Ground state energy of a spinor field in the background of a finite radius flux tube

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We develop a formalism for the calculation of the ground state energy of a spinor field in the background of a cylindrically symmetric magnetic field. The energy is expressed in terms of the Jost function of the associated scattering problem. Uniform asymptotic expansions needed are obtained from the Lippmann-Schwinger equation. The general results derived are applied to the background of a finite radius flux tube with a homogeneous magnetic field inside and the ground state energy is calculated numerically as a function of the radius and the flux. It turns out to be negative, remaining smaller by a factor of α than the classical energy of the background except for very small values of the radius which are outside the range of applicability of QED.
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I. INTRODUCTION

The ground state energy of the spinor field in the background of a magnetic field has been investigated since the early days of quantum electrodynamics (QED). So, for instance, the effective potential in a strong magnetic field is well known. For a weak field it takes increasing positive values, for stronger fields it turns down, and for $B\to\infty$ it becomes

$$V_{\rm eff}\sim-\frac{(eB)^2}{12\pi^2}\ln\sqrt{\frac{2eB}{m_{\rm e}}}, \tag{1}$$

where $m_{\rm e}$ is the electron mass. For details see, e.g., Ref. [1].

Actual interest in this topic results, for example, from the symmetry restoration due to a magnetic field in electroweak theory or from the influence of that field on the character of the phase transition [2]. While most work has been done in homogeneous magnetic fields, mainly because in that case explicit formulas exist, the extension to inhomogeneous fields is of interest.

Some work is this direction has already been done. For example for a flux where the magnetic field is concentrated on the surface of the tube, the fermion determinant was calculated in Ref. [3]. However, in that case the classical energy is infinite. In Ref. [4], in 2+1 dimensions a magnetic field homogeneous in the y direction and with a special shape in the x direction allowing for explicit formulas had been considered. The result for the ground state energy per unit area of the planes was expressed in quite elementary functions, which allowed the discussion of the total energy within the family of fields considered. It was shown that the system is driven towards a uniform magnetic field. An extension of these formulas to the (3+1)-dimensional case was given in Ref. [5].

Furthermore, there exists a number of investigations of the density per unit volume $\epsilon(r)$ of the ground state energy, where r is the radial coordinate in the perpendicular plane, in the background of an infinitely thin magnetic flux tube. Partly, this is motivated by the close relation to the Aharonov-Bohm effect. The first investigation of this kind was done in Ref. [6] for the ground state energy density of a scalar field, later reconsidered and generalized to the spinor case in Ref. [7], see also Ref. [8]. Extensively investigated is the (2+1)-dimensional case, see Ref. [9], and papers cited therein. Also, there are similar investigations in the background of an infinitely thin cosmic string [10]. The calculations in the background of an infinitely thin magnetic flux tube have the drawback that the energy density per unit volume cannot be integrated to get the energy density $\mathcal{E}$ per unit length due to the singular behavior near the string, $\epsilon(r)\sim r^{-4}$, which follows already from dimensional reasons.

In addition one might consider the combined effect of boundaries and background fields. This has been started in Ref. [11], where imposing spectral boundary conditions the spinor field was considered in a finite region of space in the background of an Aharonov-Bohm flux string.

In the present paper we continue the consideration of inhomogeneous magnetic background fields and calculate the ground state energy of the spinor field in the background of a straight magnetic flux tube of *finite* radius R, more exactly, the energy density per unit length. The reason to consider a flux tube of finite radius R is that the associated classical energy is finite and the dependence of the total energy when R varies while the flux is fixed can be analyzed. The interesting question in this context is if some radius R_m exists where the complete energy, i.e., the sum of classical enegy of the magnetic field and the ground state energy of the spinor field, is minimized and the magnetic string gets stable. In doing this analysis we use and generalize the formalism developed in Refs. [12] and [13] for a smooth scalar background field. In Sec. II we start discussing in detail the renormalization of the ground state energy. We will normalize the energy in such a way that it vanishes for the electron mass $m_e\to\infty$, the resulting massless limit will also be considered. The needed counterterms and the subtraction employed is then elegantly described using the heat-kernel language. After having explained in detail the renormalization procedure

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we express the ground state energy in terms of the Jost function of the associated scattering problem. The procedure developed in Refs. [12,13] consists of adding and subtracting the uniform asymptotic expansion of the Jost function. This is done in Sec. IV using a perturbative expansion of the Lippmann-Schwinger equation. Various details of this calculation are relegated to the Appendixes A and B. In the remaining part of the paper, Secs. V and VI, an analytical as well as numerical description of the ground state energy for spinors in the presence of a finite radius flux tube is performed. Some details can be found in Appendix C. In the Conclusion we summarize the main results of the paper.

II. BASIC FORMULAS AND THE RENORMALIZATION

The considered background is a straight magnetic flux tube of finite radius R, i.e., the magnetic field

$$\vec{B}(\vec{x})=\frac{\phi}{2\pi}h(r)\vec{e}_z, \tag{2}$$

where $h(r)$ is a profile function with compact support in the radial variable $r=\sqrt{x^2+y^2}$ in the plane perpendicular to the tube. By the normalization $\int_0^\infty dr r h(r)=1$, ϕ has the meaning of the flux inside the tube. The corresponding vector potential can be chosen to be

$$\vec{A}(\vec{x})=\frac{\phi}{2\pi}\frac{a(r)}{r}\vec{e}_\varphi. \tag{3}$$

The profile functions $h(r)$ and $a(r)$ are connected by the relation $h(r)=a'(r)/r$. Below, we will use the special case of a homogenoeus magnetic field inside the tube, where these functions read

$$h(r)=\frac{2}{R^2}\Theta(R-r),\quad a(r)=\frac{r^2}{R^2}\Theta(R-r)+\Theta(r-R). \tag{4}$$

In this case the solutions of the field equations can be expressed in terms of Bessel and hypergeometric functions.

Another choice could be a magnetic field concentrated on the surface of the cylinder, $h(r)=\delta(r-R)/R$, where the solutions can solely be expressed in terms of Bessel functions. But in that case the classical energy of the background is infinite.

The classical energy of the background (per unit length of the string) is

$$\mathcal{E}^{\text{class}}\equiv\frac{1}{2}\int d\vec{x}\vec{B}^2=\frac{\phi^2}{4\pi}\int_0^\infty dr\, rh(r)^2. \tag{5}$$

The ground state energy of the spinor field in that background is given by

$$\mathcal{E}=-\frac{\mu^{2s}}{2}\sum_{(n,\epsilon)}e_{(n,\epsilon)}{}^{1-2s}. \tag{6}$$

Here, the minus sign in front of the rhs. accounts for the spinor obeying anticommutation relations s ($s>2$ and $s\to 0$ in the end) is the regularization parameter in the zeta functional regularization which we use here and μ is the arbitrary dimensional parameter entering this regularization. In fact, due to the translational invariance along the axis of the flux tube, we have to consider the energy density per unit length. This will be taken into account below.

The $e_{(n,\epsilon)}$ are the eigenvalues of the Hamiltonian

$$\mathcal{H}=-i\gamma^0\gamma^j[\partial/\partial x^j-ieA_j(x)]+\gamma^0 m_{\mathrm{e}} \tag{7}$$

which follows from the Dirac equation. Here, $\epsilon=\pm 1$ is the sign of the one particle energies $e_{(n,\epsilon)}$ for the particle respectively the antiparticle which themselves are chosen to be positive. All other quantum numbers are included into (n). Furthermore, the index j denotes spatial indices only and summation over it is included.

For the renormalization we follow the standard procedure using the heat kernel expansion. The ground state energy can be expressed by the heat kernel $K(t)$ of $\mathcal{H}^2$,

$$\mathcal{E}=-\frac{\mu^{2s}}{2}\int_0^\infty \frac{dt t^{s-3/2}}{\Gamma(s-1/2)}\; K(t)$$

with the asymptotic expansion for $t\to 0$

$$K(t)\sim\frac{\mathrm{e}^{-tm^2}}{(4\pi t)^{d/2}}\sum_{n\geqslant 0}a_n t^n,$$

where d is the dimension of the manifold, $d=3$ in our case.

The heat-kernel coefficients a_n for the operator under consideration are well known. The relevant operator $\mathcal{H}^2$ reads explicitly

$$\mathcal{H}^2=-\nabla^j\nabla_j+\frac{1}{2}\sigma^{ij}F_{ij}+m^2,$$

with $\sigma^{\mu\nu}=(i/2)[\gamma^\mu\gamma^\nu]$ and the leading coefficients can be found, for example, in Ref. [14].

For $n=0$ we note that the coefficient is independent on the background field and corrresponds to the contribution of the empty Minkowski space. We drop this contribution without further comment. The coefficient a_1 is zero and for a_2 the general formula reads

$$a_2=\mathrm{Tr}\int d\vec{x}\left(-\frac{1}{12}F_{ij}^2+\frac{1}{8}(\sigma^{ij}F_{ij})^2\right). \tag{8}$$

Here, the trace is over the spinors. The integration along the axis of the flux tube gives the corresponding volume by which we have to divide. So the following formulas have to be understood always as densities with respect to this axis.

The trace in Eq. (8) can be carried out and by means of Eq. (3) we arrive at

$$a_2=\frac{8\pi}{3}\delta^2\int_0^\infty dr\, rh(r)^2, \tag{9}$$

where the notation

$$\delta = \frac{e\phi}{2\pi} \tag{10}$$

is introduced. Here, e is the electron charge, and we can rewrite this relation as $\delta = \sqrt{\alpha/\pi}\phi$, where α is the fine structure constant.

Using the heat kernel expansion it can be shown that the divergent part of the ground state energy results from the contribution of the heat kernel coefficients a_n with $n \leqslant 2$. We define

$$\mathcal{E}^{\text{div}} = \frac{a_2}{32\pi^2}\left(\frac{1}{s} - 2 + \ln\frac{4\mu^2}{m_e^2}\right). \tag{11}$$

Then the renormalized ground state energy is given by

$$\mathcal{E}^{\text{ren}} = \mathcal{E} - \mathcal{E}^{\text{div}}. \tag{12}$$

Here the limit $s \to 0$ can be performed because the pole part is subtracted. In general, the definition of $\mathcal{E}^{\text{div}}$ is not unique. By the definition (11), the normalization condition

$$\mathcal{E}^{\text{ren}} \to 0 \quad \text{for} \quad m_e \to \infty \tag{13}$$

is assured. This normalization condition is natural as it implies that a very massive field should not show quantum fluctuations. On the other hand, it fixes the arbitrariness which came in with the parameter μ in Eq. (6).

Along with the subtraction of $\mathcal{E}^{\text{div}}$ from $\mathcal{E}$ it must be added to $\mathcal{E}^{\text{class}}$. This is equivalent to a renormalization of the flux according to

$$\phi^2 \to \phi^2 + \frac{(e\phi)^2}{12\pi^2}\left(\frac{1}{s} - 2 + \ln\frac{4\mu^2}{m_e^2}\right). \tag{14}$$

From this renormalization procedure it is possible to determine the leading asymptotic behavior of the renormalized ground state energy when the radius R of the flux tube tends to zero. In fact, this could have been done using the arguments given in Ref. [15] where the general scaling behavior of the Casimir energy was investigated. The point is simply that the regularized ground state energy (6) has a series expansion with respect to powers of the mass m_e. Note that this is not affected by the zero modes, which are present here (see Ref. [16]), because we consider the ground state energy and not the determinant of the operator $\mathcal{P}$. Then, by means of Eqs. (12) and (11) we subtract a contribution containing $\ln m_e$. Therefore, the renormalized ground state energy $\mathcal{E}^{\text{ren}}$ becomes for $m_e \to 0$ proportional to $\ln m_e$. Now, for dimensional reasons it can be written as

$$\mathcal{E}^{\text{ren}} = \frac{f(Rm_e)}{R^2},$$

where f is some function of the dimensionless combination Rm_e. Consequently, for $R \to 0$ the behavior must be

$$\mathcal{E}^{\text{ren}} \sim \frac{-a_2}{16\pi^2}\ln\frac{1}{Rm_e}$$

and, for instance, with the background (4)

$$\mathcal{E}^{\text{ren}} \sim \frac{-1}{3\pi}\frac{\delta^2}{R^2}\ln\frac{1}{Rm_e}. \tag{15}$$

This behavior will be confirmed below in the course of the explicit calculations. In fact, this behavior follows already from the heat kernel expansion and has the same origin as the asymptotics (1). For fixed flux ϕ, small R correspond to large B and, using Eq. (10) and $\phi = \pi R^2 B$, formula (15) turns into Eq. (1) [note V_{eff} (1) is the energy density per unit volume]. For the same reason it can obviously be improved using the renormalization group. Because the theory is infrared free and the limit $R \to 0$ corresponds to high momenta it is impossible to make R too small.

When comparing Eq. (15) with the classical energy of this background,

$$\mathcal{E}^{\text{class}} = \frac{\phi^2}{2\pi R^2}, \tag{16}$$

one could think that the complete energy can be made negative for sufficiently small R. However, this would require $Rm_e < \exp[-3\pi/(2\alpha)]$, which is far outside the range of applicability of QED and also ruled out by the renormalization group argument.

Note that in the massless case the renormalization scheme must be modified. As discussed in Ref. [19], for $a_2 \neq 0$ which is the case here generically [$a_2 = 0$ means a vanishing background, cf. Eq. (9)] and which represents the conformal anomaly, there is no transition for the renormalized ground state energy $\mathcal{E}^{\text{ren}}$ from $m_e \neq 0$ to $m_e = 0$. Usually, in the massless case in the definition of $\mathcal{E}^{\text{div}}$ similar to Eq. (11) instead of the mass, another parameter, the renormalization scale Λ, is used. Therefore the renormalized ground state energy contains a nonuniqueness proportional to the heat kernel coefficient a_2, i.e., proportional to the classical energy [see formulas (5) and (9)].

III. GROUND STATE ENERGY EXPRESSED IN TERMS OF THE JOST FUNCTION

We express the regularized ground state energy (6) in terms of the Jost function of the scattering problem associated with the operator $\mathcal{H}$, Eq. (7). Because the background is translationally invariant along the third axis, by means of

$$\Psi_{(n,\epsilon)}(\vec{x}) = e^{ip_3x^3}\begin{pmatrix}\Phi \\ \psi\end{pmatrix},$$

we rewrite the corresponding Dirac equation using the spiral representation of the gamma matrices in the form

$$\begin{pmatrix} p_0+\hat{L}-m_e\sigma_3 & p_3\sigma_3 \\ p_3\sigma_3 & p_0+\hat{L}+m_e\sigma_3 \end{pmatrix} \begin{pmatrix} \Phi \\ \psi \end{pmatrix} = 0, \tag{17}$$

with $\hat{L}=i\Sigma_{i=1}^{2}\sigma_i(\partial/\partial x^i+ieA_i)$. It is sufficient to consider $p_3=0$ and one of the two decoupled equations. Then by means of

$$\Phi = \begin{pmatrix} ig_1(r) & e^{-i(m+1)\varphi} \\ g_2(r) & e^{-im\varphi} \end{pmatrix}$$

$(m=-\infty,\infty)$ we arrive at the equation

$$\begin{pmatrix} p_0-m_e & \frac{\partial}{\partial r}-\frac{m-\delta a(r)}{r} \\ -\frac{\partial}{\partial r}-\frac{m+1-\delta a(r)}{r} & p_0+m_e \end{pmatrix} \Phi(r)=0, \tag{18}$$

where we introduced the notation

$$\Phi(r) = \begin{pmatrix} g_1(r) \\ g_2(r) \end{pmatrix} \tag{19}$$

for the solution $\Phi(r)$.

The solutions in the exterior space, $r>R$, are

$$\Phi_J^0(r) = \begin{cases} \begin{pmatrix} \sqrt{p_0+m_e} & J_{m-\delta+1}(kr) \\ \sqrt{p_0-m_e} & J_{m-\delta}(kr) \end{pmatrix} & \text{for} \quad m+1-\delta>0, \\ \begin{pmatrix} \sqrt{p_0+m_e} & J_{\delta-m-1}(kr) \\ -\sqrt{p_0-m_e} & J_{\delta-m}(kr) \end{pmatrix} & \text{for} \quad m-\delta<0. \end{cases} \tag{20}$$

The Jost solution of Eq. (17) is the solution which behaves for $r\to 0$ as the free solution (20). Its asymptotics for $r\to\infty$ can be written as

$$\Phi(r) \sim \frac{1}{2}[f_m(k)\Phi^0_{H^{(2)}}(r)+\overline{f}_m(k)\Phi^0_{H^{(1)}}(r)], \tag{21}$$

where $\Phi_{H^{(1,2)}}$ are the solutions (20) with the Hankel functions instead of the Bessel function. The coefficient $f_m(k)$ is the Jost function and $\overline{f}_m(k)$ its complex conjugate.

The ground state energy can be expressed in terms of the Jost function much in the same way as in the scalar case [13]. However, due to the translational invariance in the direction parallel to the flux tube, we have the energy density

$$\mathcal{E} = -\frac{\mu^{2s}}{2}\int_{-\infty}^{\infty}\frac{dk_3}{2\pi}\sum_{(n,\epsilon)}(k_3^2+e^2_{(n,\epsilon)})^{1/2-s}, \tag{22}$$

instead of the general formula (6). After carrying out the integration over k_3 we arrive at

$$\mathcal{E} = -\frac{\mu^{2s}}{2}\frac{1}{2\sqrt{\pi}}\frac{\Gamma(s-1)}{\Gamma(s-1/2)}\sum_{(n)}(e^2_{(n)})^{1-s},$$

where (n) denotes the remaining quantum numbers in the plane perpendicular to the axis of the tube. Now the sum over these (n) can be expressed through the Jost function and we get to the relevant order in s,

$$\mathcal{E} = C_s\sum_{m=-\infty}^{\infty}\int_{m_e}^{\infty}dk(k^2-m_e^2)^{1-s}\frac{\partial}{\partial k}\ln f_m(ik) \tag{23}$$

with $C_s=\{1+s[-1+2\ln(2\mu)]\}/(2\pi)$. This representation can be obtained in much the same way as done in the scalar case in Ref. [13]. One has to take into account the known analytical properties of the Jost function (which differ from that in the scalar case). One has to use as an intermediate step a finite quantization volume with appropriate bound conditions (bag conditions work well). Then, in the course of tending this volume to infinity, the translational invariant contribution from the Minkowski space must be dropped and the remaining finite part, after a deformation of the integration contour, just delivers Eq. (23).

We remark that in the considered problem there are zero modes [16] (at $k=0$). In the sum in Eq. (22) they have to be taken into account. Just in the same way as shown in detail in the scalar case in Ref. [13] for the bound states they do not show up explicitly in representation Eq. (23).

Here we have taken into accout that both signs of the one particle energies as well as both signs of the spin projection give equal contributions to the ground state energy thus resulting in a factor of 4 which is included into C_s. This expression will be used in the calculations below.

The renormalization of the ground state energy is defined by Eq. (12). The remaining task is to perform the analytical continuation as $s\to 0$. However, this is not immediately possible using representation (23) for $\mathcal{E}$. To continue we use the uniform asymptotic expansion $\ln f_m^{as}(ik)$ of the logarithm of the Jost function, $\ln f_m(ik)$, defined in such a way that the difference

$$\ln f_m(ik)-\ln f_m^{as}(ik)=O\left(\frac{1}{m^4}\right) \tag{24}$$

is of the order m^{-4} in the limit $m\to\infty$, $k\to\infty$ for m/k fixed. Then we split the renormalized ground state energy by adding and subtracting $\ln f_m^{as}(ik)$ to get

$$\mathcal{E}^{ren}=\mathcal{E}^{f}+\mathcal{E}^{as} \tag{25}$$

with the ''finite'' part

$$\mathcal{E}^{f}=\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{m_e}^{\infty}dk(k^2-m_e^2)\frac{\partial}{\partial k}[\ln f_m(ik)-\ln f^{as}(ik)], \tag{26}$$

where it was possible to put $s=0$, and the ''asymptotic'' part

$$\mathcal{E}^{as}=C_s\sum_{m=-\infty}^{\infty}\int_{m_e}^{\infty}dk(k^2-m_e^2)^{1-s}\frac{\partial}{\partial k}\ln f_m^{as}(ik)-\mathcal{E}^{div}. \tag{27}$$

Here, we included the subtraction of $\mathcal{E}^{\mathrm{div}}$ according to Eq. (12). The continuation to $s=0$ will yield a finite result because the pole contributions cancel. This continuation will be done in Sec. V. But because the expression for $\ln f^{\mathrm{as}}$ is quite simple, this task can be done analytically.

The subdivision (25) of $\mathcal{E}$ is not unique, only the condition (24) has to be satisfied. The inclusion of higher orders into $\ln f^{\mathrm{as}}$ would speed up the convergence of the momentum sum in $\mathcal{E}^{\mathrm{f}}$, for instance. But we will use $\ln f^{\mathrm{as}}$ in the minimal form obeying Eq. (24).

IV. THE UNIFORM ASYMPTOTIC EXPANSION OF THE JOST FUNCTION

The uniform asymptotic expansion of the Jost function can be obtained from the Lippmann-Schwinger equation in much the same way as it was done in the scalar case [13]. We rewrite Eq. (18) in the form

$$\begin{pmatrix} p_0-m_{\mathrm{e}} & \frac{\partial}{\partial r}-\frac{m}{r} \\ -\frac{\partial}{\partial r}-\frac{m+1}{r} & p_0+m_{\mathrm{e}} \end{pmatrix}\Phi(r) = \frac{-\delta a(r)}{r}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Phi(r)\equiv\Delta\mathcal{P}(r)\Phi(r).$$

The operator on the left-hand side can be inverted using the free solutions (20) and we get the integral equation

$$\Phi(r)=\Phi^0(r)+\int_0^r dr'\,r'\,g(r,r')\Delta\mathcal{P}(r')\Phi(r') \tag{28}$$

with

$$g(r,r')=-\frac{\pi}{2i}[\Phi_J(r)\Phi^T_{H(1)}(r')-\Phi_J(r')\Phi^T_{H(1)}(r)], \tag{29}$$

where Φ^T means the transposed of Φ. Inserting Φ_J^0 (20) for $\Phi^0(r)$ in the right-hand side of Eq. (28), this equation determines just the Jost solution. Using the asymptotic expansion of the Hankel functions and comparing with Eq. (21) we obtain for the Jost function the representation

$$f_m(k)=1-\frac{\pi}{2i}\int_0^\infty dr\,r\,\Phi^T_{H(1)}(r)\Delta\mathcal{P}(r)\Phi(r). \tag{30}$$

Equation (28) can be iterated. It turns out that we need all contributions up to the fourth power in $\Delta\mathcal{P}$ in order to satisfy condition (24). Note that in the scalar case the second power had been sufficient. Iterating Eq. (28) we obtain

$$\begin{aligned}\Phi(r)=&\Phi_J^0(r)+\int_0^r dr'\,r'\,g(r,r')\Delta\mathcal{P}(r')\Phi_J^0(r')\\ &+\int_0^r dr'\,r'\int_0^{r'}dr''\,r''\,g(r,r')\Delta\mathcal{P}(r')\\ &\times g(r',r'')\Delta\mathcal{P}(r'')\Phi_J^0(r'')\\ &+\int_0^r dr'\,r'\int_0^{r'}dr''\,r''\int_0^{r''}dr'''\,r'''\,g(r,r')\Delta\mathcal{P}(r')\\ &\times g(r',r'')\Delta\mathcal{P}(r'')g(r'',r''')\Delta\mathcal{P}(r''')\Phi_J^0(r''')\\ &+\mathcal{O}[(\Delta\mathcal{P})^4].\end{aligned} \tag{31}$$

This expression has to be inserted into Eq. (30). In fact, we need the logarithm of the Jost function. Therefor the appearing expression must be expanded again. We write the result as $\ln f_m(k)=\Sigma_{n\geq 1}\ln f_m^{(n)}(k)$ where n denotes the power of the operator $\Delta\mathcal{P}$. Up to the fourth order we obtain (for several details see Appendix A)

$$\ln f_m^{(1)}(k)=-\left(\frac{\pi}{2i}\right)\int_0^\infty dr\,r\,\Phi^T_{H(1)}(r)\Delta\mathcal{P}(r)\Phi_J(r), \tag{32}$$

$$\begin{aligned}\ln f_m^{(2)}(k)=&-\left(\frac{\pi}{2i}\right)^2\int_0^\infty dr\,r\int_0^r dr'\,r'\,\Phi^T_{H(1)}(r)\Delta\mathcal{P}(r)\\ &\times\Phi_{H(1)}(r)\Phi_J^T(r')\Delta\mathcal{P}(r')\Phi_J(r'),\end{aligned} \tag{33}$$

$$\begin{aligned}\ln f_m^{(3)}(k)=&-2\left(\frac{\pi}{2i}\right)^3\int_0^\infty dr\,r\int_0^r dr'\,r'\\ &\times\int_0^{r'}dr''\,r''\,\Phi^T_{H(1)}(r)\Delta\mathcal{P}(r)\Phi_{H(1)}(r)\\ &\times\Phi^T_{H(1)}(r')\Delta\mathcal{P}(r')\Phi_J(r')\Phi_J^T(r'')\Delta\mathcal{P}(r'')\Phi_J(r''),\end{aligned} \tag{34}$$

$$\begin{aligned}\ln f_m^{(4)}(k)=&-\left(\frac{\pi}{2i}\right)^4\int_0^\infty dr\,r\int_0^r dr'\,r'\int_0^{r'}dr''\,r''\int_0^{r''}dr'''\,r'''\\ &\times[4\Phi^T_{H(1)}(r)\Delta\mathcal{P}(r)\Phi_{H(1)}(r)\Phi^T_{H(1)}(r')\\ &\times\Delta\mathcal{P}(r')\Phi_J(r')\Phi^T_{H(1)}(r'')\Delta\mathcal{P}(r'')\Phi_J(r'')\\ &\times\Phi_J^T(r''')\Delta\mathcal{P}(r''')\Phi_J(r''')+2\Phi^T_{H(1)}(r)\Delta\mathcal{P}(r)\\ &\times\Phi_{H(1)}(r)\Phi^T_{H(1)}(r')\Delta\mathcal{P}(r')\Phi_{H(1)}(r')\Phi_J^T(r'')\\ &\times\Delta\mathcal{P}(r'')\Phi_J(r'')\Phi_J^T(r''')\Delta\mathcal{P}(r''')\Phi_J(r''')],\end{aligned} \tag{35}$$

where rearrangings of the integration domains had been made.

Now, because we are interested in the Jost function for imaginary momentum, we turn from the Bessel functions to the corresponding modified ones. Then we have to perform the uniform asymptotic expansion of these expressions.

Before doing this we note that it turned out to be more convenient not to use the orbital momentum m as the expansion parameter, but instead

$$\nu=\begin{cases} m+\dfrac{1}{2} & \text{for } m=0,1,2,\ldots, \\ -m-\dfrac{1}{2} & \text{for } m=-1,-2,\ldots, \end{cases} \tag{36}$$

with $\nu=\frac{1}{2},\frac{3}{2},\ldots,$ in both cases.

Then we need uniform asymptotic expansions of the modified Bessel functions for $\nu\to\infty$, z fixed, of the following type:

$$\left.\begin{matrix} K_{\nu+1/2}(\nu z) \\ I_{\nu+1/2}(\nu z)\end{matrix}\right\}\sim\sqrt{\pi}^{-\epsilon}\sqrt{\frac{t}{2\nu}}\left(\frac{1-t}{1+t}\right)^{\epsilon/4} e^{\epsilon\nu\eta(z)}\exp\left\{\frac{-6t^2-5\epsilon t^3}{24\nu}+\frac{-4\epsilon t^3-4t^4+5\epsilon t^5+5t^6}{16\nu^2}\right.$$
$$\left.+\frac{-2160t^4-2304\epsilon t^5+7440t^6+7695\epsilon t^7-5400t^8-5525\epsilon t^9}{5760\nu^3}+\ldots\right\}$$

with $\epsilon=\mp1$ for K_ν respectively I_ν, $\eta=\sqrt{1+z^2}+\ln[z/(1+\sqrt{1+z^2})]$ and $t=(1+z^2)^{-1/2}$ which may be derived from the commonly known one [17] by a corresponding reexpansion. Similar expansion for $K_{\nu-1/2}$ and $I_{\nu-1/2}$ are also used.

Now we insert this expansion into the logarithm of the Jost function. Then the integrations over r''', r'' and r' can be carried out successively by the saddle point method [only equal arguments in the functions $\eta(z)$ yield contributions which do not exponentially decrease for $\nu\to\infty$] as done in Ref. [13]. In doing so it becomes apparant that terms up to the fourth power in $\Delta\mathcal{P}$ contribute to the asymptotic expansion in ν up to the order ν^{-3}. The relevant saddle point expansion is presented in the Appendix B, see Eq. (B1). Finally we collect all contributions up to this order and define

$$\ln f^{\mathrm{as}}(ik)=\sum_{n=1}^{3}\sum_{j=n}^{3n}\int_0^\infty\frac{dr}{r}\;X_{n,j}\frac{t^j}{\nu^n} \tag{37}$$

with the notation $t=[1+(rk/\nu)^2]^{-1/2}$. The coefficients turn out to be

$$X_{1,1}=\frac{(a\delta)^2}{2},\quad X_{1,3}=-\frac{(a\delta)^2}{2},$$

$$X_{2,2}=\frac{1}{4}\delta^2(a^2-raa'),$$

$$X_{2,4}=\frac{1}{4}\delta^2(-3a^2+raa'),\quad X_{2,6}=\frac{1}{2}(a\delta)^2$$

$$X_{3,3}=\frac{1}{4}\delta^2\left(a^2-raa'+\frac{1}{2}r^2aa''-\frac{1}{2}\delta^2a^4\right),$$

$$X_{3,5}=\frac{1}{8}\delta^2\left(-\frac{39}{2}a^2+7raa'-r^2aa''+6\delta^2a^4\right),$$

$$X_{3,7}=\frac{1}{8}\delta^2(35a^2-5raa'-5\delta^2a^4),$$

$$X_{3,9}=\frac{-35}{16}\delta^2a^2. \tag{38}$$

Here, a means the profile function $a(r)$ in Eq. (3). Below, when inserting this expansion into $\mathcal{E}^{\mathrm{as}}$, the sum over the orbital momentum must be performed. There some contributions cancel, for instance, those which are proportional to δ and δ^3. They are not shown in formula (38).

V. THE ASYMPTOTIC PART OF THE GROUND STATE ENERGY

The asymptotic part of the ground state energy is given by Eq. (27) and the expression (37) for the asymptotic expansion of the logarithm of the Jost function. We rewrite it in the form

$$\mathcal{E}^{\mathrm{as}}=2C_s\sum_{\nu=1/2,3/2,\ldots}\int_{m_{\mathrm{e}}}^\infty dk(k^2-m_{\mathrm{e}}^2)^{1-s}$$
$$\times\frac{\partial}{\partial k}\int_0^\infty\frac{dr}{r}\sum_{n=1}^{3}\sum_{j=n}^{3n}X_{n,j}\frac{t^j}{\nu^n}-\mathcal{E}^{\mathrm{div}}. \tag{39}$$

Here, the sum over the orbital momentum m in Eq. (27) is rewritten as a sum over ν, Eq. (36). By means of Eq. (C1) it will be replaced by two integrals. The contribution resulting from the first integral can be calculated explicitly using formula (C2). Together with the explicit expressions for $X_{n,i}$ (38), after a straightforward calculation, it can be seen to cancel exactly $\mathcal{E}^{\mathrm{div}}$ [for arbitrary profile function $a(r)$].

So we are left with the contribution resulting from the second integral in Eq. (C1). There the integration over k can be carried out by means of formula (C3). Then defining

$$\Sigma_{n,j}(x)=\frac{\Gamma(s+j/2-1)}{\Gamma(j/2)}\frac{-i}{x^j}\int_0^\infty\frac{d\nu}{1+\exp(2\pi\nu)}\times\left(\frac{(i\nu)^{j-n}}{[1+(i\nu/x)^2]^{s+j/2-1}}-\frac{(-i\nu)^{j-n}}{[1+(-i\nu/x)^2]^{s+j/2-1}}\right) \tag{40}$$

we arrive at

$$\mathcal{E}^{\rm as}=-\frac{1}{\pi}m_{\rm e}^2\sum_{n=1}^{3}\sum_{j=n}^{3n}\int_0^\infty\frac{dr}{r}X_{n,j}\Sigma_{n,j}(rm_{\rm e}). \tag{41}$$

In the functions $\Sigma_{n,j}(x)$ the analytic continuation to $s=0$ has to be performed. For this reason one has to integrate by parts several times to get rid of the singular denominator. The resulting expressions are shown in the Appendix C, formula (C4). Now in the integration over r, it is useful to integrate by parts which reduces to the substitutions $raa'\to-\frac{1}{2}a^2r\partial_r$ and $r^2aa''\to-a'^2+\frac{1}{2}a^2r\partial_r^2r$. We note that after doing this the contribution resulting from $n=2$ vanishes, already before the integration over r will be carried out. Then the integrations over r and ν can be interchanged and after rescaling $\nu\to\nu rm_{\rm e}$ we get

$$\mathcal{E}^{\rm as}=\frac{-16}{\pi}\int_0^\infty\frac{dr}{r^3}\{a(r)^2g_1(rm_{\rm e})-r^2a(r)'^2g_2(rm_{\rm e})+a(r)^4g_3(rm_{\rm e})\}, \tag{42}$$

with

$$g_i(x)=\int_x^\infty d\nu\sqrt{\nu^2-x^2}f_i(\nu)\quad(i=1,2,3).$$

The functions f_i are displayed in Appendix C, Eq. (C5). This is the final formula for $\mathcal{E}^{\rm as}$ for an arbitrary profile function $a(r)$.

For the homogeneous magnetic field inside the flux tube, i.e., for the profile function (4), the integration over r can be performed explicitly. After elementary calculations we get

$$\begin{aligned}\mathcal{E}^{\rm as}=\frac{-4}{\pi R^2}\Bigg\{&\int_0^{Rm_{\rm e}}d\nu\frac{\nu^3}{3(Rm_{\rm e})^2}\delta^2\left[f_1(\nu)-4f_2(\nu)+\frac{8}{35}\delta^2f_3(\nu)\left(\frac{\nu}{m_eR}\right)^4\right]+\int_{Rm_{\rm e}}^\infty d\nu\Bigg[f_1(\nu)\delta^2\Bigg(\frac{\nu^3-\sqrt{\nu^2-(Rm_{\rm e})^2}^{\,3}}{3(Rm_{\rm e})^2}\\&+\frac{\sqrt{\nu^2-(Rm_{\rm e})^2}}{2}-\frac{(Rm_{\rm e})^2}{2\nu}\ln\frac{[\nu+\sqrt{\nu^2-(Rm_{\rm e})^2}]}{m_eR}\Bigg)-4f_2(\nu)\delta^2\frac{\nu^3-\sqrt{\nu^2-(Rm_{\rm e})^2}^{\,3}}{3(Rm_{\rm e})^2}\\&+f_3(\nu)\delta^4\Bigg(\frac{8\nu^7-\sqrt{\nu^2-(Rm_{\rm e})^2}[8\nu^6+4\nu^4(Rm_{\rm e})^2+3\nu^2(Rm_{\rm e})^4-15(Rm_{\rm e})^6]}{105(Rm_{\rm e})^6}+\frac{\sqrt{\nu^2-(Rm_{\rm e})^2}}{2}\\&-\frac{(Rm_{\rm e})^2}{2\nu}\ln\frac{(\nu+\sqrt{\nu^2-(Rm_{\rm e})^2})}{m_eR}\Bigg)\Bigg]\Bigg\}.\end{aligned} \tag{43}$$

This expression consists of two parts which we write in the form

$$\mathcal{E}^{\rm as}=\delta^2\frac{e_1(Rm_{\rm e})}{R^2}+\delta^4\frac{e_2(Rm_{\rm e})}{R^2}. \tag{44}$$

Here e_1, respectively, e_2 describe the contributions proportional to the second, respectively, fourth power of the coupling δ to the background. They are shown in Fig. 1. Their behavior for $x\to0$ can be calculated from Eq. (43) and we have

$$e_1(x)\sim\frac{\ln x}{3\pi}+0.1348+O(x),$$

$$e_2(x)\sim-0.0354+O(x). \tag{45}$$

The logarithmic contribution is just that which was to be expected from the heat kernel expansion (15).

VI. THE "FINITE" PART OF THE GROUND STATE ENERGY AND NUMERICAL RESULTS

The finite part of the ground state energy is defined by Eq. (26) together with the asymptotic expansion of the Jost func-

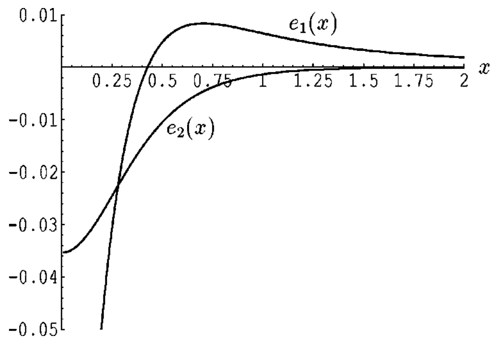

FIG. 1. The functions e_1 and e_2 appearing in the asymptotic part of the ground state energy.

tion, Eq. (37). In general, these quantities can be calculated only numerically. We consider here the case of a homogeneous magnetic field inside the tube as given by Eq. (4). In that case the solutions of the field equations are known, they are hypergeometric functions inside and Bessel functions outside. As these formulas are in general quite well known and easy to derive, we give here only the result. The notations are close to that in the paper [18]. For positive orbital momentum [$m=0,1,\ldots,$ in formula (36)] we have

$$f_\nu(ik)=2\left(\frac{kR}{2}\right)^{\nu+1/2}\frac{\exp(-\delta/2)}{\Gamma(\nu+3/2)}\left\{\frac{kR}{2}K_{\nu-1/2-\delta}(kR)\times{}_1F_1\left(1+\frac{(kR)^2}{4\delta},\nu+\frac{3}{2};\delta\right)+\left(\nu+\frac{1}{2}\right)\times K_{\nu+1/2-\delta}(kR)\,{}_1F_1\left(\frac{(kR)^2}{4\delta},\nu+\frac{1}{2};\delta\right)\right\},\quad (46)$$

and for negative m ($m=-1,-2,\ldots$)

$$f_\nu(ik)=2\left(\frac{kR}{2}\right)^{\nu+1/2}\frac{\exp(-\delta/2)}{\Gamma(\nu+3/2)}\left\{\frac{kR}{2}K_{\nu-1/2+\delta}(kR)\times{}_1F_1\left(\nu+\frac{1}{2}+\frac{(kR)^2}{4\delta},\nu+\frac{3}{2};\delta\right)+\left(\nu+\frac{1}{2}\right)\times K_{\nu+1/2+\delta}(kR)\,{}_1F_1\left(\nu+\frac{1}{2}+\frac{(kR)^2}{4\delta},\nu+\frac{1}{2};\delta\right)\right\}.\quad (47)$$

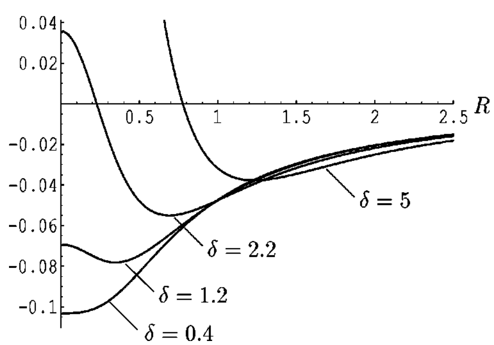

FIG. 2. The function $R^2\delta^{-2}\mathcal{E}^{\mathrm{f}}(R)$ for several values of δ.

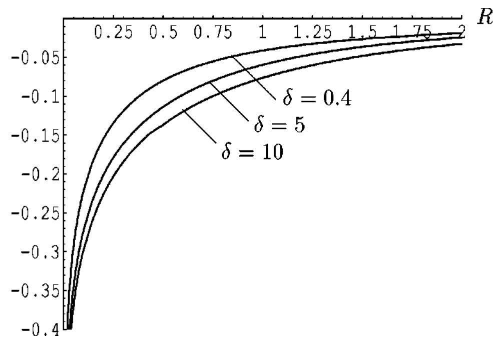

FIG. 3. The complete ground state energy multiplied by $R^2\delta^{-2}$ for several values of δ.

Thereby the pure Aharonov-Bohm phase is dropped as it does not contribute to $\mathcal{E}^f$.

The asymptotic part of the Jost function can be obtained explicitly by carrying out the elementary integrations over r in Eq. (37). Now, having given all ingredients in the integrand of $\mathcal{E}^f$, the remaining task is to perform numerical computations for several values of the parameters. For this task it turned out to be useful to integrate by parts and to substitute $k=\sqrt{x}/R$. Then we have

$$\mathcal{E}^{\mathrm{f}}=\frac{-1}{2\pi}\frac{1}{R^2}\sum_{\nu=1/2,3/2,\ldots}\int_{(Rm_{\mathrm{e}})^2}^{\infty}dx[\ln f_\nu^+(ik)+\ln f_\nu^-(ik)-2\ln f^{\mathrm{as}}(ik)]_{|k=\sqrt{x}/R}.\quad (48)$$

This expression can be calculated numerically. The integration over x is quite quickly convergent, the sum over ν not. So, in order to achieve a satisfactory precision for the plots, ν must be summed up to 15, for large x up to even higher values.

The general behavior of $\mathcal{E}^{\mathrm{f}}$ as a function of the radius R of the flux tube is quite smooth. For $R\to 0$ it is proportional to R^{-2}. This can be seen analytically from Eq. (48). For $R\to\infty$ it is proportional to R^{-3} which we observed numerically. Having in mind that the behavior for $R\to\infty$ is determined by the next heat kernel coefficient after a_2 we conclude from this that $a_{5/2}$ is nonvanishing. This seems in contradiction with the general results saying that for manifolds without boundary half-integer coefficients vanish. But one has to remember that higher coefficients contain (at least) squares of derivatives of the background field which for the presented example leads to undefined expressions. Thus, for the higher coefficients the general formulas do not apply and there is no contradiction at all.

In Fig. 2 the function $\mathcal{E}^{\mathrm{f}}(R)$ is shown multiplied by $R^2\delta^{-2}$ as a function of R for several values of δ. In Fig. 3 the complete ground state energy, $\mathcal{E}^{\mathrm{ren}}(R)$, multiplied by $R^2\delta^{-2}$ is shown for several values of δ. In general, this function takes only negative values, relatively weakly depending on the flux δ. For small R, the logarithmic contribution is dominating.

The complete energy is the sum of $\mathcal{E}^{\mathrm{class}}$ (16) and $\mathcal{E}^{\mathrm{ren}}$ (25). In Fig. 3, the classical energy would be a straight horizontal line at $2\pi/\alpha$. From this it is clear that the complete energy, remaining a monotone decreasing function of the radius, deviates only slighly from the classical energy for all values of the radius R except for very small ones as mentioned at the end of Sec. II.

For large δ, in $\mathcal{E}^{\rm f}$ and $\mathcal{E}^{\rm as}$ the contributions proportional to δ^4 dominate, giving (at last for $0\leqslant R\lesssim 1$) $\mathcal{E}^{\rm f}$ ($\mathcal{E}^{\rm as}$) large positive (negative) values. But these contributions cancel each other. This was seen in the numerical calculations. Also, this corresponds to the procedure of adding and subtracting $\ln f^{\rm as}$ in Sec. III which contains terms proportional to δ^4. However, we did not perform a complete investigation of the behavior for large δ.

VII. CONCLUSIONS

In this article we have provided a full analysis of the ground state energy of the spinor field in the background of a straight magnetic flux tube of finite radius R. The formalism developed applies in principle to any magnetic field with cylindrical symmetry. Assuming that the Jost function is known or can be determined numerically, Eqs. (42) and (48) give the final formulas for this case. We have applied these formulas to the case of the magnetic field (2). The final result consists of a very explicit ''asymptotic'' part, Eq. (43), and a part to be determined numerically, Eq. (48). A detailed numerical analysis shows, that the ground state energy turns out be negative, remaining for almost all values of the radius R by a factor proportional to the fine structure constant α smaller than the classical energy. As a result, in the range of applicability of our results, the total energy remains positive and, furthermore, does not show a minimum for finite values of R given a fixed flux. The magnetic string thus remains unstable also when including quantum corrections into the total energy.

The behavior of the ground state energy for large R at fixed flux can be understood in terms of the heat kernel expansion because it is for dimensional reasons equivalent to the large mass behavior. We found numerically for $R\to\infty$ a behavior $\sim R^{-3}m_{\rm e}^{-1}$ which corresponds to $a_{5/2}$. This coefficient with a half integer number may be present because the background (4) has a step. In a smooth background the first nonvanishing coefficient (after a_n with $n\leqslant 2$ which were subtracted out by the renormalization) may be a_3 with the resulting behavior $\sim R^{-4}m_{\rm e}^{-2}$. In a homogeneous magnetic background this coefficient is zero because it containes derivatives. In a nonhomogeneous background it may be nonzero and delivers the leading asymptotics. The next coefficient in sequence is a_4 delivering terms $\sim R^{-6}m_{\rm e}^{-4}$. This coefficient is nonzero in a homogeneous field and must deliver the Euler-Heisenberg contribution.

An interesting question is on the general dependence of the ground state energy on the specific background chosen here. First we remark, that we are in agreement with Ref. [4] with respect to the ground state energy beeing a small addendum to the classical energy (as long as R is not too small). Secondly, from explicit formulas such as Eq. (42) for $\mathcal{E}^{\rm as}$ we expect that the dependence on the shape of the magnetic field inside the flux tube will be weak. This will be so at last for sufficiently smooth background fields. When, however, the background becomes singular, this may change. As an example one can consider the ground state energy of a massive spinor field with bag boundary conditions on a sphere calculated in Ref. [20] showing as function of the radius even changes of the sign. Another consequence of our calculations is that it seems impossible to shrink the radius of the string to zero bcause of the logarithmic singularity (15) appearing in that case. In view of this, it would be interesting to reconsider earlier investigations in the background of the infinitely thin string whereby we admit that it might well happen that this singularity can be absorbed into some counterterm.

Even if we have found a negative answer within the class of examples (2), the results presented can be a starting point to consider further the question if inhomogeneous magnetic fields can minimize the energy for fixed flux. Furthermore, it seems possible to include other aspects as for example external electric fields and the anomalous magnetic moment. Also, the techniques developed here, are suited for the calculation of the fermionic contribution to the vacuum polarization in the background of the Nielsen-Olesen vortex, the Z string, or in a chromomagnetic background.

Finally we note that the ground state energy found here has a quite similar behavior as that in a homogeneous magnetic background insofar as both quantities are completely negative [in the definition of the effective potential as in Eq. (1) usually the classical energy is included, in the notion of the ground state energy it is not]. Of course, these quantities differ in details, for instance in their asymptotic behavior as mentioned above.

ACKNOWLEDGMENT

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APPENDIX A: PERTURBATION THEORY FOR THE LOGARITHM OF THE JOST FUNCTION

In this appendix we will derive the expansion for the logarithm of the Jost function, Eq. (30). The first step is to use Eq. (31) in Eq. (30). The Jost function itself up to the fourth power in the perturbation $\Delta(\mathcal{P})$ reads

$$f_m(k)=:1+x_1+x_2+x_3+x_4+\mathcal{O}[(\Delta\mathcal{P})^5], \tag{A1}$$

with the definitions

$$x_1=-\left(\frac{\pi}{2i}\right)\int_0^\infty dr_1 r_1 \Phi^T_{H^1}(r_1)\Delta\mathcal{P}(r_1)\Phi_J(r_1), \tag{A2}$$

$$x_2=-\left(\frac{\pi}{2i}\right)\int_0^\infty dr_1 r_1\int_0^{r_1}dr_2 r_2\Phi^T_{H^1}(r_1)\Delta\mathcal{P}(r_1)g(r_1,r_2)\Delta\mathcal{P}(r_2)\Phi_J(r_2), \tag{A3}$$

$$x_3=-\left(\frac{\pi}{2i}\right)\int_0^\infty dr_1 r_1\int_0^{r_1}dr_2 r_2\int_0^{r_2}dr_3 r_3\Phi^T_{H^1}(r_1)\Delta\mathcal{P}(r_1)g(r_1,r_2)\Delta\mathcal{P}(r_2)g(r_2,r_3)\Delta\mathcal{P}(r_3)\Phi_J(r_3), \tag{A4}$$

$$x_4=-\left(\frac{\pi}{2i}\right)\int_0^\infty dr_1 r_1\int_0^{r_1}dr_2 r_2\int_0^{r_2}dr_3 r_3\int_0^{r_3}dr_4 r_4\Phi^T_{H^1}(r_1)\Delta\mathcal{P}(r_1)g(r_1,r_2)\Delta\mathcal{P}(r_2)g(r_2,r_3)\Delta\mathcal{P}(r_3)g(r_3,r_4)\Delta\mathcal{P}(r_4)\Phi_J(r_4). \tag{A5}$$

We will need the combinations

$$\ln f_m(k)=\ln f_m^{(1)}(k)+\ln f_m^{(2)}(k)+\ln f_m^{(3)}(k)+\ln f_m^{(4)}(k)+\mathcal{O}[(\Delta(\mathcal{P})^5], \tag{A6}$$

with

$$\ln f_m^{(1)}=x_1, \tag{A7}$$

$$\ln f_m^{(2)}=x_2-\frac{1}{2}x_1^2, \tag{A8}$$

$$\ln f_m^{(3)}=\frac{1}{3}x_1^3-x_1x_2+x_3, \tag{A9}$$

$$\ln f_m^{(4)}=-\frac{1}{4}x_1^4+x_1^2x_2-\frac{1}{2}x_2^2-x_1x_3+x_4. \tag{A10}$$

Let us consider $\ln f_m(k)$ order by order. The first order $\ln f_m^{(1)}(k)$ is already given by its definition, Eqs. (A7) and (A2). For the calculation of $\ln f_m^{(2)}(k)$ some manipulations are needed. The main trick, also for the calculation of the higher orders, is the rearrangement of integration domains. At the beginning we will give details, later on only an idea of the single steps is given.

Using Eq. (29) one obtains

$$\ln f_m^{(2)}(k)=\left(\frac{\pi}{2i}\right)^2\Bigg\{\int_0^\infty dr_1 r_1\int_0^{r_1}dr_2 r_2\Phi^T_{H^1}(r_1)\Delta\mathcal{P}(r_1)\Phi_J(r_1)\Phi^T_{H^1}(r_2)\Delta\mathcal{P}(r_2)\Phi_J(r_2)$$
$$-\int_0^\infty dr_1 r_1\int_0^{r_1}dr_2 r_2\Phi^T_{H^1}(r_1)\Delta\mathcal{P}(r_1)\Phi_{H^1}(r_1)\Phi^T_J(r_2)\Delta\mathcal{P}(r_2)\Phi_J(r_2)-\frac{1}{2}\left[\int_0^\infty dr_1 r_1\Phi^T_{H^1}(r_1)\Delta\mathcal{P}(r_1)\Phi_J(r_1)\right]^2\Bigg\}.$$

The first and third terms combine to give

$$\frac{1}{2}\left(\frac{\pi}{2i}\right)^2\int_0^\infty dr_1 r_1\Phi^T_{H^1}(r_1)\Delta\mathcal{P}(r_1)\Phi_J(r_1)\left\{\int_0^{r_1}dr_2 r_2\Phi^T_{H^1}(r_2)\Delta\mathcal{P}(r_2)\Phi_J(r_2)-\int_{r_1}^\infty dr_2\Phi^T_{H^1}(r_2)\Delta\mathcal{P}(r_2)\Phi_J(r_2)\right\}.$$

Next the integration domains may be rearranged,

$$\int_0^\infty dr_1 r_1\int_{r_1}^\infty dr_2=\int_0^\infty dr_2\int_0^{r_2}dr_1$$

and changing finally the name of the variable, $r_1\leftrightarrow r_2$, one arrives at Eq. (33).

When calculating the higher orders it is extremely helpful to systematically use the lower orders already obtained. So for the next order we start with

$$\ln f_m^{(3)}(k)=x_3-\frac{1}{6}(\ln f_m^{(1)})^3-(\ln f_m^{(1)})(\ln f_m^{(2)}),$$

where

$$x_3=x_{3,1}+x_{3,2}+x_{3,3}+x_{3,4}$$

consists of the pieces

$$x_{3,1}=-\left(\frac{\pi}{2i}\right)^3\int_0^\infty dr_1r_1\int_0^{r_1}dr_2r_2\int_0^{r_2}dr_3r_3\Phi_{H^1}^T(r_1)\Delta\mathcal{P}(r_1)\Phi_J(r_1)\Phi_{H^1}^T(r_2)\Delta\mathcal{P}(r_2)\Phi_J(r_2)\Phi_{H^1}^T(r_3)\Delta\mathcal{P}(r_3)\Phi_J(r_3),$$

$$x_{3,2}=\left(\frac{\pi}{2i}\right)^3\int_0^\infty dr_1r_1\int_0^{r_1}dr_2r_2\int_0^{r_2}dr_3r_3\Phi_{H^1}^T(r_1)\Delta\mathcal{P}(r_1)\Phi_J(r_1)\Phi_{H^1}^T(r_2)\Delta\mathcal{P}(r_2)\Phi_{H^1}(r_2)\Phi_J^T(r_3)\Delta\mathcal{P}(r_3)\Phi_J(r_3),$$

$$x_{3,3}=\left(\frac{\pi}{2i}\right)^3\int_0^\infty dr_1r_1\int_0^{r_1}dr_2r_2\int_0^{r_2}dr_3r_3\Phi_{H^1}^T(r_1)\Delta\mathcal{P}(r_1)\Phi_{H^1}(r_1)\Phi_J^T(r_2)\Delta\mathcal{P}(r_2)\Phi_J(r_2)\Phi_{H^1}^T(r_3)\Delta\mathcal{P}(r_3)\Phi_J(r_3),$$

$$x_{3,4}=-\left(\frac{\pi}{2i}\right)^3\int_0^\infty dr_1r_1\int_0^{r_1}dr_2r_2\int_0^{r_2}dr_3r_3\Phi_{H^1}^T(r_1)\Delta\mathcal{P}(r_1)\Phi_{H^1}(r_1)\Phi_J^T(r_2)\Delta\mathcal{P}(r_2)\Phi_{H^1}(r_2)\Phi_J^T(r_3)\Delta\mathcal{P}(r_3)\Phi_J(r_3).$$

Cancellations occur due to the identity

$$\int_0^\infty dr_1r_1\int_0^{r_1}dr_2r_2\ldots\int_0^{r_n}dr_{n+1}f(r_1)\ldots f(r_{n+1})=\frac{1}{(n+1)!}\left[\int_0^\infty drf(r)\right]^{n+1}, \tag{A11}$$

which can be proven by induction. It shows

$$x_{3,1}-\frac{1}{6}(\ln f_m^{(1)})^3=0.$$

To manipulate the contribution $(\ln f_m^{(1)})(\ln f_m^{(2)})$ integrals are spitted according to

$$\int_0^\infty dr_3=\int_0^{r_2}dr_3+\int_{r_2}^\infty dr_3 \tag{A12}$$

and identities of the kind

$$\int_0^r dr_1\int_{r_1}^\infty dr_2=\int_0^r dr_2\int_0^{r_2}dr_1+\int_r^\infty dr_2\int_0^r dr_1 \tag{A13}$$

are used. One arrives at

$$-(\ln f_m^{(1)})(\ln f_m^{(2)})=-x_{3,2}-x_{3,3}+x_{3,4}$$

ending up with Eq. (35).

Finally, the last order we will need can be written as

$$\ln f_m^{(4)}(k)=x_4-\frac{1}{2}[\ln f_m^{(2)}(k)]^2-\frac{1}{2}[\ln f_m^{(1)}(k)]^2[\ln f_m^{(2)}(k)]-[\ln f_m^{(1)}(k)][\ln f_m^{(3)}(k)]-\frac{1}{24}(\ln f_m^{(1)})^4.$$

The contribution

$$x_4=\sum_{i=1}^{8}x_{4,i}$$

consists of

$$x_{4,1}=\left(\frac{\pi}{2i}\right)^4\int_0^\infty dr_1r_1\int_0^{r_1}dr_2r_2\int_0^{r_2}dr_3r_3\int_0^{r_3}dr_4r_4\Phi_{H^1}^T(r_1)\Delta\mathcal{P}(r_1)\Phi_J(r_1)$$
$$\times\Phi_{H^1}^T(r_2)\Delta\mathcal{P}(r_2)\Phi_J(r_2)\Phi_{H^1}^T(r_3)\Delta\mathcal{P}(r_3)\Phi_J(r_3)\Phi_{H^1}^T(r_4)\Delta\mathcal{P}(r_4)\Phi_J(r_4),$$

$$x_{4,2}=-\left(\frac{\pi}{2i}\right)^4\int_0^\infty dr_1r_1\int_0^{r_1}dr_2r_2\int_0^{r_2}dr_3r_3\int_0^{r_3}dr_4r_4$$
$$\times\Phi_{H^1}^T(r_1)\Delta\mathcal{P}(r_1)\Phi_J(r_1)\Phi_{H^1}^T(r_2)\Delta\mathcal{P}(r_2)\Phi_J(r_2)\Phi_{H^1}^T(r_3)\Delta\mathcal{P}(r_3)\Phi_{H^1}(r_3)\Phi_J^T(r_4)\Delta\mathcal{P}(r_4)\Phi_J(r_4),$$

$$x_{4,3}=-\left(\frac{\pi}{2i}\right)^4\int_0^\infty dr_1r_1\int_0^{r_1}dr_2r_2\int_0^{r_2}dr_3r_3\int_0^{r_3}dr_4r_4\Phi_{H^1}^T(r_1)\Delta\mathcal{P}(r_1)$$
$$\times\Phi_J(r_1)\Phi_{H^1}^T(r_2)\Delta\mathcal{P}(r_2)\Phi_{H^1}(r_2)\Phi_J^T(r_3)\Delta\mathcal{P}(r_3)\Phi_J(r_3)\Phi_{H^1}^T(r_4)\Delta\mathcal{P}(r_4)\Phi_J(r_4),$$

$$x_{4,4}=\left(\frac{\pi}{2i}\right)^4\int_0^\infty dr_1r_1\int_0^{r_1}dr_2r_2\int_0^{r_2}dr_3r_3\int_0^{r_3}dr_4r_4\Phi_{H^1}^T(r_1)\Delta\mathcal{P}(r_1)\Phi_J(r_1)$$
$$\times\Phi_{H^1}^T(r_2)\Delta\mathcal{P}(r_2)\Phi_{H^1}(r_2)\Phi_J^T(r_3)\Delta\mathcal{P}(r_3)\Phi_{H^1}(r_3)\Phi_J^T(r_4)\Delta\mathcal{P}(r_4)\Phi_J(r_4),$$

$$x_{4,5}=-\left(\frac{\pi}{2i}\right)^4\int_0^\infty dr_1r_1\int_0^{r_1}dr_2r_2\int_0^{r_2}dr_3r_3\int_0^{r_3}dr_4r_4\Phi_{H^1}^T(r_1)\Delta\mathcal{P}(r_1)\Phi_{H^1}(r_1)$$
$$\times\Phi_J^T(r_2)\Delta\mathcal{P}(r_2)\Phi_J(r_2)\Phi_{H^1}^T(r_3)\Delta\mathcal{P}(r_3)\Phi_J(r_3)\Phi_{H^1}^T(r_4)\Delta\mathcal{P}(r_4)\Phi_J(r_4),$$

$$x_{4,6}=\left(\frac{\pi}{2i}\right)^4\int_0^\infty dr_1r_1\int_0^{r_1}dr_2r_2\int_0^{r_2}dr_3r_3\int_0^{r_3}dr_4r_4\Phi_{H^1}^T(r_1)\Delta\mathcal{P}(r_1)\Phi_{H^1}(r_1)$$
$$\times\Phi_J^T(r_2)\Delta\mathcal{P}(r_2)\Phi_J(r_2)\Phi_{H^1}^T(r_3)\Delta\mathcal{P}(r_3)\Phi_{H^1}(r_3)\Phi_J^T(r_4)\Delta\mathcal{P}(r_4)\Phi_J(r_4),$$

$$x_{4,7}=\left(\frac{\pi}{2i}\right)^4\int_0^\infty dr_1r_1\int_0^{r_1}dr_2r_2\int_0^{r_2}dr_3r_3\int_0^{r_3}dr_4r_4\Phi_{H^1}^T(r_1)\Delta\mathcal{P}(r_1)\Phi_{H^1}(r_1)$$
$$\times\Phi_J^T(r_2)\Delta\mathcal{P}(r_2)\Phi_{H^1}(r_2)\Phi_J^T(r_3)\Delta\mathcal{P}(r_3)\Phi_J(r_3)\Phi_{H^1}^T(r_4)\Delta\mathcal{P}(r_4)\Phi_J(r_4),$$

$$x_{4,8}=-\left(\frac{\pi}{2i}\right)^4\int_0^\infty dr_1r_1\int_0^{r_1}dr_2r_2\int_0^{r_2}dr_3r_3\int_0^{r_3}dr_4r_4\Phi_{H^1}^T(r_1)\Delta\mathcal{P}(r_1)\Phi_{H^1}(r_1)\Phi_J^T(r_2)$$
$$\times\Delta\mathcal{P}(r_2)\Phi_{H^1}(r_2)\Phi_J^T(r_3)\Delta\mathcal{P}(r_3)\Phi_{H^1}(r_3)\Phi_J^T(r_4)\Delta\mathcal{P}(r_4)\Phi_J(r_4).$$

Equation (A11) shows

$$x_{4,1}-\frac{1}{24}[\ln f_m^{(1)}(k)]^4=0.$$

With the help of rearrangements as Eqs. (A12), (A13), and similar ones, it can be shown that

$$-[\ln f_m^{(1)}(k)][\ln f_m^{(3)}(k)]=-2x_{4,7}+4x_{4,8}-2x_{4,4},$$

$$-\frac{1}{2}[\ln f_m^{(1)}(k)]^2[\ln f_m^{(2)}(k)]=-x_{4,5}+x_{4,7}-x_{4,3}-x_{4,8}+x_{4,4}-2x_{4,2},$$

and

$$-\frac{1}{2}(\ln f_m^{(2)}(k))^2=-x_{4,6}-2\int_0^\infty dr_1r_1\int_0^{r_1}dr_2r_2\int_0^{r_2}dr_3r_3\times\int_0^{r_3}dr_4r_4\Phi_{H^1}^T(r_1)\Delta\mathcal{P}(r_1)\Phi_{H^1}(r_1)\times\Phi_{H^1}^T(r_2)\Delta\mathcal{P}(r_2)\Phi_{H^1}(r_2)\Phi_J^T(r_3)\times\Delta\mathcal{P}(r_3)\Phi_J(r_3)\times\Phi_J^T(r_4)\Delta\mathcal{P}(r_4)\Phi_J(r_4).$$

Putting all pieces together one arrives at Eq. (35).

APPENDIX B: SADDLE POINT EXPANSION OF INTEGRALS

For the derivation of Eqs. (37) and (38) repeated use of saddle point expansions was made. The relevant result is stated in this appendix.

For $\nu\to\infty$ one obtains the following asymptotic expansion

$$\int_0^r dr'\,\phi(r')e^{\nu\varphi(r')}=e^{\nu\varphi(r)}\sum_{k=1}^\infty h_{k-1}\nu^{-k}, \qquad \text{(B1)}$$

where the needed leading terms of the expansion are

$$h_0=\frac{\phi(r)}{\varphi'(r)},$$

$$h_1=\frac{\phi(r)\varphi''(r)}{[\varphi'(r)]^3}-\frac{\phi'(r)}{[\varphi'(r)]^2},$$

$$h_2=\frac{\phi''(r)}{[\varphi'(r)]^3}-\frac{3\phi'(r)\varphi''(r)}{[\varphi'(r)]^4}+\frac{3\phi(r)[\varphi''(r)]^2}{[\varphi'(r)]^5}-\frac{\phi(r)\varphi'''(r)}{[\varphi'(r)]^4}.$$

APPENDIX C: REPRESENTATION OF SUMS AS INTEGRALS

Here we display some formulas used in this paper, where sums are replaced by integrals.

The sum over half integer numbers can be represented by two integrals, the necessary analytical properties of the function $f(\nu)$ being assumed:

$$\sum_{l=0}^{\infty} f\left(l+\frac{1}{2}\right)=\int_0^{\infty} d\nu \; f(\nu)+\int_0^{\infty}\frac{d\nu}{1+\mathrm{e}^{2\pi\nu}}\,\frac{f(i\nu)-f(-i\nu)}{i}. \tag{C1}$$

The first part of $\mathcal{E}^{\mathrm{as}}$ in Sec. V can be calculated using

$$\int_0^{\infty} d\nu\int_m^{\infty} dk(k^2-m^2)^{1-s}\frac{\partial}{\partial k}\frac{t^i}{\nu^n} =-\frac{m^{2-2sn}}{2}\times\frac{\Gamma(2-s)\Gamma[(1+i-n)/2]\Gamma\{[s+(n-3)/2]\}}{(rm)^{n-1}\Gamma(i/2)}. \tag{C2}$$

The integration over k in formula (39) can be done using

$$\int_m^{\infty} dk(k^2-m^2)^{1-s}\frac{\partial}{\partial k}t^i =-m^{2-2s}\frac{\Gamma(2-s)\Gamma(s+i/2-1)}{\Gamma(i/2)}\times\frac{(\nu/mr)^i}{[1+(\nu/mr)^2]^{s+i/2-1}}. \tag{C3}$$

The functions $\Sigma_{n,j}(x)$, Eq. (40), can be written as

$$\Sigma_{n,j}(x)=\frac{4}{x^2}\int_x^{\infty} d\nu\sqrt{\nu^2-x^2}f_{n,j}(\nu), \tag{C4}$$

for $n=1,\ j=1{,}3$ and $n=3,\ j=3{,}5{,}7{,}9$ with

$$f_{1,1}(\nu)=-\frac{1}{1+\exp(2\pi\nu)},$$

$$f_{1,3}(\nu)=-\left(\frac{\nu}{1+\exp(2\pi\nu)}\right)',$$

$$f_{3,3}(\nu)=\left(\frac{1}{\nu}\,\frac{1}{1+\exp(2\pi\nu)}\right)',$$

$$f_{3,5}(\nu)=\frac{1}{3}\left[\frac{1}{\nu}\left(\frac{\nu}{1+\exp(2\pi\nu)}\right)'\right]',$$

$$f_{3,7}(\nu)=\frac{1}{15}\left\{\frac{1}{\nu}\left[\frac{1}{\nu}\left(\frac{\nu^3}{1+\exp(2\pi\nu)}\right)'\right]'\right\}',$$

$$f_{3,9}(\nu)=\frac{1}{105}\left(\frac{1}{\nu}\left\{\frac{1}{\nu}\left[\frac{1}{\nu}\left(\frac{\nu^5}{1+\exp(2\pi\nu)}\right)'\right]'\right\}'\right)'.$$

For $n=2$ the formulas are slightly more explicit. They read

$$\Sigma_{2,2}=-\frac{1}{x^2}\ln(1+e^{-2\pi x}),$$

$$\Sigma_{2,4}=\frac{\pi}{x}\,\frac{1}{1+e^{2\pi x}},$$

$$\Sigma_{2,6}=\frac{3\pi}{4x}\,\frac{1}{1+e^{2\pi x}}-\frac{\pi^2}{2}\,\frac{e^{2\pi x}}{(1+e^{2\pi x})^2}.$$

The functions $f_{i,j}$ build the ingredients for the function $g_i(x)$ in Eq. (42). Explicitly we find

$$\begin{aligned}
f_1(x)=&\frac{1}{2}f_{1,1}(x)-\frac{1}{2}f_{1,3}(x)+\frac{1}{4}f_{3,3}(x)-\frac{39}{16}f_{3,5}(x)\\
&+\frac{35}{8}f_{3,7}(x)-\frac{35}{16}f_{3,9}(x)\\
&-\frac{1}{2}x\partial_x\left(-\frac{1}{4}f_{3,3}(x)+\frac{7}{8}f_{3,5}(x)-\frac{5}{8}f_{3,7}(x)\right)\\
&+\frac{1}{2}x\partial_x^2\left(\frac{x}{8}f_{3,3}(x)-\frac{x}{8}f_{3,5}(x)\right),\\
f_2(x)=&\frac{1}{8}[f_{3,3}(x)-f_{3,5}(x)],\\
f_3(x)=&-\frac{1}{8}[f_{3,3}(x)-6f_{3,5}(x)+5f_{3,7}(x)].
\end{aligned} \tag{C5}$$

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