Dyson equation for a mass condensate. It was shown in [5] that, in the presence of a magnetic field, the critical coupling moves from some finite value to zero — even an infinitesimal coupling will break the chiral symmetry. That observation is consistent with our finding here.

An interesting test of this idea would be to examine the effect on system with the interaction

$$L_{\text{int}}=\frac{\kappa}{2}(\bar{\Psi}\Psi)^2, \tag{24}$$

which can break parity by generating a parity violating fermion mass if κ has the appropriate sign and is of large enough magnitude [18]. Our analysis seems to suggest that a large magnetic field should in fact tend to increase the magnitude of the critical coupling.

III. STRATEGY IN $D=3+1$

In $3+1$ dimensions, the situation is more complicated. The zero modes of the Dirac Hamiltonian in a background magnetic field still have some dynamics which is non-trivial and there are more possibilities for interactions.

To understand the general strategy that we will take, begin with the Lagrangian which describes Dirac fermions interacting with an external electromagnetic field,

$$\mathcal{S}_0=\int d^4x\, i\bar{\Psi}\gamma^\mu D_\mu\Psi, \tag{25}$$

where $D_\mu=\partial_\mu+ieA_\mu(x)$. For concreteness, we choose the chiral representation of the Dirac matrices,

$$\gamma^0=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \gamma^i=i\begin{pmatrix}0 & \sigma^i\\ \sigma^i & 0\end{pmatrix}, \tag{26}$$

and we shall use the notation

$$\Psi=\begin{pmatrix}\Psi_L\\ \Psi_R\end{pmatrix}. \tag{27}$$

The Lagrangian has the form

$$\mathcal{S}_0=\int d^4x[i\Psi_L^\dagger(D_0+\vec{\sigma}\cdot\vec{D})\Psi_L+i\Psi_R^\dagger(D_0-\vec{\sigma}\cdot\vec{D})\Psi_R]. \tag{28}$$

This action has a global $U(1)_L\times U(1)_R$ chiral symmetry. In the following, we must keep in mind that when there are both background electric and magnetic fields, or if the gauge field is dynamical, this symmetry is reduced to the vector $U(1)$ by the axial anomaly. For simplicity, in this paper we will assume that this is not the case. It would be straightforward to extend the present work to models with dynamical $U(1)$ fields, by introducing more species of fermions so that there exist chiral symmetries which are unaffected by the axial anomaly. For now, we will consider the case of a constant non-dynamical background magnetic field,

$$A_0=0,\quad A_i=-\frac{B}{2}\epsilon_{ij3}x^j. \tag{29}$$

In this background field, it is convenient to make use of the mixed spacetime-momentum representation as follows:

$$\mathcal{S}_0=\int\frac{d\omega dk}{(2\pi)^2}d^2z[\Psi_L^\dagger(\omega-k\sigma^3+\mathcal{D})\Psi_L+i\Psi_R^\dagger(\omega+k\sigma^3-\mathcal{D})\Psi_R], \tag{30}$$

where, by definition, $z=x^1+ix^2, d^2z=dx^1dx^2$ and

$$\mathcal{D}=i\begin{pmatrix}0 & 2\frac{\partial}{\partial z}+\frac{eB}{2}\bar{z}\\ 2\frac{\partial}{\partial\bar{z}}-\frac{eB}{2}z & 0\end{pmatrix}. \tag{31}$$

The spectrum of $(k\sigma^3-\mathcal{D})$ is well known. The equation

$$(k\sigma^3-\mathcal{D})\phi_\lambda(\bar{z},z)=\lambda\phi_\lambda(\bar{z},z) \tag{32}$$

has eigenvalues

$$\lambda=\pm\sqrt{k^2+2n|eB|}, \tag{33}$$

for all integers $n=0,1,2,\ldots$. When $n\geqslant 1$, these are dispersion relations of $(1+1)$-dimensional Dirac fermions with masses given by $\sqrt{2n|eB|}$. [Of course, this mass gap is not Lorentz invariant from a four-dimensional point of view. However, it is Lorentz invariant in $1+1$ dimensions. The reason for this is that there is a subgroup of the Lorentz group which survives in the background magnetic field — it is invariant under boosts in the direction of the field lines.

Here this will mean that the effective theory for Landau levels must be invariant under the (1+1)-dimensional Lorentz transformations.]

Zero modes of $\mathcal{D}$ are solutions of the equation

$$\mathcal{D}\phi_m(z,\bar{z})=0, \tag{34}$$

and are given by the infinite set of ortho-normal functions

$$\phi_m(z,\bar{z})=\frac{\bar{z}^m}{\sqrt{\pi\Gamma(m+1)}}\left(\frac{eB}{2}\right)^{(m+1)/2}\exp\left(-\frac{eB}{4}z\bar{z}\right)\begin{pmatrix}0\\1\end{pmatrix}, \tag{35}$$

with $m=0,1,2,\ldots$. Here, without loss of generality, we assumed that $eB>0$. In the (1+1)-dimensional theory, the zero modes of $\mathcal{D}$ correspond to massless fermions. The Lagrangian is (restoring the space dependence)

$$\mathcal{S}_0=\int dx^0dx^3\sum_{m=0}^{\infty}\left(i\psi_m^{(L)*}\partial_+\psi_m^{(L)}+i\psi_m^{(R)*}\partial_-\phi_m^{(R)}\right)+(\text{massive modes}), \tag{36}$$

where $\partial_\pm\equiv\partial_0\mp\partial_3$ and the new fields are defined as the coefficient functions in the expansions of Ψ_L and Ψ_R over the complete set of eigenstates:

$$\Psi_L=\sum_{m=0}^{\infty}\psi_m^{(L)}(x^0,x^3)\phi_m(z,\bar{z})+\sum(\text{massive modes}), \tag{37}$$

$$\Psi_R=\sum_{m=0}^{\infty}\psi_m^{(R)}(x^0,x^3)\phi_m(z,\bar{z})+\sum(\text{massive modes}). \tag{38}$$

Note that the kinetic term in the action (36) for the massless modes appears to have a $U(N)_R\times U(N)_L$ with $N\to\infty$ symmetry. This is just the unitary symmetry which mixes the different wave functions of the degenerate zero modes, and which is actually there for every Landau level. This effective symmetry is not preserved by interactions.

We will analyze the possibility that interactions that are added to this field theory drive a spontaneous breaking of the $U(1)_R\times U(1)_L$ chiral symmetry by generating a mass gap for the fermions in Eq. (36). We expect that this spontaneous symmetry breaking takes place at very long wave lengths. Our approach to the problem in the following sections can be summarized as follows:

(i) We consider the theory of four-dimensional fermions in a magnetic field as described above, with some interactions which should be local and respect (3+1)-dimensional Poincaré and chiral symmetry but are otherwise unspecified.

(ii) We then consider the effective field theory which is obtained by integrating out all of the massive modes of the fermions in Eq. (36) and momentum states of the massless modes above an ultraviolet cutoff.

(iii) We assume that the resulting effective Lagrangian is local. To guarantee locality, we generally have to assume that the interactions between fermions in four dimensions are short ranged.[3] Furthermore, the ultraviolet cutoff for the effective theory should be less than the mass gap of the lightest massive mode, $|eB|$.

(iv) We consider all relevant operators which could be added to the effective Lagrangian which are consistent with the symmetries of the theory. Since they should be relevant in the sense of two dimensional field theory, these are all possible four-fermion operators which are consistent with symmetry. There are an infinite number of such operators.

(v) We compute the beta function for the coupling constants of the relevant operators and look for infrared stable fixed points. These fixed points should govern the behavior of the very long-wavelength degrees of freedom of the theory.

(vi) If the coupling constants flow to an infrared stable fixed point, then an infrared limit of the massless theory exists and there is no symmetry breaking. If, on the other hand, the coupling constant flow is not in the domain of attraction of any infrared fixed point, so that it flows to strong coupling in the infrared, we postulate that this implies the dynamical generation of a mass gap — and spontaneously broken chiral symmetry. We shall find examples of both kinds of behavior.

An example of a local four dimensional interaction which preserves the $U(1)_R\times U(1)_L$ chiral symmetry is the Nambu–Jona-Lasinio interaction [22],

$$\mathcal{S}_{int}=\frac{G}{2}\int d^4x[(\bar{\Psi}\Psi)^2+(\bar{\Psi}i\gamma_5\Psi)^2]. \tag{39}$$

A renormalizable version of this interaction would be one which is mediated by a massive scalar mesons.

IV. GENERAL STRUCTURE OF THE LOW-ENERGY THEORY

The constraints of (1+1)-dimensional Lorentz invariance and $U(1)_R\times U(1)_L$ chiral symmetry allow four-fermion coupling constants as in the effective theory,

$$\mathcal{L}_{eff}=\sum_{n=0}^{\infty}\left(\psi_n^{(L)*}i\partial_+\psi_n^{(L)}+\psi_n^{(R)*}i\partial_-\psi_n^{(R)}\right)+\sum_{n_1,n_2,m_1,m_2=0}^{\infty}g_0\begin{pmatrix}n_1 & n_2\\ m_1 & m_2\end{pmatrix}\psi_{n_1}^{(L)*}\psi_{n_2}^{(R)}\psi_{m_1}^{(R)*}\psi_{m_2}^{(L)}. \tag{40}$$

The coupling constants

$$g_0\begin{pmatrix}n_1 & n_2\\ m_1 & m_2\end{pmatrix}$$

[3]If there are long-ranged interactions which are mediated by massless fields, the correct procedure would be to retain the long wavelength modes of the massless fields in the effective Lagrangian.

obey further constraints from charge conjugation, parity, and time reversal (CPT) symmetry and the symmetry of the underlying theory under translations and rotations about the axis of the magnetic field which are summarized in Appendix A. A general solution of those constraints would yield the most general allowed structure of the low-energy effective action in Eq. (40).

Some particular solutions of those constraints are of interest. For example, the constraints allow the maximally symmetric solution with $U(N)_R \times U(N)_L (N \to \infty)$ symmetry,

$$g\begin{pmatrix} n_1 & n_2 \\ m_1 & m_2 \end{pmatrix} = g_{LR}\delta_{n_1,m_2}\delta_{n_2,m_1}, \tag{41}$$

where g_{LR} is real. Similarly, there is a solution with $U(N)_V$ (with $N \to \infty$) symmetry,

$$g\begin{pmatrix} n_1 & n_2 \\ m_1 & m_2 \end{pmatrix} = g_V \delta_{n_1,n_2}\delta_{m_1,m_2}, \tag{42}$$

where g_V is real again.

In addition a large class of solutions could be found by the reduction of the interactions in the original $(3+1)$-dimensional model to the lowest Landau level. After such a reduction of the Nambu–Jona-Lasinio interaction (39), we arrive at

$$g\begin{pmatrix} n_1 & n_2 \\ m_1 & m_2 \end{pmatrix} = \frac{G}{2^{(n_1+m_1+n_2+m_2)/2}} \times \sqrt{\frac{\Gamma(n_1+m_1+1)\Gamma(n_2+m_2+1)}{\Gamma(n_1+1)\Gamma(m_1+1)\Gamma(n_2+1)\Gamma(m_2+1)}}. \tag{43}$$

Since it descends from a Lorentz and chirally invariant interaction in $3+1$ dimensions, it must necessarily satisfy the constraints of symmetry [see Eqs. (A6), (A7), (A10), (A11) and (A12)]. Unlike the first two solutions in Eqs. (41) and (42), this last one in Eq. (43) does not seem to have any extra symmetry in addition to the required $U(1)_R \times U(1)_L$ flavor symmetry.

Now, we envisage having obtained the effective action (40) by integrating out all modes in the higher Landau levels, as well as all momentum modes of the fermions in the lowest Landau level which are above a certain cutoff. Of the many interactions that this procedure would produce, we have kept only the local four-fermion operators. This procedure is legitimate only if the ultraviolet cutoff of this model is lower than the lowest mass gap of the fields which have been eliminated, i.e. $|eB|$. By chiral symmetry and Lorentz invariance, the effective action cannot contain mass terms for the fermions. Furthermore, the only Lorentz invariant four-fermion operator is of the form given in Eq. (40).

The renormalization group procedure examines how the coupling constants in Eq. (40) change as we further lower the cutoff to isolate the very long wavelength excitations. This information is encoded in the beta function.

The β function for the general coupling constant in Eq. (40) is computed in $2+\epsilon$ dimensions to two loop order in Appendix B. The result is

$$\begin{aligned}
\beta(g)\begin{pmatrix} N_1 & N_2 \\ M_1 & M_2 \end{pmatrix} =& 2\varepsilon g\begin{pmatrix} N_1 & N_2 \\ M_1 & M_2 \end{pmatrix} - \frac{1}{2\pi}\sum_{k_1}\left[g\begin{pmatrix} N_1 & N_2 \\ k_1 & N_1-N_2+k_1 \end{pmatrix} g\begin{pmatrix} N_1-N_2+k_1 & k_1 \\ M_1 & M_2 \end{pmatrix} \right. \\
&\left. - g\begin{pmatrix} N_1 & k_1 \\ M_1 & N_1+M_1-k_1 \end{pmatrix} g\begin{pmatrix} N_1+M_1-k_1 & N_2 \\ k_1 & M_2 \end{pmatrix}\right] \\
&- \frac{1}{8\pi^2}\sum_{k_1,k_2}\left[g\begin{pmatrix} N_1 & k_2 \\ k_1 & N_1+k_1-k_2 \end{pmatrix} g\begin{pmatrix} N_1+k_1-k_2 & N_2 \\ M_1 & M_2+k_1-k_2 \end{pmatrix} g\begin{pmatrix} M_2+k_1-k_2 & k_1 \\ k_2 & M_2 \end{pmatrix}\right. \\
&\left. + g\begin{pmatrix} k_1 & k_2 \\ M_1 & M_1+k_1-k_2 \end{pmatrix} g\begin{pmatrix} N_1 & N_2-M_1+k_2 \\ k_2 & M_2 \end{pmatrix} g\begin{pmatrix} M_1+k_1-k_2 & N_2 \\ N_2-M_1+k_2 & k_1 \end{pmatrix}\right] \\
&+ \frac{2}{(4\pi)^2} g\begin{pmatrix} N_1 & N_2 \\ M_1 & M_2 \end{pmatrix} \sum_{k_1,k_2}\left[g\begin{pmatrix} N_1 & k_2+N_1 \\ k_1 & k_1-k_2 \end{pmatrix} g\begin{pmatrix} k_1-k_2 & k_1 \\ k_2+N_1 & N_1 \end{pmatrix} + g\begin{pmatrix} k_1 & k_2+N_2 \\ N_2 & k_1-k_2 \end{pmatrix} g\begin{pmatrix} k_1-k_2 & N_2 \\ k_2+N_2 & k_1 \end{pmatrix}\right. \\
&\left. + g\begin{pmatrix} k_1 & k_2+M_1 \\ M_1 & k_1-k_2 \end{pmatrix} g\begin{pmatrix} k_1-k_2 & M_1 \\ k_2+M_1 & k_1 \end{pmatrix} + g\begin{pmatrix} M_2 & k_2+M_2 \\ k_1 & k_1-k_2 \end{pmatrix} g\begin{pmatrix} k_1-k_2 & k_1 \\ k_2+M_2 & M_2 \end{pmatrix}\right].
\end{aligned} \tag{44}$$

Finally, before we proceed to the next section, we note that we may have to occasionally cut off the summation over modes in the first Landau level. We do this by summing modes up to some maximum number, $n=0,1,\ldots,N$. Such a situation could be produced by considering the fermions in an external magnetic field with finite magnetic flux, or with uniform field and finite transverse area. In that case, N which can be thought of as the number of ''flavors'' which goes to infinity as $|eB|S_{12}/2\pi$ where $S_{12}\to\infty$ is the area of the two-dimensional perpendicular (with respect to the direction of the magnetic field) subspace.

V. SYMMETRIC FIXED POINTS

In Appendix B, we have computed the two-loop β function (44) for the most general coupling constants, subject to the restriction in Eq. (A12). The next natural step would be locate infrared stable fixed points of this beta function. This problem is far more complicated than we can solve at present. Instead, in our analysis below, we shall restrict our attention to the beta function for coupling constants with close to maximal symmetry.

For some scalar field theories, where there are many components of a scalar field and renormalizable interactions which couple them, it is known that the only infrared fixed points of the renormalization group flow of relevant couplings are those with maximal symmetry [23,24]. To our knowledge, there is no similar theorem for Gross-Neveu like models. On the other hand, we consider it plausible that similar arguments can be applied: in particular, if no maximally symmetric infrared stable fixed point exists, then there are no infrared stable fixed points at all.

Maximally symmetric couplings can easily be shown to be contained in other combinations of coupling constants. In Appendix C we show that the Nambu–Jona-Lasinio (NJL) coupling contains the maximally symmetric ones.

Consider the renormalization group flow in the special case when $U(N)_R\times U(N)_L$ and $U(N)_V$ couplings, as defined in Eqs. (41) and (42), are the only non-zero couplings in the effective action. Then, from our general result in the previous section, we extract the following expressions for the β functions of interest:

$$\beta_{LR}=2\varepsilon g_{LR}+\frac{1}{2\pi}g_V^2+\frac{1}{4\pi^2}(Ng_{LR}^3+2g_Vg_{LR}^2+Ng_V^2g_{LR}-g_V^3), \tag{45}$$

$$\beta_V=2\varepsilon g_V-\frac{N}{2\pi}g_V^2+\frac{1}{4\pi^2}[2Ng_V^3-(N-3)\times g_V^2g_{LR}-(N-2)g_Vg_{LR}^2]. \tag{46}$$

Note that we have been forced to introduce a cutoff on N. In order to get a sensible result when taking the limit $N\to\infty$, it is convenient to rescale the couplings as follows: $g_{LR}\to g_{lr}/\sqrt{N}$ and $g_V\to g_v/N$. As is clear from the definition, the β functions will also get rescaled accordingly, $\beta_{LR}\to\beta_{lr}/\sqrt{N}$ and $\beta_V\to\beta_v/N$. After taking this into account and performing the limit $N\to\infty$, we arrive at

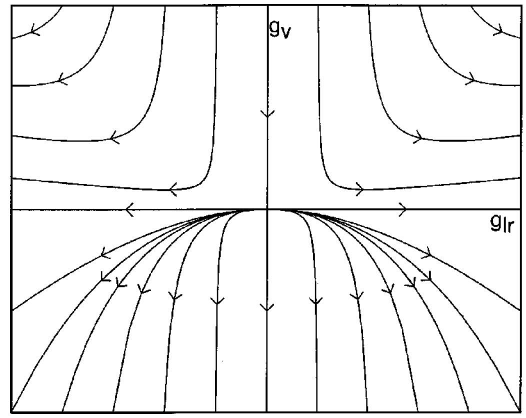

FIG. 1. The renormalization group flow in the (g_{lr},g_v) plane.

$$\beta_{lr}=2\varepsilon g_{lr}+\frac{1}{4\pi^2}g_{lr}^3, \tag{47}$$

$$\beta_v=2\varepsilon g_v-\frac{1}{2\pi}g_v^2+\frac{1}{4\pi^2}g_vg_{lr}^2. \tag{48}$$

It must be emphasized that we have adopted the rescaling of the most symmetric g_{LR} coupling by $1/\sqrt{N}$ rather than $1/N$. Since the rescaling performed above led to the well-defined β functions, we conclude that the $(1+1)$-dimensional action in Eq. (40) with g_{LR} of order $\sim 1/\sqrt{N}$ describes a consistent and non-trivial interacting theory in the limit $N\to\infty$.[4]

When $\varepsilon=0$ we can solve the renormalization group equations (47) and (48) explicitly, and the analytical solution reads

[4]As one can see, the rescaling of g_{LR} by $1/N$ is also meaningful. The resulting β functions are

$$\beta_{lr}=2\varepsilon g_{lr}, \tag{49}$$

$$\beta_v=2\varepsilon g_v-\frac{1}{2\pi}g_v^2. \tag{50}$$

The corresponding theory is less interesting. Indeed, the expressions for the β functions in Eq. (49) and (50) in the limit $\epsilon=0$ describe the situation when g_{lr} coupling does not run at all, while g_v experiences asymptotic freedom from the $g_v>0$ side and infrared freedom from the $g_v<0$ side. We shall see in a moment that this picture corresponds to a special case ($g_{lr}=0$) of the flow described by Eqs. (47) and (48).

$$g_{lr}(t)=\frac{g_{lr}(0)}{\sqrt{1-\frac{g_{lr}^2(0)t}{2\pi^2}}}, \tag{51}$$

$$g_v(t)=\frac{g_v(0)}{\left(1+\frac{2\pi g_v(0)}{g_{lr}^2(0)}\right)\sqrt{1-\frac{g_{lr}^2(0)t}{2\pi^2}}-\frac{2\pi g_v(0)}{g_{lr}^2(0)}\left(1-\frac{g_{lr}^2(0)t}{2\pi^2}\right)}. \tag{52}$$

This flow is presented graphically in Fig. 1 where the arrows show the flow direction toward ultraviolet.

In the upper half-plane $g_v>0$, the simple analysis of the flow given by Eqs. (51) and (52) reveals infrared (with $g_v \to +\infty$) and ultraviolet (with $g_{lr}\to\pm\infty$ and $g_v\to+\infty$) Landau poles at

$$t_{IR}=-\left(\frac{2\pi}{g_v(0)}+\frac{g_{lr}^2(0)}{2g_v^2(0)}\right), \tag{53}$$

$$t_{UV}=\frac{2\pi^2}{g_{lr}^2(0)}, \tag{54}$$

respectively. We argue that the strong infrared dynamics (with $g_v\to+\infty$) in this half-plane of couplings is an indication of a mass generation and breaking of the chiral $U(1)$ symmetry. Indeed, the generation of the fermion mass in the infrared region seems to be the only way one can avoid running into the problem of the physical Landau pole.

The generation of the fermion mass in the $g_v>0$ half-plane, in its turn, is consistent with the expectation of the universality of the magnetic catalysis in a wide range of $(3+1)$-dimensional models (such as the NJL) with a short range interaction. Indeed, as we established, the low energy dynamics in such models is described by the $(1+1)$-dimensional effective action in Eq. (40) with the coupling satisfying the set of restriction in Eqs. (A10), (A11) and (A12). The generic coupling [say, like that in Eq. (43) coming from the interaction of the lowest Landau level modes] which does not have any extra symmetry is still expected to have the $U(N)_R\times U(N)_L$ and $U(N)_V$ contribution [see Eqs. (C7a) and (C7b)]. Then, if this $U(N)_V$ contribution is positive, $g_v>0$, it is going to drive the system to the generation of mass.

Now let us study the flow in the lower half-plane $g_v\leqslant 0$. As is easy to see, there is an infrared fixed point at $(g_{lr},g_v)=(0,0)$ and the ultraviolet Landau pole (with $g_v\to -\infty$ while g_{lr} is either fixed or approaches $\pm\infty$) at the following values of t:

$$t_{UV}=\left(\frac{2\pi}{|g_v(0)|}-\frac{g_{lr}^2(0)}{2g_v^2(0)}\right), \quad \text{if} \quad |g_v(0)|\geqslant\frac{g_{lr}^2(0)}{2\pi}, \tag{55}$$

$$t_{UV}=\frac{2\pi^2}{g_{lr}^2(0)}, \quad \text{if} \quad |g_v(0)|<\frac{g_{lr}^2(0)}{2\pi}. \tag{56}$$

Since the infrared fixed point (0,0) corresponds to weakly coupled dynamics, there is apparently no mass generation in this half-plane of the coupling space. This is also in full agreement with our general expectation. Indeed, the negative values of g_v correspond to the repulsion rather than attraction in the fermion-antifermion channel which is responsible for the generation of mass, breaking chiral $U(1)$ symmetry.

Our conclusion here is that whether this model breaks chiral symmetry or not is entirely dependent on the sign of the coupling constant g_v.

The reader might be puzzled by the fact that a $U(1)$ chiral symmetry can be broken in an effectively two dimensional system. We emphasize here that this phenomenon is identical to that in the chiral Gross-Neveu model. Strictly speaking, chiral symmetry can only be broken in the large N limit. The finite N system should still be chirally symmetric, as the low dimensionality of the system would not allow for spontaneous breaking of a continuous symmetry. Indeed, if we consider the case of a large but finite value of N, an infrared stable fixed point appears for $g_v>0$ at $(g_{lr},g_v)\simeq(-2\pi,\pi N)$ which goes to infinity with $N\to\infty$. This fixed point is at strong coupling, so a conclusion based on perturbation theory is speculative at best, but its appearance is consistent with the expectation that the massless limit of this model is well defined (albeit strongly coupled) when $g_v>0$ and N is finite and chiral symmetry breaking need not take place. When $N\to\infty$ this fixed point moves to infinite coupling and chiral symmetry breaking is possible.

VI. CONCLUSION

In this paper we have shown that the effective low-energy dynamics of the $U(1)$ chirally symmetric models with a short range interaction in a background magnetic field is described by a $(1+1)$-dimensional Gross-Neveu like model with an infinite number of flavors [see Eq. (40)]. Here we established that different flavors come out as the representation space of the magnetic translations in the original $(3+1)$-dimensional model. The number of flavors is infinite and proportional to the area of the two dimensional space perpendicular to the magnetic field.

Based exclusively on the arguments of symmetry, we established a set of conditions, given by Eqs. (A6), (A7), (A10), (A11) and (A12), that the couplings of the effective theory have to satisfy. To show that they allow a non-trivial solution, we presented a few (out of infinitely many possible) examples of couplings that satisfy all the constrains. Among them, there are, in particular, the highly symmetric $U(N)_R \times U(N)_L$ and $U(N)_V$ (with $N\to\infty$) couplings. These latter are of special interest because their renormalization group flow is self-contained and allows an analytical solution [see Eqs. (51) and (52)].

At the level of the effective theory, we calculated the two-loop β function and analyzed the renormalization group flow in the two-dimensional subspace of the $U(N)_R \times U(N)_L$ and $U(N)_V$ couplings. The general result is argued to indicate the generation of the fermion mass in the $g_v > 0$ half-plane of couplings. In the other half-plane, the infrared dynamics is weakly coupled and there is no mass generation. This mass generation pattern is consistent with the earlier suggested conjecture of the universality of the so-called magnetic catalysis [5] in, at least, the models from the same universality class as the chiral $U(1)$ NJL model.

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APPENDIX A: CONSTRAINTS ON COUPLING CONSTANTS

Let us clarify the origin of the action as well as the meaning of different flavors in Eq. (40) in terms of the original $(3+1)$-dimensional model defined in Eqs. (28) and (39).

We start from the analysis of the space-time symmetries in the model. Notice that, due to the presence of the background field, the standard translations in the two-dimensional plane perpendicular to the magnetic field are not symmetries of the original model in Eq. (28). Nevertheless, there are other transformations, the so-called magnetic translations, which leave the action invariant. In contrast to the case of the ordinary translations, the two generators of the magnetic translations do not commute. In the symmetric gauge given in Eq. (29), the explicit representation of the generators and their commutation relation read

$$X=\sqrt{eB}\left(x^1+i\frac{D_2}{eB}\right), \qquad \text{(A1a)}$$

$$Y=\sqrt{eB}\left(x^2-i\frac{D_1}{eB}\right), \qquad \text{(A1b)}$$

$$[X,Y]=i, \qquad \text{(A1c)}$$

where we assume that $eB>0$. It is easy to check that the operators X and Y commute with the Hamiltonian of our model. Instead of these X and Y, it is convenient to introduce the ''creation'' and ''annihilation'' operators

$$a=\frac{X+iY}{\sqrt{2}}, \qquad \text{(A2a)}$$

$$a^\dagger=\frac{X-iY}{\sqrt{2}}, \qquad \text{(A2b)}$$

$$[a,a^\dagger]=1. \qquad \text{(A2c)}$$

Having introduced these operators in the problem, we realize that the Fock space is spanned by the set of states $|q,m\rangle$ where the quantum number $m=0,1,2,\ldots$ denotes the eigenvalue of $a^\dagger a$ operator and the multi-index q represents all the other quantum numbers (say, the Landau level number, the fermion spin projection and the chirality). In the absence of any vacuum rearrangement (symmetry breaking), the above set of states (in coordinate representation) reads

$$\langle x|m,n,\sigma,\chi,p_\parallel\rangle=\frac{1}{\sqrt{2\pi}l}\frac{1}{(\sqrt{2}l)^{m-n}} \times\sqrt{\frac{n!}{m!}}e^{-ix_\parallel p_\parallel}\bar{z}^{m-n}L_n^{(m-n)}\left(\frac{z\bar{z}}{2l^2}\right) \times\exp\left(-\frac{z\bar{z}}{4l^2}\right)\phi_{\sigma,\chi}, \qquad \text{(A3)}$$

where $l=1/\sqrt{eB}$ is the magnetic length, $z=x^1+ix^2, \bar{z}=x^1-ix^2, x_\parallel p_\parallel=x^0p_0-x^3p_3$ and $\phi_{\sigma,\chi}$ is the spinor with a given spin projection σ and chirality χ. Note that the expression in Eq. (A3) is well defined even for the case when $m<n$ (both numbers are positive) due to the Rodrigues formula for the generalized Laguerre polynomials,

$$z^{m-n}L_n^{(m-n)}(z)=\frac{1}{n!}e^z\frac{d^n}{dz^n}(e^{-z}z^m). \qquad \text{(A4)}$$

One has to remember also that all the modes in the lowest Landau level $(n=0)$ have the same projection of the spin, while the modes in the higher Landau levels $(n\geqslant 1)$ have both projections.

Since we are interested in the structure of the effective action, describing the infrared dynamics $(p_0\ll\sqrt{eB})$, it is sufficient to take into account only those degrees of freedom that originate from the lowest Landau level modes $(n=0)$. These modes freely propagate in the $(1+1)$-dimensional parallel (x^0,x^3) subspace, and are classified by the chirality and the eigenvalue of $a^\dagger a$ [see Eq. (A3)]. In what follows, we denote the effective degrees of freedom accordingly, $\psi_m^{(L,R)}(x_\parallel)$, where the superscript (L) or (R) denotes the states that result from the $(3+1)$-dimensional states of definite chirality. In the parallel subspace (x^0,x^3), $\psi_m^{(L)}(x_\parallel)$ and $\psi_m^{(R)}(x_\parallel)$ have the interpretation of the left and right moving along the x^3 axis modes, respectively.

While restricting the kinetic term in the NJL model (39) to the lowest Landau level modes,

$$\Psi(x_\parallel,x_\perp)\to\sum_{m=0}^{\infty}\frac{\phi_{+,+}\psi_m^{(R)}(x_\parallel)+\phi_{+,-}\psi_m^{(L)}(x_\parallel)}{\sqrt{2\pi m!}\,l}\times\left(\frac{\bar{z}}{\sqrt{2}l}\right)^m\exp\left(-\frac{z\bar{z}}{4l^2}\right), \tag{A5}$$

we check that $\gamma^\mu\Pi_\mu\to\gamma_\parallel^\mu p_\mu^\parallel$. After performing the integration over the perpendicular space coordinates in the original NJL action, we arrive at the effective model as in Eq. (40). Remarkably, while the NJL coupling G is dimensionful, the effective coupling in Eq. (40) is dimensionless, $g\sim G/l^2\equiv G|eB|$.

From the derivation above, we see that the fields of different flavors in the effective $(1+1)$-dimensional theory (40) correspond to different eigenstates of $a^\dagger a$ operator. This simple observation, as we show below, has far reaching consequences.

Now, let us establish the allowed structure of coupling constants in Eq. (40). The most general restriction on the couplings comes from the condition of reality of the action. This requires that

$$g^*\begin{pmatrix} n_1 & n_2 \\ m_1 & m_2 \end{pmatrix}=g\begin{pmatrix} m_2 & m_1 \\ n_2 & n_1 \end{pmatrix}. \tag{A6}$$

Similarly, the invariance under the parity $(x^3\to -x^3)$ leads to another restriction:

$$g\begin{pmatrix} n_1 & n_2 \\ m_1 & m_2 \end{pmatrix}=g\begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix}. \tag{A7}$$

These two conditions are too general and so are not of great interest or of great power by themselves. It turns out, however, that there are other, more restrictive conditions.

By recalling the origin of flavors in Eq. (40), we realize that the effective theory should enjoy some kind of flavor symmetry that results from the symmetry under magnetic translations of the original $(3+1)$-dimensional theory. This flavor symmetry, as will become clear in a moment, puts further restrictions on the allowed structure of the four-index coupling in Eq. (40).

The infinitesimal transformations of the magnetic translations and the related rotation in the perpendicular plane are given by the following operators:

$$U_1=1+i\varepsilon_1 X\equiv 1+\frac{i\varepsilon_1}{\sqrt{2}}(a+a^\dagger), \tag{A8a}$$

$$U_2=1+i\varepsilon_2 Y\equiv 1+\frac{\varepsilon_2}{\sqrt{2}}(a-a^\dagger), \tag{A8b}$$

$$U_{12}=1+i\varepsilon_{12}a^\dagger a. \tag{A8c}$$

To determine the transformation properties of the fields of different flavors, we again recall that, by construction, these fields are the eigenstates of the $a^\dagger a$ operator. Then, by doing a simple exercise, we find that the action of the creation and annihilation operators, $a^\dagger$ and a, on the properly normalized fields should read

$$a^\dagger\psi_n=\sqrt{n+1}\,\psi_{n+1},\qquad a\psi_n=\sqrt{n}\,\psi_{n-1}. \tag{A9}$$

Making use of these properties, we check that the kinetic term in the effective action is invariant under the transformations in Eqs. (A8a), (A8b) and (A8c). In fact, if we had started with a more general, non-diagonal kinetic term in the effective action [which is not forbidden by the chiral $U(1)$ symmetry], the requirement of invariance under the magnetic translations would lead us back to the diagonal form as in Eq. (40).

The invariance of the four-fermion interaction g in Eq. (40) under the set of transformations in Eqs. (A8a), (A8b) and (A8c) leads to

$$\sqrt{n_1}g\begin{pmatrix} n_1-1 & n_2 \\ m_1 & m_2 \end{pmatrix}-\sqrt{n_2+1}g\begin{pmatrix} n_1 & n_2+1 \\ m_1 & m_2 \end{pmatrix}+\sqrt{m_1}g\begin{pmatrix} n_1 & n_2 \\ m_1-1 & m_2 \end{pmatrix}-\sqrt{m_2+1}g\begin{pmatrix} n_1 & n_2 \\ m_1 & m_2+1 \end{pmatrix}=0, \tag{A10}$$

$$\sqrt{n_1+1}g\begin{pmatrix} n_1+1 & n_2 \\ m_1 & m_2 \end{pmatrix}-\sqrt{n_2}g\begin{pmatrix} n_1 & n_2-1 \\ m_1 & m_2 \end{pmatrix}+\sqrt{m_1+1}g\begin{pmatrix} n_1 & n_2 \\ m_1+1 & m_2 \end{pmatrix}-\sqrt{m_2}g\begin{pmatrix} n_1 & n_2 \\ m_1 & m_2-1 \end{pmatrix}=0, \tag{A11}$$

$$g\begin{pmatrix} n_1 & n_2 \\ m_1 & m_2 \end{pmatrix}=0,\quad \text{unless}\quad n_1-n_2+m_1-m_2=0. \tag{A12}$$

APPENDIX B: CALCULATION OF THE TWO-LOOP β FUNCTION

To derive the two-loop β function of the effective spinor theory in Eq. (40), we apply the method of Ref. [24] that was used for the φ^4 scalar theory in $d=4-\varepsilon$ dimensions. In dimensional regularization (with $D=2+2\varepsilon$), our renormalized Lagrangian density reads

$$\mathcal{L}_{NJL}=\sum_{n=0}^{\infty}\,(Z_n^{(L)}\psi_n^{(L)*}i\partial_+\psi_n^{(L)}+Z_n^{(R)}\psi_n^{(R)*}i\partial_-\psi_n^{(R)})+\sum_{n_1,n_2,m_1,m_2=0}^{\infty}\mu^{-2\varepsilon}G\begin{pmatrix}n_1 & n_2\\ m_1 & m_2\end{pmatrix}\times\psi_{n_1}^{(L)*}\psi_{n_2}^{(R)}\psi_{m_1}^{(R)*}\psi_{m_2}^{(L)}$$

$$=\sum_{n=0}^{\infty}\,(\psi_n^{(L)*}i\partial_+\psi_n^{(L)}+\psi_n^{(R)*}i\partial_-\psi_n^{(R)})+\sum_{n_1,n_2,m_1,m_2=0}^{\infty}\mu^{-2\varepsilon}g\begin{pmatrix}n_1 & n_2\\ m_1 & m_2\end{pmatrix}\psi_{n_1}^{(L)*}\psi_{n_2}^{(R)}\psi_{m_1}^{(R)*}\psi_{m_2}^{(L)}+(\cdots), \quad \text{(B1)}$$

where the coupling G includes the coupling constant renormalization, $G=Z_g g$, and the ellipsis denotes the counterterms. We remind the reader that the four-index coupling g is non-zero only for $n_1-n_2+m_1-m_2=0$. This means that the four-fermion term in Eq. (B1) contains the sum only over three indices (say, n_1, n_2, and m_1), while the fourth one (m_2) is superfluous.

Before proceeding with the actual calculation of the β function, we need to specify how to handle the infrared divergences that appear in the calculation of the Feynman diagrams [24,25]. Such divergences usually come from the propagators of the massless fermions. If treated improperly, they could easily obscure the calculation of the relevant diagrams and eventually lead to a wrong result. To avoid the problem, in what follows, we modify the infrared region by changing the fermion propagators as follows:

$$S^{(L)}(p)=\frac{p_-}{p^2}\to\frac{p_-}{p^2-m^2}, \quad \text{(B2a)}$$

$$S^{(R)}(p)=\frac{p_+}{p^2}\to\frac{p_+}{p^2-m^2}. \quad \text{(B2b)}$$

This infrared regularization procedure respects all the symmetries of the model and does not change the ultraviolet region.

Now, let us calculate the β function. First of all, we recall that the relation between the bare and the renormalized couplings, in the dimensional regularization ($D=2+2\varepsilon$), reads

$$g_0\begin{pmatrix}N_1 & N_2\\ M_1 & M_2\end{pmatrix}=\frac{\mu^{-2\varepsilon}}{\sqrt{Z_{N_1}^{(L)}Z_{M_1}^{(R)}Z_{N_2}^{(R)}Z_{M_2}^{(L)}}}G\begin{pmatrix}N_1 & N_2\\ M_1 & M_2\end{pmatrix}. \quad \text{(B3)}$$

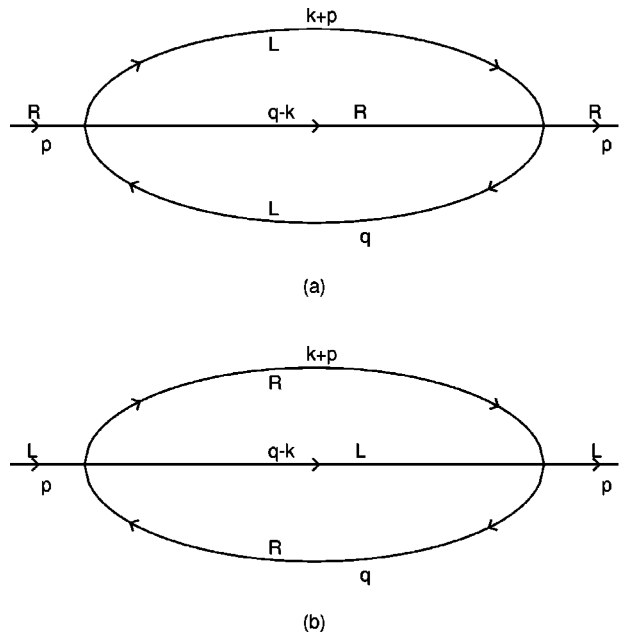

FIG. 2. Diagrams contributing to $\Gamma^{(2R)}$ and $\Gamma^{(2L)}$, respectively.

This is going to be used in the definition of the β function. In calculation, we impose the following renormalization conditions:

$$\frac{\partial}{\partial p_-}\Gamma_{NN}^{(2R)}\Big|_{p=0}=\frac{\partial}{\partial p_+}\Gamma_{NN}^{(2L)}\Big|_{p=0}=1, \quad \text{(B4)}$$

$$\Gamma^{(4)}\begin{pmatrix}N_1 & N_2\\ M_1 & M_2\end{pmatrix}\Big|_{p=0}=g\begin{pmatrix}N_1 & N_2\\ M_1 & M_2\end{pmatrix}. \quad \text{(B5)}$$

Note that the introduction of the effective infrared cutoff m in the fermion propagators earlier allows us to use the renormalization point at $p=0$.

The Feynman diagrams of the relevant contributions to the two-point functions at two-loop order are given in Figs. 2a and 2b. By extracting the divergent (of order $1/\varepsilon$) terms of these two-loop corrections, we arrive at the equations

$$\frac{\partial}{\partial p_-}\Gamma_{NN}^{(2R)}\Big|_{p=0}=Z_N^{(R)}+\frac{c^{(r)}}{(4\pi)^2\varepsilon}\times\sum_{k_1,k_2}G\begin{pmatrix}k_1 & k_2+N\\ N & k_1-k_2\end{pmatrix}G\begin{pmatrix}k_1-k_2 & N\\ k_2+N & k_1\end{pmatrix}, \quad \text{(B6)}$$

$$\frac{\partial}{\partial p_+}\Gamma_{NN}^{(2L)}\Big|_{p=0}=Z_N^{(L)}+\frac{c^{(l)}}{(4\pi)^2\varepsilon}\times\sum_{k_1,k_2}G\begin{pmatrix}N & k_2+N\\ k_1 & k_1-k_2\end{pmatrix}G\begin{pmatrix}k_1-k_2 & k_1\\ k_2+N & N\end{pmatrix}, \quad \text{(B7)}$$

where, by definition,

$$(\text{Fig. 2a})\rightarrow \frac{c^{(r)}}{(4\pi)^2\varepsilon}+O(1)\equiv -\frac{1}{\mu^{2D-4}}\frac{\partial}{\partial p_-}\int \frac{d^Dkd^Dq}{(2\pi)^{2D}}\frac{(k+p)_-(q-k)_+q_-}{[(k+p)^2-m^2][(q-k)^2-m^2](q^2-m^2)}\bigg|_{p=0}$$

$$=-\frac{1}{(4\pi)^2\varepsilon}+O(1), \tag{B8}$$

$$(\text{Fig. 2b})\rightarrow \frac{c^{(l)}}{(4\pi)^2\varepsilon}+O(1)\equiv -\frac{1}{\mu^{2D-4}}\frac{\partial}{\partial p_+}\int \frac{d^Dkd^Dq}{(2\pi)^{2D}}\frac{(k+p)_+(q-k)_-q_+}{[(k+p)^2-m^2][(q-k)^2-m^2](q^2-m^2)}\bigg|_{p=0}$$

$$=-\frac{1}{(4\pi)^2\varepsilon}+O(1). \tag{B9}$$

Thus, we see that $c^{(r)}=c^{(l)}=-1$.

In a similar way, the perturbative expansion for the four-point function reads

$$\begin{aligned}
\Gamma^{(4)}\begin{pmatrix} N_1 & N_2\\ M_1 & M_2\end{pmatrix}\bigg|_{p=0} = {} & G\begin{pmatrix} N_1 & N_2\\ M_1 & M_2\end{pmatrix}+\frac{a_0+a_1\varepsilon}{4\pi\varepsilon}\sum_{k_1}\left[G\begin{pmatrix} N_1 & N_2\\ k_1 & N_1-N_2+k_1\end{pmatrix}G\begin{pmatrix} N_1-N_2+k_1 & k_1\\ M_1 & M_2\end{pmatrix}\right.\\
&\left.-G\begin{pmatrix} N_1 & k_1\\ M_1 & N_1+M_1-k_1\end{pmatrix}G\begin{pmatrix} N_1+M_1-k_1 & N_2\\ k_1 & M_2\end{pmatrix}\right]\\
&+\frac{b_0+b_1\varepsilon}{(4\pi\varepsilon)^2}\sum_{k_1,k_2}\left[G\begin{pmatrix} N_1 & N_2\\ k_1 & N_1-N_2+k_1\end{pmatrix}G\begin{pmatrix} N_1-N_2+k_1 & k_1\\ k_2 & N_1-N_2+k_2\end{pmatrix}G\begin{pmatrix} N_1-N_2+k_2 & k_2\\ M_1 & M_2\end{pmatrix}\right.\\
&\left.+G\begin{pmatrix} N_1 & k_1\\ M_1 & N_1+M_1-k_1\end{pmatrix}G\begin{pmatrix} N_1+M_1-k_1 & k_2\\ k_1 & N_1+M_1-k_2\end{pmatrix}G\begin{pmatrix} N_1+M_1-k_2 & N_2\\ k_2 & M_2\end{pmatrix}\right]\\
&+\frac{c_0}{(4\pi)^2\varepsilon}\sum_{k_1,k_2}\left[G\begin{pmatrix} N_1 & k_2\\ k_1 & N_1+k_1-k_2\end{pmatrix}G\begin{pmatrix} N_1+k_1-k_2 & N_2\\ M_1 & M_2+k_1-k_2\end{pmatrix}G\begin{pmatrix} M_2+k_1-k_2 & k_1\\ k_2 & M_2\end{pmatrix}\right.\\
&\left.+G\begin{pmatrix} k_1 & k_2\\ M_1 & M_1+k_1-k_2\end{pmatrix}G\begin{pmatrix} N_1 & N_2-M_1+k_2\\ k_2 & M_2\end{pmatrix}G\begin{pmatrix} M_1+k_1-k_2 & N_2\\ N_2-M_1+k_2 & k_1\end{pmatrix}\right]\\
&+\frac{d_0+d_1\varepsilon}{(4\pi\varepsilon)^2}\sum_{k_1,k_2}\left[G\begin{pmatrix} N_1 & k_2\\ k_1 & N_1+k_1-k_2\end{pmatrix}G\begin{pmatrix} N_1-N_2+k_1 & k_1\\ M_1 & M_2\end{pmatrix}G\begin{pmatrix} N_1+k_1-k_2 & N_2\\ k_2 & N_1-N_2+k_1\end{pmatrix}\right.\\
&+G\begin{pmatrix} k_1 & k_2\\ M_1 & k_1-k_2+M_1\end{pmatrix}G\begin{pmatrix} N_1 & N_2\\ N_2-N_1+k_1 & k_1\end{pmatrix}G\begin{pmatrix} k_1-k_2+M_1 & N_2-N_1+k_1\\ k_2 & M_2\end{pmatrix}\\
&+G\begin{pmatrix} N_1 & N_1+k_1-k_2\\ k_1 & k_2\end{pmatrix}G\begin{pmatrix} M_1+k_2-k_1 & N_2\\ N_1+k_1-k_2 & M_2\end{pmatrix}G\begin{pmatrix} k_2 & k_1\\ M_1 & M_1+k_2-k_1\end{pmatrix}\\
&\left.+G\begin{pmatrix} N_1+M_1-k_2 & N_2\\ k_1+k_2-M_2 & k_1\end{pmatrix}G\begin{pmatrix} N_1 & k_2\\ M_1 & N_1+M_1-k_2\end{pmatrix}G\begin{pmatrix} k_1 & k_1+k_2-M_2\\ k_2 & M_2\end{pmatrix}\right].
\end{aligned} \tag{B10}$$

Note that the restriction $n_1-n_2+m_1-m_2=0$, is satisfied for each four-index coupling that appears here. The coefficients a_i, b_i,c_i and d_i are defined by the following expressions:

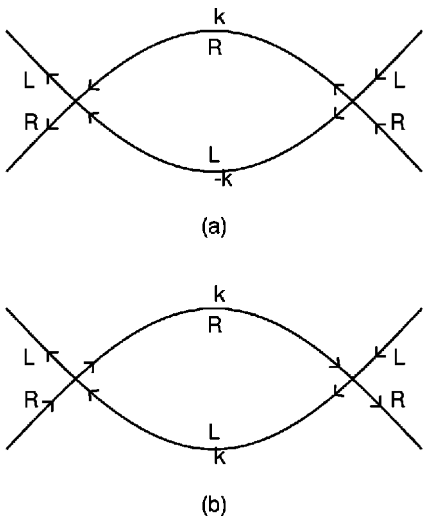

FIG. 3. Diagrams contributing to $\Gamma^{(4)}$ at one loop which determine a_0 and a_1.

$$
\begin{aligned}
(\text{Fig. }3)\rightarrow \frac{a_0+a_1\varepsilon}{4\pi\varepsilon}+O(\varepsilon)
&\equiv \frac{i}{\mu^{D-2}}\int \frac{d^Dk}{(2\pi)^D}\,\frac{k^2}{(k^2-m^2)^2}\\
&=-\frac{1}{4\pi\varepsilon}\left(1+\varepsilon(1+\gamma)+\varepsilon\,\ln\frac{m^2}{4\pi\mu^2}\right)+O(\varepsilon),
\end{aligned}
\tag{B11}
$$

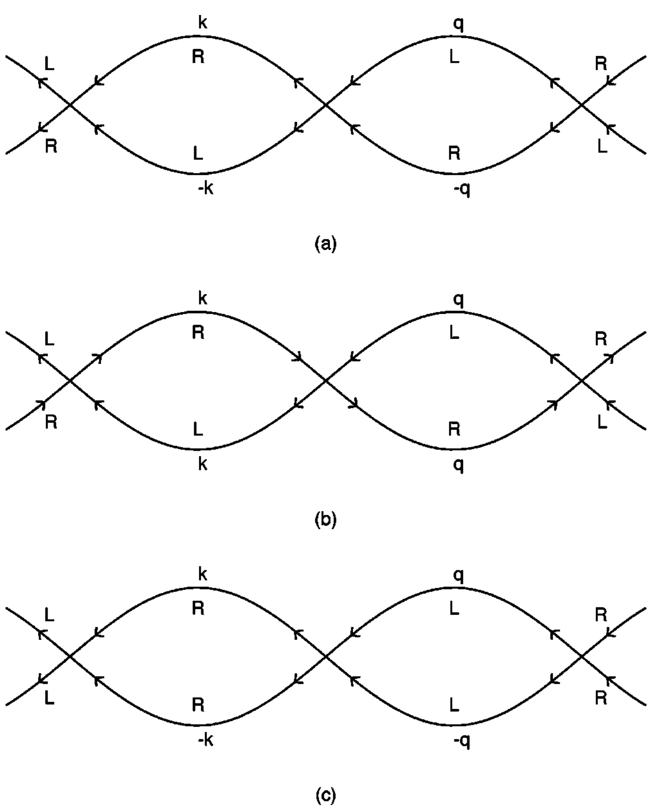

FIG. 4. Diagrams contributing to $\Gamma^{(4)}$ at two loops which determine b_0 and b_1. The diagram (c) is finite.

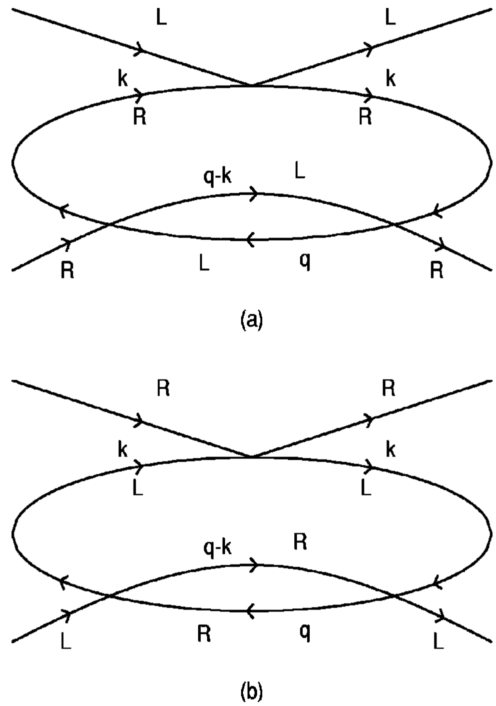

FIG. 5. Diagrams contributing to $\Gamma^{(4)}$ at two loops which determine c_0.

$$
\begin{aligned}
(\text{Fig. }4)\rightarrow \frac{b_0+b_1\varepsilon}{(4\pi\varepsilon)^2}+O(1)
&\equiv \frac{i^2}{\mu^{2D-4}}\int \frac{d^Dk\,d^Dq}{(2\pi)^{2D}}\,\frac{q^2k^2}{(q^2-m^2)^2(k^2-m^2)^2}\\
&=\frac{1}{(4\pi\varepsilon)^2}\left(1+2\varepsilon(1+\gamma)+2\varepsilon\ln\frac{m^2}{4\pi\mu^2}\right)+O(1),
\end{aligned}
\tag{B12}
$$

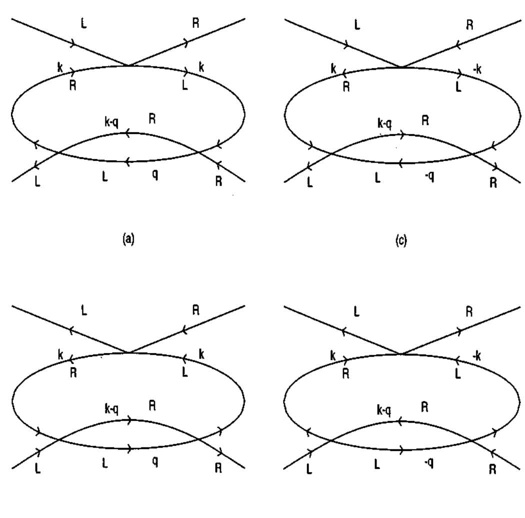

FIG. 6. Diagrams contributing to $\Gamma^{(4)}$ at two loops which determine d_0 and d_1.

$$(\text{Fig. }5)\to\frac{c_0}{(4\pi)^2\varepsilon}+O(\varepsilon)$$
$$\equiv\frac{i^2}{\mu^{2D-4}}\int\frac{d^Dkd^Dq}{(2\pi)^{2D}}\frac{k_+^2q_-(q-k)_-}{(k^2-m^2)^2(q^2-m^2)[(q-k)^2-m^2]}$$
$$=-\frac{1}{2(4\pi)^2\varepsilon}+O(\varepsilon),\tag{B13}$$

and

$$(\text{Fig. }6)\to\frac{d_0+d_1\varepsilon}{(4\pi\varepsilon)^2}+O(1)\equiv\frac{i^2}{\mu^{2D-4}}\int\frac{d^Dkd^Dq}{(2\pi)^{2D}}\times\frac{k^2q_-(k-q)_+}{(k^2-m^2)^2(q^2-m^2)[(k-q)^2-m^2]}$$
$$=-\frac{1}{(4\pi\varepsilon)^2}\left(\frac{1}{2}+\varepsilon(1+\gamma)+\varepsilon\ln\frac{m^2}{4\pi\mu^2}\right)+O(1).\tag{B14}$$

Here $\gamma\approx0.577$ is the Euler constant.

From Eqs. (B11) – (B14), we obtain

$$a_0=-1,\quad a_1=-(1+\gamma)-\ln\frac{m^2}{4\pi\mu^2},\tag{B15a}$$

$$b_0=1,\quad b_1=2(1+\gamma)+2\ln\frac{m^2}{4\pi\mu^2},\tag{B15b}$$

$$c_0=-\frac{1}{2},\tag{B15c}$$

$$d_0=-\frac{1}{2},\quad d_1=-(1+\gamma)-\ln\frac{m^2}{4\pi\mu^2}.\tag{B15d}$$

After expressing the function G in terms of g, we arrive at

$$G\begin{pmatrix}N_1 & N_2\\ M_1 & M_2\end{pmatrix}=g\begin{pmatrix}N_1 & N_2\\ M_1 & M_2\end{pmatrix}-\frac{a_0+a_1\varepsilon}{4\pi\varepsilon}\sum_{k_1}\left[g\begin{pmatrix}N_1 & N_2\\ k_1 & N_1-N_2+k_1\end{pmatrix}g\begin{pmatrix}N_1-N_2+k_1 & k_1\\ M_1 & M_2\end{pmatrix}\right.$$
$$\left.-g\begin{pmatrix}N_1 & k_1\\ M_1 & N_1+M_1-k_1\end{pmatrix}g\begin{pmatrix}N_1+M_1-k_1 & N_2\\ k_1 & M_2\end{pmatrix}\right]-\frac{b_0-2a_0^2+(b_1-4a_1a_0)\varepsilon}{(4\pi\varepsilon)^2}$$
$$\times\sum_{k_1,k_2}\left[g\begin{pmatrix}N_1 & N_2\\ k_1 & N_1-N_2+k_1\end{pmatrix}g\begin{pmatrix}N_1-N_2+k_1 & k_1\\ k_2 & N_1-N_2+k_2\end{pmatrix}g\begin{pmatrix}N_1-N_2+k_2 & k_2\\ M_1 & M_2\end{pmatrix}\right.$$
$$\left.+g\begin{pmatrix}N_1 & k_1\\ M_1 & N_1+M_1-k_1\end{pmatrix}g\begin{pmatrix}N_1+M_1-k_1 & k_2\\ k_1 & N_1+M_1-k_2\end{pmatrix}g\begin{pmatrix}N_1+M_1-k_2 & N_2\\ k_2 & M_2\end{pmatrix}\right]$$
$$-\frac{c_0}{(4\pi)^2\varepsilon}\sum_{k_1,k_2}\left[g\begin{pmatrix}N_1 & k_2\\ k_1 & N_1+k_1-k_2\end{pmatrix}g\begin{pmatrix}N_1+k_1-k_2 & N_2\\ M_1 & M_2+k_1-k_2\end{pmatrix}g\begin{pmatrix}M_2+k_1-k_2 & k_1\\ k_2 & M_2\end{pmatrix}\right.$$
$$\left.+g\begin{pmatrix}k_1 & k_2\\ M_1 & M_1+k_1-k_2\end{pmatrix}g\begin{pmatrix}N_1 & N_2-M_1+k_2\\ k_2 & M_2\end{pmatrix}g\begin{pmatrix}M_1+k_1-k_2 & N_2\\ N_2-M_1+k_2 & k_1\end{pmatrix}\right]$$
$$-\frac{d_0+a_0^2+(d_1+2a_0a_1)\varepsilon}{(4\pi\varepsilon)^2}\sum_{k_1,k_2}\left[g\begin{pmatrix}N_1 & k_2\\ k_1 & N_1+k_1-k_2\end{pmatrix}g\begin{pmatrix}N_1-N_2+k_1 & k_1\\ M_1 & M_2\end{pmatrix}\right.$$
$$\times g\begin{pmatrix}N_1+k_1-k_2 & N_2\\ k_2 & N_1-N_2+k_1\end{pmatrix}+g\begin{pmatrix}k_1 & k_2\\ M_1 & k_1-k_2+M_1\end{pmatrix}g\begin{pmatrix}N_1 & N_2\\ N_2-N_1+k_1 & k_1\end{pmatrix}$$
$$\times g\begin{pmatrix}k_1-k_2+M_1 & N_2-N_1+k_1\\ k_2 & M_2\end{pmatrix}+g\begin{pmatrix}N_1 & N_1+k_1-k_2\\ k_1 & k_2\end{pmatrix}g\begin{pmatrix}M_1+k_2-k_1 & N_2\\ N_1+k_1-k_2 & M_2\end{pmatrix}g\begin{pmatrix}k_2 & k_1\\ M_1 & M_1+k_2-k_1\end{pmatrix}$$
$$\left.+g\begin{pmatrix}N_1+M_1-k_2 & N_2\\ k_1+k_2-M_2 & k_1\end{pmatrix}g\begin{pmatrix}N_1 & k_2\\ M_1 & N_1+M_1-k_2\end{pmatrix}g\begin{pmatrix}k_1 & k_1+k_2-M_2\\ k_2 & M_2\end{pmatrix}\right].\tag{B16}$$

The β function is defined as follows [24]:

$$\sum_{n_1,n_2,m_1,m_2} \beta(g)\begin{pmatrix} n_1 & n_2 \\ m_1 & m_2 \end{pmatrix} \frac{\partial}{\partial g\begin{pmatrix} n_1 & n_2 \\ m_1 & m_2 \end{pmatrix}} \left[\frac{G\begin{pmatrix} N_1 & N_2 \\ M_1 & M_2 \end{pmatrix}}{\sqrt{Z^{(L)}_{N_1} Z^{(R)}_{M_1} Z^{(R)}_{N_2} Z^{(L)}_{M_2}}} \right] = 2\varepsilon \frac{G\begin{pmatrix} N_1 & N_2 \\ M_1 & M_2 \end{pmatrix}}{\sqrt{Z^{(L)}_{N_1} Z^{(R)}_{M_1} Z^{(R)}_{N_2} Z^{(L)}_{M_2}}}. \tag{B17}$$

By making use of this definition, we calculate the two-loop β function,

$$\begin{aligned}
\beta(g)\begin{pmatrix} N_1 & N_2 \\ M_1 & M_2 \end{pmatrix} &= 2\varepsilon g\begin{pmatrix} N_1 & N_2 \\ M_1 & M_2 \end{pmatrix} + \frac{a_0}{2\pi}\sum_{k_1}\left[g\begin{pmatrix} N_1 & N_2 \\ k_1 & N_1-N_2+k_1 \end{pmatrix} g\begin{pmatrix} N_1-N_2+k_1 & k_1 \\ M_1 & M_2 \end{pmatrix} - g\begin{pmatrix} N_1 & k_1 \\ M_1 & N_1+M_1-k_1 \end{pmatrix}\right. \\
&\quad \left.\times g\begin{pmatrix} N_1+M_1-k_1 & N_2 \\ k_1 & M_2 \end{pmatrix}\right] + \frac{b_1-2a_0a_1}{4\pi^2}\sum_{k_1,k_2}\left[g\begin{pmatrix} N_1 & N_2 \\ k_1 & N_1-N_2+k_1 \end{pmatrix}\right. \\
&\quad \times g\begin{pmatrix} N_1-N_2+k_1 & k_1 \\ k_2 & N_1-N_2+k_2 \end{pmatrix} g\begin{pmatrix} N_1-N_2+k_2 & k_2 \\ M_1 & M_2 \end{pmatrix} \\
&\quad \left. + g\begin{pmatrix} N_1 & k_1 \\ M_1 & N_1+M_1-k_1 \end{pmatrix} g\begin{pmatrix} N_1+M_1-k_1 & k_2 \\ k_1 & N_1+M_1-k_2 \end{pmatrix} g\begin{pmatrix} N_1+M_1-k_2 & N_2 \\ k_2 & M_2 \end{pmatrix}\right] \\
&\quad + \frac{c_0}{4\pi^2}\sum_{k_1,k_2}\left[g\begin{pmatrix} N_1 & k_2 \\ k_1 & N_1+k_1-k_2 \end{pmatrix} g\begin{pmatrix} N_1+k_1-k_2 & N_2 \\ M_1 & M_2+k_1-k_2 \end{pmatrix} g\begin{pmatrix} M_2+k_1-k_2 & k_1 \\ k_2 & M_2 \end{pmatrix}\right. \\
&\quad \left. + g\begin{pmatrix} k_1 & k_2 \\ M_1 & M_1+k_1-k_2 \end{pmatrix} g\begin{pmatrix} N_1 & N_2-M_1+k_2 \\ k_2 & M_2 \end{pmatrix} g\begin{pmatrix} M_1+k_1-k_2 & N_2 \\ N_2-M_1+k_2 & k_1 \end{pmatrix}\right] \\
&\quad + \frac{d_1+a_0a_1}{4\pi^2}\sum_{k_1,k_2}\left[g\begin{pmatrix} N_1 & k_2 \\ k_1 & N_1+k_1-k_2 \end{pmatrix} g\begin{pmatrix} N_1-N_2+k_1 & k_1 \\ M_1 & M_2 \end{pmatrix} g\begin{pmatrix} N_1+k_1-k_2 & N_2 \\ k_2 & N_1-N_2+k_1 \end{pmatrix}\right. \\
&\quad + g\begin{pmatrix} k_1 & k_2 \\ M_1 & k_1-k_2+M_1 \end{pmatrix} g\begin{pmatrix} N_1 & N_2 \\ N_2-N_1+k_1 & k_1 \end{pmatrix} g\begin{pmatrix} k_1-k_2+M_1 & N_2-N_1+k_1 \\ k_2 & M_2 \end{pmatrix} \\
&\quad + g\begin{pmatrix} N_1 & N_1+k_1-k_2 \\ k_1 & k_2 \end{pmatrix} g\begin{pmatrix} M_1+k_2-k_1 & N_2 \\ N_1+k_1-k_2 & M_2 \end{pmatrix} g\begin{pmatrix} k_2 & k_1 \\ M_1 & M_1+k_2-k_1 \end{pmatrix} \\
&\quad \left. + g\begin{pmatrix} N_1+M_1-k_2 & N_2 \\ k_1+k_2-M_2 & k_1 \end{pmatrix} g\begin{pmatrix} N_1 & k_2 \\ M_1 & N_1+M_1-k_2 \end{pmatrix} g\begin{pmatrix} k_1 & k_1+k_2-M_2 \\ k_2 & M_2 \end{pmatrix}\right] \\
&\quad + \frac{2}{(4\pi)^2} g\begin{pmatrix} N_1 & N_2 \\ M_1 & M_2 \end{pmatrix}\sum_{k_1,k_2}\left[g\begin{pmatrix} N_1 & k_2+N_1 \\ k_1 & k_1-k_2 \end{pmatrix} g\begin{pmatrix} k_1-k_2 & k_1 \\ k_2+N_1 & N_1 \end{pmatrix} + g\begin{pmatrix} k_1 & k_2+N_2 \\ N_2 & k_1-k_2 \end{pmatrix} g\begin{pmatrix} k_1-k_2 & N_2 \\ k_2+N_2 & k_1 \end{pmatrix}\right. \\
&\quad \left. + g\begin{pmatrix} k_1 & k_2+M_1 \\ M_1 & k_1-k_2 \end{pmatrix} g\begin{pmatrix} k_1-k_2 & M_1 \\ k_2+M_1 & k_1 \end{pmatrix} + g\begin{pmatrix} M_2 & k_2+M_2 \\ k_1 & k_1-k_2 \end{pmatrix} g\begin{pmatrix} k_1-k_2 & k_1 \\ k_2+M_2 & M_2 \end{pmatrix}\right],
\end{aligned} \tag{B18}$$

where we already used the fact that $a_0^2 = b_0 = -2d_0$ which, by the way, is the necessary condition for the renormalizability of the model. Note that the last term in Eq. (B18) appears due to the renormalization of the two-point function. After taking into account the values of constants in Eqs. (B15a) – (B15d), we arrive at our final result for the β function:

$$\begin{aligned}
\beta(g)\begin{pmatrix} N_1 & N_2 \\ M_1 & M_2 \end{pmatrix} &= 2\varepsilon g\begin{pmatrix} N_1 & N_2 \\ M_1 & M_2 \end{pmatrix} - \frac{1}{2\pi}\sum_{k_1}\left[g\begin{pmatrix} N_1 & N_2 \\ k_1 & N_1-N_2+k_1 \end{pmatrix}\right. \\
&\quad \left. \times g\begin{pmatrix} N_1-N_2+k_1 & k_1 \\ M_1 & M_2 \end{pmatrix} - g\begin{pmatrix} N_1 & k_1 \\ M_1 & N_1+M_1-k_1 \end{pmatrix} g\begin{pmatrix} N_1+M_1-k_1 & N_2 \\ k_1 & M_2 \end{pmatrix}\right] \\
&\quad - \frac{1}{8\pi^2}\sum_{k_1,k_2}\left[g\begin{pmatrix} N_1 & k_2 \\ k_1 & N_1+k_1-k_2 \end{pmatrix} g\begin{pmatrix} N_1+k_1-k_2 & N_2 \\ M_1 & M_2+k_1-k_2 \end{pmatrix} g\begin{pmatrix} M_2+k_1-k_2 & k_1 \\ k_2 & M_2 \end{pmatrix}\right.
\end{aligned}$$

$$+g\begin{pmatrix} k_1 & k_2 \\ M_1 & M_1+k_1-k_2 \end{pmatrix} g\begin{pmatrix} N_1 & N_2-M_1+k_2 \\ k_2 & M_2 \end{pmatrix} g\begin{pmatrix} M_1+k_1-k_2 & N_2 \\ N_2-M_1+k_2 & k_1 \end{pmatrix}\Bigg]$$

$$+\frac{2}{(4\pi)^2} g\begin{pmatrix} N_1 & N_2 \\ M_1 & M_2 \end{pmatrix} \sum_{k_1,k_2} \Bigg[g\begin{pmatrix} N_1 & k_2+N_1 \\ k_1 & k_1-k_2 \end{pmatrix} g\begin{pmatrix} k_1-k_2 & k_1 \\ k_2+N_1 & N_1 \end{pmatrix} + g\begin{pmatrix} k_1 & k_2+N_2 \\ N_2 & k_1-k_2 \end{pmatrix} g\begin{pmatrix} k_1-k_2 & N_2 \\ k_2+N_2 & k_1 \end{pmatrix}$$

$$+g\begin{pmatrix} k_1 & k_2+M_1 \\ M_1 & k_1-k_2 \end{pmatrix} g\begin{pmatrix} k_1-k_2 & M_1 \\ k_2+M_1 & k_1 \end{pmatrix} + g\begin{pmatrix} M_2 & k_2+M_2 \\ k_1 & k_1-k_2 \end{pmatrix} g\begin{pmatrix} k_1-k_2 & k_1 \\ k_2+M_2 & M_2 \end{pmatrix}\Bigg]. \quad \text{(B19)}$$

APPENDIX C: NJL COUPLINGS CONTAIN MAXIMALLY SYMMETRIC ONES

While having less symmetry, the coupling in Eq. (43) still could contain contributions of highly symmetric solutions as in Eqs. (41) and (42). In order to extract such contributions, we introduce the projection operators to the corresponding subspaces in the space of couplings,

$$P^{(LR)}[\cdots] = \lim_{N\to\infty} I^{(LR)} \frac{N\,\mathrm{Tr}^{(LR)}[\cdots] - \mathrm{Tr}^{(V)}[\cdots]}{N(N^2-1)}, \quad \text{(C1)}$$

$$P^{(V)}[\cdots] = \lim_{N\to\infty} I^{(V)} \frac{N\,\mathrm{Tr}^{(V)}[\cdots] - \mathrm{Tr}^{(LR)}[\cdots]}{N(N^2-1)}, \quad \text{(C2)}$$

where, by definition,

$$I^{(LR)}\begin{pmatrix} n_1 & n_2 \\ m_1 & m_2 \end{pmatrix} = \delta_{n_1,m_2}\delta_{n_2,m_1}, \quad \text{(C3)}$$

$$\mathrm{Tr}^{(LR)}\left[g\begin{pmatrix} n_1 & n_2 \\ m_1 & m_2 \end{pmatrix}\right] = \sum_{n,m} g\begin{pmatrix} n & m \\ m & n \end{pmatrix}, \quad \text{(C4)}$$

$$I^{(V)}\begin{pmatrix} n_1 & n_2 \\ m_1 & m_2 \end{pmatrix} = \delta_{n_1,n_2}\delta_{m_1,m_2}, \quad \text{(C5)}$$

$$\mathrm{Tr}^{(V)}\left[g\begin{pmatrix} n_1 & n_2 \\ m_1 & m_2 \end{pmatrix}\right] = \sum_{n,m} g\begin{pmatrix} n & n \\ m & m \end{pmatrix}. \quad \text{(C6)}$$

By applying the projection operators in Eqs. (C1) and (C2) to the coupling in Eq. (43), we easily extract the symmetric contributions,

$$P^{(LR)}\left[g\begin{pmatrix} n_1 & n_2 \\ m_1 & m_2 \end{pmatrix}\right] = \frac{2G}{N+1} I^{(LR)}, \quad \text{(C7a)}$$

$$P^{(V)}\left[g\begin{pmatrix} n_1 & n_2 \\ m_1 & m_2 \end{pmatrix}\right] = \frac{2G}{N+1} I^{(V)}. \quad \text{(C7b)}$$

[1] Y.J. Ng and Y. Kikuchi, in *Vacuum Structure in Intense Fields*, edited by H. M. Fried and B. Müller (Plenum, New York, 1991).

[2] D.M. Gitman, E.S. Fradkin and Sh. M. Shvartsman, in *Quantum Electrodynamics with Unstable Vacuum*, edited by V.L. Ginzburg (Nova Science, Commack, NY, 1995).

[3] J.K. Jain, in *Perspectives in Quantum Hall Effects*, edited by S.D. Sarma and A. Pinczuk (Wiley, New York, 1997).

[4] S.W. MacDowell and O. Tornkvist, Phys. Rev. D **45**, 3833 (1992); K. Enqvist, Int. J. Mod. Phys. D **7**, 331 (1998); R. Fiore, A. Tiesi, L. Masperi, and A. Megevand, Mod. Phys. Lett. A **14**, 407 (1999); V. Skalozub and M. Bordag, ''Electroweak phase transition in a strong magnetic field,'' hep-ph/9807510; K. Kajantie *et al.*, Nucl. Phys. **B544**, 357 (1999); T. Inagaki, T. Muta, and S.D. Odintsov, Prog. Theor. Phys. Suppl. **127**, 93 (1997).

[5] V.P. Gusynin, V.A. Miransky, and I.A. Shovkovy, Phys. Rev. Lett. **73**, 3499 (1994); Phys. Rev. D **52**, 4718 (1995); Phys. Lett. B **349**, 477 (1995); Phys. Rev. D **52**, 4747 (1995); Nucl. Phys. **B462**, 249 (1996).

[6] C.N. Leung, Y.J. Ng, and A.W. Ackley, Phys. Rev. D **54**, 4181 (1996); D.-S. Lee, C.N. Leung, and Y.J. Ng, *ibid.* **55**, 6504 (1997).

[7] D.K. Hong, Y. Kim, and S.-J. Sin, Phys. Rev. D **54**, 7879 (1996); D.K. Hong, *ibid.* **57**, 3759 (1998).

[8] I.A. Shushpanov and A.V. Smilga, Phys. Lett. B **402**, 351 (1997); S. Kanemura, H.-T. Sato, and H. Tochimura, Nucl. Phys. **B517**, 567 (1998); A.Yu. Babansky, E.V. Gorbar, and G.V. Shchepanyuk, Phys. Lett. B **419**, 272 (1998); T. Itoh and H. Kato, Phys. Rev. Lett. **81**, 30 (1998).

[9] V. Elias, D.G.C. McKeon, V.A. Miransky, and I.A. Shovkovy, Phys. Rev. D **54**, 7884 (1996).

[10] V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, Phys. Rev. Lett. **83**, 1291 (1999).

[11] S.P. Klevansky and R.H. Lemmer, Phys. Rev. D **39**, 3478 (1989).

[12] K.G. Klimenko, Theor. Math. Phys. **89**, 1161 (1992); Z. Phys. C **54**, 323 (1992).

[13] S. Schramm, B. Müller, and A.J. Schramm, Mod. Phys. Lett. A **A7**, 973 (1992).

[14] G.W. Semenoff, I.A. Shovkovy, and L.C.R. Wijewardhana, Mod. Phys. Lett. A **A13**, 1143 (1998).

[15] K. Farakos and N.E. Mavromatos, Int. J. Mod. Phys. B **12**, 809 (1998); N.E. Mavromatos and A. Momen, Mod. Phys. Lett. A **A13**, 1765 (1998); K. Farakos, G. Koutsoumbas, N.E. Mavromatos, and A. Momen, hep-ph/9905272.
[16] K. Farakos, G. Koutsoumbas, and N.E. Mavromatos, Phys. Lett. B **431**, 147 (1998); Int. J. Mod. Phys. B **12**, 2475 (1998); W.V. Liu, ''Parity breaking and phase transition induced by a magnetic field in superconductors,'' cond-mat/9808134.
[17] E.J. Ferrer and V. de la Incera, Phys. Rev. D **58**, 065008 (1998); ''Yukawa coupling contribution to magnetic field induced dynamical mass,'' hep-ph/9810473.
[18] G.W. Semenoff and L.C.R. Wijewardhana, Phys. Rev. Lett. **63**, 2633 (1989); Phys. Rev. D **45**, 1342 (1992); H. Hamidian, S. Jaimungal, G.W. Semenoff, P. Suranyi, and L.C.R. Wijewardhana, *ibid.* **53**, 5886 (1996).
[19] G.W. Semenoff, Phys. Rev. Lett. **53**, 2449 (1984).
[20] A.J. Niemi and G.W. Semenoff, Nucl. Phys. **B269**, 131 (1986); Phys. Rep. **135**, 99 (1986).
[21] G.W. Semenoff, Phys. Rev. D **37**, 2838 (1988).
[22] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).
[23] E. Brezin, J.C. Le Guillou, and J. Zinn-Justin, Phys. Rev. B **10**, 892 (1974).
[24] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 1996).
[25] J.A. Gracey, Nucl. Phys. **B341**, 403 (1990); **B367**, 657 (1991); Phys. Rev. D **50**, 2840 (1994).