Covariant technique of derivative expansion of the one-loop effective action

N. G. Pletnev* and A. T. Banin
Institute of Mathematics, Novosibirsk, Prosp. Koptiug 4, RU-630090, Russia
(Received 29 January 1999; published 22 October 1999)

A simple systematic method for calculating derivative expansions of the one-loop effective action is presented. This method is based on using symbols of operators and well known deformation quantization theory. To demonstrate its advantages we present several examples of application for scalar theory, Yang-Mills theory, and scalar electrodynamics. The superspace formulation of the method is considered for Kählerian and non-Kählerian quantum corrections for Wess-Zumino and for Heisenberg-Euler Lagrangians in super QED models. [S0556-2821(99)04918-8]

PACS number(s): 03.70.+k, 04.60.Gw, 11.15.Tk, 12.60.Jv

I. INTRODUCTION

The low energy effective action (EA) (see Ref. [1]) contains all predictions of quantum field theory and is a central object of research in physical situations, when we are interested in phenomena at an energy scale which is smaller then some cutoff Λ. Then fundamental heavy degrees of freedom of the underlying theory appear in loops only as virtual states and an integration over both these mass states and all massless excitations above the scale we are interested in leads, generally speaking, to nonlocal quantities.

Unfortunately, the straightforward calculations even for one-loop EA, determined by the spectrum of the operator $H=\delta^2 S/\delta\phi\delta\phi$ as a functional of external fields, face this essential problem. Such a problem can be precisely solved only for some very specific simple configurations of background fields, when eigenvalues of H can be found precisely. Therefore, the problem of development of the manifestly covariant methods for calculating the (nonlocal) EA as a series of local terms depending on background field derivatives has attracted much attention. The leading term, named the effective potential, is the most investigated term in the derivative expansion. It is a useful object for the determination of vacuum structure of the full theory [2].

The most known method for calculating the derivative expansion EA (DEEA) is the so-called Schwinger-DeWitt asymptotic expansion [3–5]. All interesting quantities, such as EA, Green function, energy-momentum tensor, currents, and anomalies are expressed in this approach in terms of asymptotic coefficient heat kernel decomposition, so-called Hadamard-Minakshisundaram-DeWitt-Seeley (HMDS) coefficients. Various effective covariant methods for calculating HMDS coefficients has been developed by many authors. Schwinger-DeWitt decomposition gives the good description of vacuum polarization effects of mass fields on a background of weak background fields. However, the description of such physical phenomena as Hawking radiation or the anomalous magnetic moment of the electron involves consideration of the nonlocal structure of the effective action. Among various methods for investigation of nonlocal effective dynamics, such as direct summation of the terms with higher derivatives [6] and integration of anomalies, the most preferable is the covariant method of the theory of perturbations [7].

On the other hand, it has been known for a long time, that one-loop EA can be, at least formally, rewritten as, first, a quantized path integral for a fictive particle, which correctly describes the behavior of spin and color degrees of freedom in external fields [8]. This representation and its modifications are used for calculation of derivative expansion in QFT, for research of complicated Feynman diagrams and for application of stationary phase method [9]. Unfortunately, the application of the quantum-mechanical path integral in curved phase space meets the difficulties of its correct definition. This is related to the time-slicing procedure [10], because it is not covariant itself. Moreover, in order to get a sense of the path integral, it is necessary to add some designations in the way of finite-dimentional approximation. The ambiguities arising from such procedure have the same source as the quantization procedure [11].

The powerful nonrenormalization theorems in the supersymmetrical theories [12] do not prohibit the quantum corrections for superpotential. So far perturbative calculations determine the effective Kählerian potential. When the supersymmetry is unbroken, this potential determines both the effective potential of the theory and the kinetic terms. The problem of calculating the Kählerian potential was developed by many authors both on a component level [13], and with the use of the supergraph technique [14]. The generalization of the operator Schwinger–DeWitt representation for an appropriate heat kernel is also developed (see, for example, Ref. [15]).

The principe of manifest covariance is crucial for effective theories constructing. It means that any physical theory, which possesses some symmetry, must be formulated in such a form where all symmetries are manifest both at classical and quantum levels. The main advantages of the background field method consist in the fact that it allows us to formulate supersymmetrical and gauge invariant theory of perturbations manifestly (see Ref. [16], and reference therein).

A lot of interest in perturbative calculations one-loop EA for $N=2,4$ super-Yang-Mills (SYM) theories has been attracted recently. It was induced by exact Seiberg-Witten results in $N=2$ supersymmetrical gauge theories without and with material hypermultiplets [17], where the Kählerian po-

*Email address: pletnev@math.nsc.ru

tential and mass of stable states are predetermined by holomorphity and duality of the prepotential in the space of quantum modules of the theory.

An undertaken test [18,19] of the forms of non-Abelian supersymmetrical EA by direct calculations indicates the presence of one-loop holomorphic functions $\mathcal{F}$ and real function $\mathcal{H}(W,\overline{W})$, which are incompatible with special geometry and consequently with $N=2$ supersymmetry. Therefore a problem of contributions of higher dimensions and their influence on the Bogomol'nyi-Prasad-Summerfield (BPS) formula of mass remains important. One of the main obstacles in the investigation of the EA in $N=2,4$ SYM models in conventional superspace is the presence of infinitely reducible structure. The formulation of the theory in harmonic superspace in terms of unconstrained superfields [20] has not quantization problems. Recently, the first examples of quantum calculations with manifest $N=2$ supersymmetry have been given within the context of harmonic superspace formalism [21].

There is an unsolved problem of how to break $N=2$ supersymmetry. The $N=2$ supersymmetry can be broken spontaneously or softly if we want to save its useful properties. The soft breaking [22] is a very practical approach to analyzing possible phenomenological applications of exact solutions. But it has a limited predictive power because of plenty of free parameters. Therefore, finding nonsupersymmetrical vacuum solutions for the scalar potential induced by quantum effects in the hypermultiplet sector of $N=2$ gauge theories is an important problem.

However, the above mentioned problems, despite active attention to them recently, do not still go beyond an approximation of constant background fields. In this paper we try to develop a scheme [23] for calculations for one-loop DEEA, equally suitable for models, which can contain internal symmetry and gauge or other background fields or superfields. In Ref. [24] the authors have offered this computing scheme for gauge theories, but really they did not go further than extraction of divergences. We want to ratify this method as very effective for some problems. It should be noted that in Refs. [23, 24] the derivative expansion method was presented as a collection of separate useful expressions and identities. At the same time the direct connection between them and the problem of deformation quantization can be easily seen [25].

We use the definition of a star (or Moyal) product [26] to give a phase-space definition of the operator trace. This allows us to get a convenient derivative expansion for the heat kernel. The star product approach to quantization is particularly adapted to such problems. First, its structure allows us to deal with the expansion in $\hbar$ in a simple way. Secondly, it is the only known general quantization scheme which allows the quantization of any symplectic manifold including those where a choice of the polarization is impossible. Extensive lists of the literature on this subject can be found in Ref. [28].

Here we present a covariant method which consists of a sequential application of the symbol operator technique for formal trace calculation of the evolution operator. In practice this leads to a normal coordinate expansion of all quantities contained in the heat kernel and using the finite translation property of momentum integrals. This property is also used in other approaches [29] to calculate quantum corrections using a modified propagator, which has all gauge invariant combinations of background fields and their derivatives already. It should be noted that this procedure does not affect the space-time relation of background fields. the proposed technique allows us to produce a derivative expansion for the effective action on the background of exact solutions for the Heisenberg equation.

Obtaining that or other specific results has demonstrated the character of basic elements of the method. We concentrate on advances in other calculation schemes, examples of the scalar theory with self-action in flat space [23,30], calculation colorless QCD correlators [29], and simple derivation of the chiral anomaly. We shall consider the problem of derivative expansion in scalar electrodynamics [31], which is laying outside the frameworks of theory of perturbations. A calculation by a manifestly supersymmetrical way of the first famous correction to Kählerian potential in Wess-Zumino model [32,19] and supergeneralization Schwinger EA in super QED [33] will be presented also.

The plan of the paper is as follows. We begin with a brief consideration of the offered method. Then we present several examples to demonstrate its scope for the mentioned problems. The paper ends with a short summary.

II. THE METHOD

The starting expression for the calculation of one-loop EA, obtained by integrating over quantum and (or) heavy fields in functional integral is [1] $\Gamma_{(1)}=-1/2\,\mathrm{Tr}\ln\hat{H}$, where operator $\hat{H}$ is the second functional derivative of the action, i.e., the inverse propagator in the presence of background fields. To give sense of this formal expression we use the known technique of symbols of operators [11]. In this approach the quantum expectation value of the operator $\hat{A}$ is

$$\mathrm{Tr}(\hat{A})=\int_X d\mu(\gamma)A(\gamma), \tag{1}$$

where X is the phase space and $A(\gamma)$ is the function on the phase space (i.e., the symbol of the corresponding operator $\hat{A}$). The symbol calculus is based on the so-called star product which corresponds to the usual product of operators. In this case the standard notation of one-loop EA in the form of the heat kernel or in the suitable for regularization ζ-function form

$$\zeta_H(s)=\frac{1}{\Gamma(s)}\int_0^\infty dT\,T^{s-1}\,\mathrm{Tr}(e^{-T\hat{H}}), \tag{2}$$

provides us a connection with Wigner-Weyl-Moyal formalism [34], since, due to Eq. (1) we can write

$$\zeta_H(s)=\frac{1}{\Gamma(s)}\int_0^\infty dT\,T^{s-1}\int_X d\mu(\gamma)e_\star^{-TH(\gamma)}, \tag{3}$$

where $e_\star^{-TH}$ is the star exponential, defined by

$$e_\star^H = \sum \frac{1}{N!} H\star H\star\cdots H\star.$$

This allows us to derive phase-space expressions for the formal trace by

$$\mathrm{Tr}(\hat{A}) = \int_X d\mu(\gamma)[A + \hbar\tau_1(A) + \hbar^2\tau_2(A) + \cdots]$$

in quasiclassical expansion form. In order to introduce some notations we will use in the paper we briefly review the phase-space formulation of ordinary quantum mechanics (originated by Weyl, Wigner, and Moyal [26] and extensively studied by Berezin [11]).

Symplectic manifold X can be viewed as a cotangent fiber bundle $(X^{2n}, X^n, T_x^* X, \omega)$ with the base space X^n, fiber $T_x^* X$ and fundamental symplectic two-form ω. In local coordinates, we have $\gamma = (p_1 \cdots p_n, x^1 \cdots x^n)$, $\gamma \in X^{2n}$, $x \in X^n$, $p \in T_x^* X$, $\omega = (1/2)\omega_{ij} d\gamma^i \wedge d\gamma^j$. In Hamilton mechanics X^{2n} plays the role of phase-space equipped by standard Poisson brackets $\{f,g\}_\omega = \omega^{ij}\partial_i f \partial_j g$.

Let us consider some dynamical system on a flat phase space. Let some quantization be chosen, i.e., linear mapping $A \leftrightarrow \hat{A}$ between functions in the phase-space (classical observations) and operators in the Hilbert space by the following recipe:

$$A(\gamma) \to \hat{A} = \int_X d\xi d\eta w(\xi,\eta)\tilde{A}(\xi,\eta) e^{(i/\hbar)(\xi\hat{p} - \eta\hat{q})}, \tag{4}$$

where $\tilde{A}$ is the inverse Fourier transform A, $(\hat{p},\hat{q}) = \hat{\Gamma}$ are operators satisfying the canonical commutational relation $[\hat{p},\hat{q}] = -i\hbar$, $w(\xi,\eta)$ is a some weight function, which depends on ordering rule, and $(\xi,\eta) = \gamma$ belongs to dual X^{2n} space. For practical calculations it is very convenient to employ a differential form of the last relation, i.e.,

$$\hat{A} = A(-i\partial_\gamma) e^{i\gamma\omega\hat{\Gamma}}|_{\gamma=0}. \tag{5}$$

Of course an operator can be characterized by function other then phase-space based symbol. A prime example is its integral kernel, i.e., the Dirac matrix element $\langle x|\hat{A}|y\rangle$ for which the following formulas are useful. Taking the matrix element of Eq. (4) leads to a construction of the kernel starting from the Weyl symbol [i.e., $w(\xi,\eta) = 1$]

$$\langle x|\hat{A}|y\rangle = \int dp e^{ip/\hbar(x-y)} A\left(\frac{x+y}{2}, p\right). \tag{6}$$

One passes in the opposite direction from the kernel to symbol via the Wigner transform

$$A(p,q) = \int dv e^{-ipv/\hbar}\left\langle q + \frac{1}{2}v|\hat{A}|q - \frac{1}{2}v\right\rangle = \frac{\langle q|\hat{A}|p\rangle}{\langle q|p\rangle}, \tag{7}$$

where $|p\rangle$ is the momentum eigenstate. The obtained asymmetric form A is suitable for calculations.

As soon as the mapping $A(\gamma) \leftrightarrow \hat{A}$ is constructed, the star product appears in phase space, which copies product of operators. This construction is essentially nonlocal, which is characteristic of the quantum uncertainty principle. For this basic structure there are again both integral and derivative based formulas, which are useful in varying circumstances

$$(A\star B)(\gamma) = \int\int d\gamma' d\gamma'' A(\gamma + \gamma') B(\gamma + \gamma'') e^{(2i/\hbar)\gamma'\omega\gamma''}, \tag{8}$$

$$\begin{aligned}(A\star B)(\gamma) &= e^{(i\hbar/2)\partial_{\gamma'}\omega\partial_{\gamma''}} A(\gamma') B(\gamma'')|_{\gamma=\gamma'=\gamma''} \\ &= AB + i\hbar\{A,B\}_{PB} + \cdots.\end{aligned} \tag{9}$$

The Groenewold formula (9) is a consequence of Eq. (8) and provides a small $\hbar$ expansion of $(A\star B)$. The fact that it may be evaluated through the translation of function arguments is the key feature

$$(A\star B)(p,q) = A\left(p - \frac{i\hbar}{2}\partial_q, q + \frac{i\hbar}{2}\partial_p\right) B(p,q). \tag{10}$$

The image of commutator in the WWM formalism is the Moyal bracket $\{A,B\}_M$, which is bilinear, skewed, and obeys the Jacobi identity.

It can be proved that different choices of star product correspond to different choices of operator ordering. Furthermore, there is a W_∞ symmetry linking the various choices of the star product.

For a dynamical system on curved phase space the above mentioned constructions assume natural generalizations [27]. Because the correspondence $A \leftrightarrow \hat{A}$ claims on an autonomous quantum mechanic statement, there has to be correspondence between physical results for particular dynamic systems. The quantum equations of motion are then obtained from the classical picture having pointwise multiplication and Poisson bracket replaced with their star analogues. It was proved that for an exactly solvable quantum-mechanical system, the corresponding star analogue of the evolution operator has a Fourier-Dirichlet expansion

$$e_\star^{TH} = \sum_{\lambda\in I} |\lambda\rangle\langle\lambda| e^{T\lambda/i\hbar}.$$

This allows us to localize a functional integral, turning it into a sum over spectrum of the operator $\hat{H}$ [25].

In a dynamical system, which does not have the exact solution for the spectrum, we have to calculate asymptotic expansion coefficients of the heat kernel. Our suggestion is that it is convenient to present the star product as the argument displacement.

Though the operator ordering is not essential, there are a number of systems having an inherit polarization. For example, if the $\hat{H} = \hat{p}^2 + V(\hat{q})$ then the qp ordering is the most preferable and

$$e_\star^{-TH} = e^{-T[p^2 + V(q - i\hbar\partial_p)]}. \tag{11}$$

This is the simplest case where $(p,q)\in X$ is a symplectic vector space. In the following sections we will demonstrate the treatment of such expressions.

In more complicated cases such as a particle in external gravitational and YM fields, which are connections on a principal bundle over the configuration space Q, the theme of strict deformation quantization was discussed in a number of works [35]. First, the gauge invariant definition of Wigner function was studied by Stratonovich [36]. The specific character of such system consists in the fact that the phase space of the particle is a Marsden-Weinstein reduction of T^*G, hence this space can also be considered as reduced phase space of a particular type of constrained dynamical system. Then the quantization corresponds to assigning quantum operators to be generators of an irreducible unitary representation of the group G. However, there is more then one such representation of the group and many different inequivalent quantum systems arise from the study of the same configuration space. Physically, this means that without a connection we can not separate the particle's external momentum from its own internal ''position'' and ''momentum'' which is associated with the motion on the coadjoint orbit. Using the connection ∇ on Q we had constructed a star product of standard ordered type $\star_s$, which is the natural generalization of the standard ordered product in flat X [37]. A surprisingly simple analogue of the operator

$$N=e^{(\hbar/2i)\partial^2/\partial_{p_k}\partial_{q^k}}$$

for any T^*Q takes the form

$$N=e^{(\hbar/2i)\Delta}.$$

Here the second order differential operator Δ is as follows:

$$\Delta=\frac{\partial^2}{\partial q^i\partial p_i}+\Gamma^i_{ik}(q)\frac{\partial}{\partial p_k}+p_k\Gamma^k_{ij}(q)\frac{\partial^2}{\partial p_i\partial p_j}+A_k(q)\frac{\partial}{\partial p_k},$$

where Γ^i_{jk} is the Cristoffel symbol and A is one-form on Q such that dA equals to the strength tensor.

The operator N is globally defined and induces the equivalence transformation, which yields a more physical star product of Weyl type having the complex conjugation as an involutive antilinear antiautomorfism

$$f\star_W g=N^{-1}[(Nf)\star_s(Ng)].$$

This equivalence is again the natural generalization of the flat case.

The above mentioned facts prescribe the following gauge invariant way to determinate the connection Weyl type symbol (related to ∇ ordering, because they are not commutative):

$$\begin{aligned}\nabla^p_\mu&=e^{i\partial_p\cdot\nabla}(ip_\mu+\nabla_\mu)e^{-i\partial_p\cdot\nabla}\\&=ip_\mu+\int_0^1 d\tau\cdot i\tau\partial^\nu_p F_{\nu\mu}(x+i\tau\partial_p)\\&=ip_\mu+\frac{i}{2}\partial^\nu_p F_{\nu\mu}-\frac{1}{3}\partial^{\lambda\nu}_p F_{\nu\mu,\lambda}-\frac{i}{8}\partial^{\sigma\lambda\nu}_p F_{\nu\mu,\lambda\sigma}+\cdots.\end{aligned}\tag{12}$$

The action of the operator $U=e^{i\partial_p\cdot\nabla}$ corresponds to a canonical transformation, which leads to the normal coordinate expansion. Here a role of tangent vector, along which implements parallel transport, plays $\partial/\partial p$, in the p representation of the normal coordinates. To find Eq. (12), we used the commutation relation $[\nabla_\mu,\nabla_\nu]=F_{\mu\nu}(q)$.

For development of the offered technique in superspace where the choice of gauge condition is not obvious, we notice that we have obtained a representation of the vector potential in the Fock-Schwinger gauge

$$A_\mu(q)=q^\nu\sum_{n=0}^{\infty}\frac{1}{n+2}\frac{1}{n!}q^{\alpha_1}\cdots q^{\alpha_n}F_{\nu\mu,\alpha_1\cdots\alpha_n},$$

without explicit solving the gauge condition $q^\mu A_\mu=0$. The potential term is presented by the expression

$$V_p=e^{i\partial_p\cdot\nabla}V(q)e^{-i\partial_p\cdot\nabla}=V(q+i\partial_p)\tag{13}$$

in the normal coordinate expansion form. Now we get a representation of the main object for calculations in the form $\mathrm{Tr}\ln(-\Box_p+V_p)$.

The main result of the technique of symbols is that already on the first stage of calculations we have found initial expression for $H(p,q)$ containing only gauge covariant quantities. The problem of obtaining $\Gamma_{(1)}$, thus, consists in calculating of the evolution operator of some quantum-mechanical problem with Hamiltonian $H=-\Box_p+V_p$. We shall calculate the result of star-product directly, order by order in T. It means that we will implement p,∂_p ordering until all terms having derivatives acting on nothing (vacuum) will disappear. This is a quite a simple procedure. Moreover, the sensible separation H on an exactly soluble Hamiltonian H_0 and a perturbation V allows us to construct expansions on the background of H_0 eigenstate.

III. DERIVATIVE EXPANSION EA ON A BACKGROUND OF SCALAR POTENTIAL

As the first example we shall consider a massive scalar field theory with the Lagrangian

$$\mathcal{L}=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-m^2\phi^2-U(\phi)$$

and the problem of the inverse mass decomposition EA for comparison of the offered method with results string-inspired technique [30] and other computing schemes [23,4]. Typical problems in which interaction through derivatives plays an important role are connected with stabilization of soliton solutions in the Skirm model, QCD, and in the Higgs sector of the standard models.

It is convenient to use a proper time representation for the trace of the logarithm of the operator $\hat{H}=-\Box+V(x),V=m^2+U''(\phi)$. According to the method we get the initial representation for one-loop EA

$$\Gamma_{(1)}=-\frac{1}{2}\int_0^\infty\frac{dT}{T}\int d^4xK(T), \quad (14)$$

where

$$K(T)=\int\frac{d^4p}{(2\pi)^4}e^{-T[p^2+V(x+i\partial p)]}$$

is the heat kernel. The expression for the effective action contains divergences and imposes renormalization. For the given representation the ζ regularization is intrinsic, which has the advantage of automatically preserving a large class of the classical symmetries.

We leave in the decomposition (13) fourth order terms in derivatives:

$$V_p=V+i\partial_p^\mu V_\mu-\frac{1}{2}V_{\mu\nu}\partial_p^{\mu\nu}-\frac{i}{3!}V_{\mu\nu\lambda}\partial_p^{\mu\nu\lambda}+\frac{1}{4!}V_{\mu\nu\lambda\tau}\partial_p^{\mu\nu\lambda\tau}. \quad (15)$$

The further problem is calculation of a trace of the evolution operator for a fictive particle in the potential (13). Using the known operator identity

$$e^{-T(p^2+V_p)}=e^{-Tp^2}\exp\int_0^T d\tau\, e^{+\tau p^2}(-V_p)e^{-\tau p^2}, \quad (16)$$

the kinetic term can be separated. As a result the argument of V_p is shifted as $V_p(x+i\partial_p-2i\tau p)$. We shall consider Eq. (13) as a perturbation and we shall decompose the T exponent up to derivatives of fourth order

$$P_T\exp\left(\int_0^T ds V_p(s)\right)=\sum_{n=0}^\infty\int_0^T ds_1\int_0^{s_1}ds_2\cdots\times\int_0^{s_{n-1}}ds_n V_p(s_1)V_p(s_2)\cdots V_p(s_n). \quad (17)$$

Expressions such as $e^{+\tau p^2}\partial_p e^{-\tau p^2}$ are replaced with the solutions of the Heisenberg equations, i.e., $\partial_p(\tau)=\partial_p-2\tau p$. The most complicated procedure is the disentangling of the star product. The partial simplification can be reached after commutation ∂_p to the left and using properties

$$\int\frac{d^4p}{(2\pi)^4}\partial_p^\mu(\cdots)=0. \quad (18)$$

All other calculations reduced to trivial integrations

$$\int\frac{d^4p}{(2\pi)^4}e^{-Tp^2}\{1,p_\mu p_\nu,p_{\mu_1},\dots,p_{\mu_4}\}=\frac{1}{(4\pi T)^2}\left\{1,\frac{1}{2T}\delta_{\mu\nu},\frac{1}{4T^2}\delta_{\mu_1\mu_2\mu_3\mu_4}\right\}. \quad (19)$$

For example,

$$\langle\partial_p^{\nu\mu}\rangle\equiv\int\frac{d^4p}{(2\pi)^4}e^{-Tp^2}\int_0^T ds\,\partial_p^\nu(s)\partial_p^\mu(s)=-\frac{1}{3}T^2\delta_{\mu\nu}, \quad (20)$$

$$\langle\partial_p^{\nu\tau\lambda\rho}\rangle=\frac{2}{15}T^3\delta^{\nu\lambda\tau\rho},\quad \langle\partial_p^{\mu_1\cdots\mu_6}\rangle=-\frac{2}{35}T^3\delta^{\mu_1\cdots\mu_6},$$

$$\langle\langle\partial_p^\nu\partial_p^\tau\rangle\rangle\equiv\int\frac{d^4p}{(2\pi)^4}e^{-Tp^2}\int_0^T ds\,\partial_p^\nu(s)\int_0^s ds'\,\partial_p^\tau(s')=-\frac{1}{12}T^3\delta^{\nu\tau},$$

where $\delta^{\mu_1\cdots\mu_{2k}}$ is a completely symmetrical tensor, consisting of $(2k-1)!!$ terms composed from Kronecker symbol products. After rearranging the results by extracting full derivatives, we obtain the known result [30]

$$K(T)=\frac{1}{(4\pi T)^2}e^{-VT}\times\left(1-\frac{1}{12}T^3V_\mu V_\mu+\frac{1}{5!}T^4V_{\mu\nu}V_{\mu\nu}-\frac{T^5}{3\times4!}V_{\mu\nu}V_\mu V_\nu+\frac{T^6}{12\times4!}V_\mu^2V_\nu^2\right). \quad (21)$$

Further integration over proper time gives gamma functions in any order of T. They have poles for some terms in DEEA, which correspond to known divergences.

Previously, it was mentioned that the local Schwinger-DeWitt expansion describes the vacuum polarization effect of massive quantum fields in weak background when all their invariants are smaller then the corresponding power of the mass parameter. However it is not a good approximation for the case of strong background fields and absolutely meaningless for massless theories and weak rapidly varying background fields. For investigation of these cases special methods are needed [6,7]. The result has an essentially nonlocal form. It is interesting to study how some nonlocal formfactors appear in our approach for this model. Let us consider the second order V term in the Duhamel expression (17). After simple manipulation we obtain

$$K(T)=\int_0^T ds_1\int_0^{s_1}ds_2 V(x)e^{(s_1-s_2)(\Box+2ip\nabla)}\times V(x)e^{-(s_1-s_2)(\Box+2ip\nabla)}. \quad (22)$$

Performing the integration over p we can see that the result can be rewritten in the form

$$K(T)=\frac{1}{(4\pi T)^2}\int_0^T ds_1\int_0^{s_1} ds_2 V e^{(s_1-s_2)[1-(s_1-s_2)/T]\Box}V.$$

From the last expression we obtain

$$\Gamma_{(1)}\sim-\frac{1}{2}\int\frac{dT}{T}e^{-m^2T}\int d^4x\frac{1}{(4\pi T)^2}\frac{T^2}{2}V\gamma(T\Box)V, \tag{23}$$

where the form factor has the known representation [6]

$$\gamma(T\Box)=\int_0^1 ds e^{[(1-s^2)/4]T\Box}. \tag{24}$$

Using a similar expression for form factors, one can analyze their analytical properties, calculate their high energy limits and their imaginary parts above the threshold, etc.

IV. NON-ABELIAN GAUGE FIELDS

Now we consider the gauge vacuum mean of gluon operators in the form $\langle F^2\rangle,\langle F^3\rangle$. In the absence of a consistent theory of the QCD vacuum, it was assumed that vacuum expectation values of local operators, in fact, play the role of fundamental constants for QCD sum rules. The necessity to calculate the coefficients in front of these gluon operators in decomposition for colorless correlation functions (which is used in a method of QCD sums rules) stimulated the development of gauge-invariant methods [29,4]. Unfortunately, in the standard Feynman diagrams technique, calculating the diagrams with emitting gluons from a loop and rearranging vector potentials in gauge invariant structures, is a rather hopeless problem. Because the determinant of the Dirac operator is determined after squaring by the Klein-Gordon operator, we limit ourselves to the consideration of a scalar loop in the external non-Abelian field with the Lagrangian

$$\mathcal{L}=\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi+m^2\phi^2,\quad \hat{H}=-\nabla^\mu\nabla_\mu+m^2.$$

According to the prescription described above, we use a representation of one-loop EA (3) with $H(p,\partial_p)=\nabla_p^\mu\nabla_p^\mu+m^2$, where ∇_p^μ is a covariant pseudodifferential operator (12). After extracting a free part from H in the form $H_0=p^2$ (which corresponds to the approximation where the particle motion between the interaction is free) and separating exponents similar to those in Eq. (16) we shall calculate vacuum mean dimensions (-3). Nonzero contributions from Eq. (17) in the procedure described in the second section will give the following results in the first and in the second order of the decomposition of the T exponent, respectively;

$$-\frac{1}{4}F_{\nu\mu}F_{\tau\mu}\langle\partial_p^{\nu\tau}\rangle+\frac{1}{72}F_{\nu\mu,\tau\lambda}F_{\sigma\mu}\langle\partial_p^{\nu\lambda\tau\sigma}\rangle,$$

$$\frac{1}{9}F_{\nu\mu,\mu}F_{\tau\alpha,\alpha}\langle\langle\partial_p^{\nu\tau}\rangle\rangle.$$

The other terms are either full derivatives or contain as a factor from the left or right $p^\mu\partial_p^\nu F_{\mu\nu}$, which leads to zero contributions. Further, performing trivial calculations similar to Eq. (20) and using the Bianchi identities we will get the known result [29,4]

$$K(T)=\frac{e^{-Tm^2}}{(4\pi T)^2}\left[1+\frac{T^2}{12}F_{\mu\nu}F_{\mu\nu}+T^3\left(\frac{F^3}{180}-\frac{1}{60}J^2\right)\right], \tag{25}$$

where $F^3=F_{\mu\nu}F_{\nu\alpha}F_{\alpha\mu}$, $J^2=F_{\mu\alpha,\alpha}F_{\mu\beta,\beta}$. Thus the first (after unit) term of the decomposition is related to renormalization of charge.

A less trivial problem is the calculation of the next HMDS coefficient $b_3\sim(F/m)^4$. The simplification is reached on free equations of motion $F_{\mu\nu,\mu}=0$. In the first order of the T-exponent decomposition we obtain the following contribution:

$$-\frac{1}{4\times 6!}F_{\alpha\mu}F_{\nu\mu\rho\sigma\tau\lambda}\langle\partial_p^{\lambda\tau\sigma\rho\nu\alpha}\rangle$$
$$=\frac{T^4}{70\times 6!}F_{\alpha\mu}F_{\nu\mu\rho\sigma\tau\lambda}\delta^{\lambda\tau\sigma\rho\nu\alpha},$$

where only 10 of 15 members are nonvanishing. After some manipulations with commutating of derivatives, using of Bianchi identities and equations of motions we get the contribution

$$\frac{5T^4}{7\times 2\times 6!}\left([F_{\alpha\beta},F_{\mu\beta}]^2+\frac{1}{10}[F_{\alpha\mu},F_{\rho\sigma}]^2\right). \tag{26}$$

The contribution of the second order of the decomposition (17) is

$$\frac{T^4}{2\times 6!}(\{F_{\alpha\mu},F_{\beta\mu}\}^2+5(F_{\nu\mu}F_{\nu\mu})^2). \tag{27}$$

The full result for b_3

$$\frac{T^4}{2\times 144}\left((F_{\nu\mu}F_{\nu\mu})^2+\frac{1}{5}\{F_{\alpha\mu},F_{\beta\mu}\}^2+\frac{1}{7}[F_{\alpha\beta},F_{\mu\beta}]^2+\frac{1}{70}[F_{\alpha\mu},F_{\rho\sigma}]^2\right) \tag{28}$$

coincides with Ref. [29]. It should be noted that the huge number of terms in the decomposition can be omitted at once, that essentially reduces body of work and demonstrates that computation of higher power corrections might be considerably simplified. This is important for the analysis of convergence for the series in $1/m^2$.

We have considered two well known examples. Less trivial application of the EA expansion was used in Ref. [40] for the investigation of axial anomaly. The problem of generalization world line path integral representation [38,39] for amplitudes involving axial vector leads to another interesting application of the derivative expansion. It is well known that if the spinor fields are coupled to background fields $A_\mu, A_{5\mu}$, and the pseudoscalar one than the axial current J^5_μ has an anomalous divergence. The Dirac operator, suitably continued to Euclidian space, is not Hermitian and the anomaly can be attributed to the phase of the functional determinant. In Ref. [41] using the integral representation of the complex power for the pseudodifferential operator, it was obtained an unambiguous definition of the determinant for the Dirac operator. The determinant is shown to be vector gauge invariant and it posseses the correct axial and scale anomalies.

Another popular starting point is the second order description for the fermionic one-loop effective action

$$\Gamma_{(1)} \sim \frac{1}{2}\{\mathrm{Tr}\log HH^\dagger + \mathrm{Tr}(\log H - \log H^\dagger)\}.$$

The derivative of the second term in the last expression with respect to the background field can be written as

$$\frac{\delta}{\delta A}\mathrm{Tr}(\log H - \log H^\dagger) = \mathrm{Tr}\left[\left(\frac{\delta H}{\delta A}H^\dagger - H\frac{\delta H^\dagger}{\delta A}\right)\frac{1}{HH^\dagger}\right],$$

which allows us to derive an elegant representation with the help of an auxiliary integration for the imaginary part of the effective action, i.e., for the phase of the fermion functional determinant. Recently [40], it was found that for the special case where the background consists only with an Abelian vector and an axial vector field there is a much simpler solution to this problem which treats both parts of the effective action equally. The price which we have to pay for this property is that the kinetic operator occurs non-Hermitian. We consider a more complicated example contained general non-Abelian fields $A^a_\mu, A^a_{5\mu}$. It is easily to establish that

$$(\not{p} + \not{A} + \gamma_5 \not{A}_5)^2 = -(\partial_\mu + i\mathcal{A}_\mu)^2 + Q, \qquad (29)$$

where $\mathcal{A}_\mu = A_\mu - \gamma_5 \sigma_{\mu\nu} A^\nu_5$, $A = A^a T^a$, and

$$Q = -\frac{i}{2}\sigma_{\mu\nu}F_{\mu\nu} + i\gamma_5 A^5_{\mu,\mu} + 2A^2_5 + \frac{1}{2}\sigma^{\mu\nu}[A_{5\mu}, A_{5\nu}], \qquad (30)$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu].$$

Using such a trick, the effective action is formally identical with the effective action for a scalar loop in non-Abelian field $\mathcal{A}$ and potential Q background. A new gauge parameter has values in the Clifford algebra. Let us apply the method described above for the calculation of the quantity

$$\Gamma_{(1)} = -\frac{1}{2}\int \frac{dT}{T} e^{-Tm^2}\int d^4x \frac{d^4p}{(2\pi)^4} e^{-T(\nabla_p^2 + Q)}.$$

Repeating the above calculation for this case, we get the known results

$$K(T) = \frac{1}{(4\pi T)^2}\mathrm{tr}\left\{1 - TQ + T^2\left(\frac{1}{2}Q^2 - \frac{1}{12}\mathcal{F}^2\right) + T^3\left(-\frac{1}{6}Q^3 - \frac{1}{12}Q_\mu Q_\mu + \frac{1}{12}Q\mathcal{F}^2 - \frac{i}{180}\mathcal{F}^3 + \frac{1}{60}J^2\right)\right\}, \qquad (31)$$

where

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} + \gamma_5 \sigma_{[\mu\lambda}A^\lambda_{5,\nu]} + i\sigma_{[\mu\lambda}\sigma_{\nu]\tau}A^\lambda_5 A^\tau_5,$$

$$A^\lambda_{5,\mu} = \partial_\mu A^\lambda_5 + i[A_\mu, A^\lambda_5].$$

We still need to perform the Dirac traces. We see that unlike the vector case, the axial contribution to the imaginary part has an additional term proportional to $m^2 A^2_5$. This term is prohibited by gauge invariance in the vector case, however this term may appear in axial theories with massive fermions since those theories violate the corresponding gauge invariance. Logarithmically diveregent terms combine automatically in the gauge invariant expressions

$$K(T) \sim T^2 \frac{2}{3}\mathrm{Tr}_c\{G^A_{\mu\nu}G^A_{\mu\nu} + G^{A_5}_{\mu\nu}G^{A_5}_{\mu\nu}\},$$

where $G^A_{\mu\nu} = F_{\mu\nu} + i[A_{5\mu}, A_{5\nu}]$. Therefore it is necessary only to introduce the kinetic and mass counterterms for the axial field in order to render the theory be finite. The third HMDS coefficient contains a lot of terms. Keeping only the contribution which comes from the three point function $\langle AAA_5\rangle$, we get the famous result

$$K(T) \sim -T^3\,\mathrm{Tr}_c\left[G^A_{\alpha\beta}G^{A*}_{\alpha\beta}A^5_{\mu,\mu} + \frac{4}{3}\{G^A_{\alpha\beta}, G^{A*}_{\beta\gamma}\}A^5_{\gamma,\alpha}\right]. \qquad (32)$$

It has shown that the effective action induced by a spinor loop can be rewritten in terms of an auxiliary non-Abelian gauge field and a potential. This allows us to discuss the chiral anomaly from a novel point of view.

V. DERIVATIVE EXPANSION OF EA IN QED

In recent years a lot of problems related to intensive fields and nonlinear processes such as photon splitting, nonlinear Compton effect, and pair production below the two photon threshold were experimentally investigated [44]. So far the problem of going beyond perturbation theory increased so much, that the description of quantum processes becomes rather urgent and gets practical goals. Really, studying the limit of a strong field we obtain the same information, as from a polarization function in the small distance limit.

Unfortunately, the validity of the famous Schwinger Lagrangian [5] calculated almost a half-century ago and the two-loop exact results [45] are limited by the constant field approximation. The generalization of the Schwinger result on strong varying fields or fields located in a small area is very interesting from the physical point of view. Recently the authors of Ref. [31] presented the next (after Schwinger term) nonperturbative term $F_{\alpha\beta,\lambda}F_{\sigma\delta,\gamma}L_1^{\lambda\alpha\beta\gamma\sigma\delta}(F_{\mu\nu})$ in the expansion of one-loop EA. Their result was obtained from the representation of the path integral. Even for electrodynamics it is a rather difficult problem.

In this section we would like to demonstrate the capabilities of the proposed method for the computation of the complete form for the first nontrivial correction to long wavelength limit of the EA. We use the representation (12) for the pseudodifferential operator ∇_p and the proper time representations for EA, induced by a scalar loop.

Let us consider this example in more detail. For the calculation of the expansion on a nonperturbative background it is necessary to split out free and perturbation terms in the expression $e^{-T(\Pi^2+\Delta^{(1)})}$, where

$$\Pi_\mu = p_\mu + \frac{1}{2}\partial_p^\nu F_{\nu\mu}, \quad [\Pi_\mu,\Pi_\nu] = -F_{\mu\nu}, \tag{33}$$

$$\Delta^{(1)} = \frac{i}{3}F_{(\nu\mu,\tau)}(2\Pi_\mu\partial_p^{\tau\nu} + \delta_\mu^\tau\partial_p^\nu) - \frac{1}{8}F_{(\nu\mu,\tau\lambda)}(2\Pi_\mu\partial_p^{\lambda\tau\nu} + \delta_\mu^{(\tau}\partial_p^{\lambda)\nu}) - \frac{1}{9}F_{(\nu\mu,\tau}F_{\rho\mu,\lambda)}\partial_p^{\tau\nu\lambda\rho}. \tag{34}$$

Here parentheses means symmetrization with the appropriate weight. The interesting terms in the expansion of T exponent (17) for the heat kernel are

$$K(T) = e^{-T\Pi^2}\Bigg\{1 + \int_0^T ds\, e^{s\Pi^2}\Bigg[\frac{1}{8}F_{\nu\mu,\tau\lambda}(2\Pi_\mu\partial_p^{\lambda\tau\nu} + \cdots) + \frac{1}{9}F_{\nu\mu,\tau}F_{\rho\mu,\lambda}\partial_p^{\tau\nu\lambda\rho}\Bigg]e^{-s\Pi^2} - \frac{1}{9}F_{\nu\mu,\tau}F_{\alpha\beta,\gamma}\int_0^T ds\int_0^s ds' \times e^{s\Pi^2}(2\Pi_\mu\partial_p^{\tau\nu} + \cdots)e^{-(s-s')\Pi^2} \times (2\Pi^\beta\partial_p^{\alpha\gamma} + \cdots)e^{s'\Pi^2}\Bigg\}. \tag{35}$$

The following step consists of replacing $e^{s\Pi^2}\Pi_\mu e^{-s\Pi^2}, e^{s\Pi^2}\partial_p^\mu e^{-s\Pi^2}$ in appropriate solutions $\Pi_\mu(s), \partial_p^\mu(s)$ of equations of motion

$$\dot{\Pi}(s) = [\Pi^2,\Pi(s)], \dot{\partial}_p^\mu(s) = [\Pi^2,\partial_p^\mu(s)],$$

i.e.,

$$\Pi_\mu(s) = \Pi^\nu P_{\nu\mu}(s), \quad \partial_p^\mu(s) = \partial_p^\mu + \Pi^\nu B_\nu^\mu(s), \tag{36}$$

where $P(s) = (e^{-2sF})$, $B(s) = (1/F)\,(e^{-2sF}-1)$. After Π, ∂_p ordering it is necessary to take integrals over p. In principle, some methods for solution of this problem has already been used by Schwinger. Recently to a similar problem have addressed the paper [47], with the reference to heat kernel calculation methods developed in [46].

To treat the first term in Eq. (35) we notice that the operator Π^2 is the Hamiltonian of the two Landau oscillators in momentum representation. The kernel of the operator $\langle p'|e^{-T\Pi^2}|p\rangle$ is well known explicit Meller formula, frequently used for direct calculation of the index of an operator. It is important, that this kernel is a well converging expression and consequently[1]

$$\int \frac{d^4p}{(2\pi)^4}\partial_p^a(e^{-T\Pi^2}\Pi_\mu\cdots) = 0. \tag{37}$$

For future convinience we define the moments

$$K_{a_1a_2\cdots a_n} = \int \frac{d^4p}{(2\pi)^4}e^{-T\Pi^2}\Pi_{a_1}\Pi_{a_2}\cdots\Pi_{a_n}.$$

In particular, for $n=2$ we have

$$0 = \int \frac{d^4p}{(2\pi)^4}\partial_p^b(e^{-T\Pi^2}\Pi_a) = \delta_a^b K + K_{ca}B_c^b.$$

The expansion of a matrix B begins with unit, therefore one can be inversed B^{-1} and

$$K_{ab} = -KB_{ba}^{-1}.$$

Similarly

$$K_{a_1\cdots a_4} = K(B_{a_2a_1}^{-1}B_{a_4a_3}^{-1} + B_{a_3a_1}^{-1}B_{a_4a_2}^{-1} + B_{a_4a_1}^{-1}B_{a_3a_2}^{-1}).$$

We also need in the relations

$$K_{a_1a_2\cdots a_6} = -(B_{a_2a_1}^{-1}K_{a_3\cdots a_6} + B_{a_3a_1}^{-1}K_{a_2a_4\cdots a_6} + \ldots).$$

The kernel $K(T)$ satisfies the differential equation

$$\frac{dK}{dT} = -K_{aa} = KB_{aa}^{-1}$$

or

$$K^{-1}\frac{dK}{dT} = \mathrm{tr}\left(\frac{F}{e^{-2TF}-1}\right) = \mathrm{tr}\left(\frac{Fe^{2TF}}{1-e^{2TF}}\right) = -\frac{1}{2}\mathrm{tr}\left((1-e^{2TF})^{-1}\frac{d}{dT}(1-e^{2TF})\right) = -\frac{1}{2}\mathrm{tr}\frac{d}{dT}\ln(1-e^{2TF})C, \tag{38}$$

[1]Using this property, the authors of Ref. [47] have reproduced the Schwinger result.

here $C=2\pi/F$ is a constant of integration, determined from a known limit $K=1/(4\pi T)^2$ for small T, when the particle can be considered as free. With such a choice we find the standard result

$$K(T)=\frac{1}{(4\pi T)^2}\left[\det\frac{FT}{\sinh(FT)}\right]^{1/2}. \qquad (39)$$

As a next step, it is necessary to calculate several functions from matrixes $F_{\mu\nu}$. It is known [48] that for any constant field $F_{\mu\nu}$ there is such a reference frame, where magnetic and electrical fields are parallel and their values in this system are relativistic invariants of the field. Or, if they are perpendicular, it is possible to find such a reference frame, in which the field is either purely magnetic or purely electrical. Therefore the canonical form F in this system has a block structure

$$F_{\mu\nu}=\begin{pmatrix} 0 & \lambda_1 & 0 & 0\\ -\lambda_1 & 0 & 0 & 0\\ 0 & 0 & 0 & -\lambda_2\\ 0 & 0 & \lambda_2 & 0\end{pmatrix}.$$

There is a connection between eigenvalues and invariants of the field

$$\mathcal{H}_\pm=(\lambda_1\pm i\lambda_2)^2=\frac{1}{2}(F^2\mp iF^*F). \qquad (40)$$

Any degree F can be decomposed over basis of linear combinations of F, F^*, F^2, and g. Thus, for the exponent from a matrix $P=e^{\alpha F}$ we get

$$P=e^{\alpha\lambda_1}A_1+e^{-\alpha\lambda_1}A_2+e^{i\alpha\lambda_2}A_3+e^{-i\alpha\lambda_2}A_4,$$

where $A^{(i)}_{\mu\nu}$ is another known basis [49]

$$A^{(1)}=\frac{1}{2(\lambda_1^2+\lambda_2^2)}(F^2+\lambda_2^2 g+\lambda_1 F-\lambda_2 F^*)=\frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0\\ -1 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$A^{(2)}=\frac{1}{2(\lambda_1^2+\lambda_2^2)}(F^2+\lambda_2^2 g-\lambda_1 F+\lambda_2 F^*)=\frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$A^{(3)}=\frac{-1}{2(\lambda_1^2+\lambda_2^2)}(F^2-\lambda_1^2 g+i\lambda_2 F+i\lambda_1 F^*)=-\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -i\\ 0 & 0 & i & 1\end{pmatrix},$$

$$A^{(4)}=\frac{-1}{2(\lambda_1^2+\lambda_2^2)}(F^2-\lambda_1^2 g-i\lambda_2 F-i\lambda_1 F^*)=-\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & i\\ 0 & 0 & -i & 1\end{pmatrix},$$

which has the useful projector properties $A_{(i)}^2=A_{(i)}$, $A_{(i)}A_{(j)}=0$ for $i\neq j$. The transposition operation translates $A_1\leftrightarrow A_2$ and $A_3\leftrightarrow A_4$.

Calculation of matrix functions B and B^{-1} leads to remarkably simple results

$$B=\sum_{i=1}^{4}A^{(i)}\frac{1}{f_i}(e^{\alpha f_i}-1),\quad B^{-1}=\sum_{i=1}^{4}A^{(i)}f_i(e^{\alpha f_i}-1)^{-1}. \qquad (41)$$

It is convenient to use the notations

$$f_{1,2}=\pm\lambda_1,\quad f_{3,4}=\pm i\lambda_2.$$

Now we can easily get

$$\frac{\sinh(\alpha F)}{\alpha F}=\frac{\sinh(\alpha\lambda_1)}{\alpha\lambda_1}\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}+\frac{\sin(\alpha\lambda_2)}{\alpha\lambda_2}\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

and for the kernel we obtain the Schwinger result

$$K(T)=\frac{1}{(4\pi T)^2}\frac{T\lambda_1}{\sinh(T\lambda_1)}\frac{T\lambda_2}{\sin(T\lambda_2)}. \qquad (42)$$

Then we can implement all necessary substitutions, Π, ∂_p ordering of the operators, and integration over momenta in the other terms of expression (35). After some manipulation, all matrix structures P, B, B^{-1} depending on s, s', T are grouped in several combinations. The main group is

$$B^T(s')B^{-1}(T)B(s)-B^T(s') = -\sum_i \frac{2}{f_i} A^{(i)} e^{f_i(s'-s)} \sinh(f_i s') \frac{\sinh[f_i(s-T)]}{\sinh(f_i T)} \tag{43}$$

which coincides with the Green function, used in the path integral method [31]

$$D(s,s') = \sum \frac{A^{(i)}}{2 f_i} \Big[-(1-e^{2 f_i(s'-s)}) + \coth(f_i T) \times (1+e^{2 f_i(s'-s)}) - \frac{1}{\sinh(f_i T)} (e^{f_i(2s'-T)} + e^{f_i(-2s+T)}) \Big]. \tag{44}$$

The other arising combinations of matrix structures are derivatives of D in s, s', that can be easily seen

$$B^T(s')B^{-1}(T)B(s')-B^T(s') = \sum_i -\frac{2}{f_i} A^{(i)} \sinh(f_i s') \frac{\sinh[f_i(s'-T)]}{\sinh(f_i T)} = D(s',s'), \tag{45}$$

$$B^T(s')B^{-1}(T)P(s) = \sum_i A^{(i)} e^{f_i(s'-2s+T)} \frac{\sinh(f_i s')}{\sinh(f_i T)} = -\frac{1}{2}\frac{\partial}{\partial s} D(s,s'),$$

$$P^T(s')B^{-1}(T)B(s)-P^T(s') = \sum_i A^{(i)} e^{f_i(2s'-s)} \frac{\sinh[f_i(s-T)]}{\sinh(f_i T)} = -\frac{1}{2}\frac{\partial}{\partial s'} D(s,s'),$$

$$B^T(s')B^{-1}(T)P(s') = \sum_i A^{(i)} e^{-f_i(s'-T)} \frac{\sinh(f_i s')}{\sinh(f_i T)} = -\frac{1}{2}\frac{\partial}{\partial s} D(s,s')\big|_{s=s'},$$

$$P^T(s')B^{-1}(T)P(s) = \sum_i -\frac{A^{(i)} f_i e^{f_i(2s'-2s+T)}}{2\sinh(f_i T)} = \left(-\frac{1}{2}\frac{\partial}{\partial s'}\right)\left(-\frac{1}{2}\frac{\partial}{\partial s}\right) D(s,s').$$

In these notations the result for the expression in the braces (35) looks as follows

$$\begin{aligned}
1 &+ \int_0^T ds \Big(\frac{1}{8} F_{\mu\nu,\tau\mu} D_{\nu\tau}(s,s) + \frac{1}{4} F_{\mu\nu,\tau\lambda} (\dot{D}_{\nu\mu} D_{\tau\lambda} + \dot{D}_{\tau\mu} D_{\nu\lambda} + \dot{D}_{\lambda\mu} D_{\nu\tau})(s,s) \\
&+ \frac{1}{9} F_{\nu\mu,\tau} F_{\rho\mu,\lambda} (D_{\nu\tau} D_{\rho\lambda} + D_{\rho\tau} D_{\lambda\nu} + D_{\rho\nu} D_{\lambda\tau})(s,s) \Big) \\
&+ \frac{4}{9} F_{\nu\mu,\tau} F_{\alpha\beta,\gamma} \int_0^T ds \int_0^s ds' \{ \dot{D}_{(\tau\mu}(s,s) [\dot{D}_{(\gamma\beta}(s',s') D_{\alpha)\nu)}(s,s') + \acute{D}_{\beta\nu)}(s,s') D_{\alpha\gamma}(s',s')] \\
&+ \dot{D}_{(\alpha\mu}(s,s') [\acute{D}_{\beta(\tau}(s,s') D_{\gamma)\nu)}(s,s') + \dot{D}_{\gamma)\beta}(s',s') D_{\nu\tau}(s,s)] + \dot{\acute{D}}_{\beta\mu}(s,s') [D_{(\gamma\tau}(s,s') D_{\alpha)\nu}(s,s') \\
&+ D_{\nu\tau}(s,s) D_{\alpha\gamma}(s',s')]\}.
\end{aligned} \tag{46}$$

The last step is the calculation of a plenty of standard integrals such as

$$\int D = T^2 \sum_i A^{(i)} L(f_i, T), \quad \int DD = T^3 \sum_{i,j} A^{(i)} \times A^{(j)} \left\{ L(Tf_i) L(Tf_j) + \frac{L(Tf_j) - L(Tf_i)}{(Tf_i)^2 - (Tf_j)^2} \right\}.$$

Because of combersome of the general result, we do not present it here. Besides it is inconvenient in a particular physical problem, where it is necessary only some terms. Let us note only, that functions of proper time T and relativistic invariants of fields setting in front of every possible contractions $F_{\mu\nu,\tau\lambda}$, $F_{\mu\nu,\tau} F_{\alpha\beta,\gamma}$ and with direct products $A^{(i)}, A^{(i)} \times A^{(j)}, A^{(i)} \times A^{(j)} \times A^{(k)}$ are combinations of Langevin functions $L(x) = (x\coth(x) - 1)/x^2$ and they are presented in Ref. [31].

Furthermore, it is necessary to implement renormalization through the subtraction based on common principle, which requires putting in zero the radiation corrections at the switched off field as in the original Schwinger paper [5], and replacing all bare charges and fields through the physical. Therefore it is easier to return to the initial expressions and to execute all manipulations with necessary accuracy.

When the mass of the scalar particle is greater than all other scales of the theory, we can limit the expansion by the next terms to unit

$$K(T)=\frac{T^3}{15}\left(\frac{1}{3}F_{\mu\nu,\lambda}F_{\mu\nu,\lambda}+\frac{1}{2}F_{\nu\mu}F_{\nu\mu,\lambda\lambda}\right). \tag{47}$$

This result agrees with Ref. [42]. Recently, similar methods for calculation of corrections to the long wavelength limit of EA on Yang-Mills background fields was used in Ref. [43].

It is obvious that the expressions (46) for the description of particular processes in nonconstant background fields are exact in mass of a charged particle and field strength. The gradient corrections are very important for the analysis of the effective potential, since they can reduce energy of the ground state.

This detail presentation evidently demonstrates possibilities to obtain the corrections on background which possess exact solution of classical problem. Because of a large number of physical set up of problems in nonconstant background fields, it is useful to have in an arsenal of tools of their solution a method, which is alternative to path integral representation.

VI. QUANTUM CORRECTIONS IN WESS-ZUMINO MODEL

We demonstrate how to apply the proposed technique to calculation DEEA for the supersymmetrical theories in the superfield approach. The doubtless advantage of the offered method is that this method does not require the determination of many various Green functions for calculation of functional trace of the appropriate heat kernel. To show it, we obtain the known Kählerian potential of the Wess-Zumino model [18,19,32] and lowest order non-Kählerian contributions to the one-loop effective potential.

The Wess-Zumino theory described by the action

$$S(\phi,\bar{\phi})=\int d^8z\,\bar{\phi}\phi+\int d^6z\left(\frac{m\phi^2}{2}+\frac{g}{3!}\phi^3\right)+\text{H.c.}$$

is a good model for test of various supersymmetric methods, since it has all specific peculiarities of the theories with chiral fields, and it enters as an inherent ingredient in many superfield theories [50,16].

It is known that a problem of a definition of a superfield EA agreed with symmetry of the theory can be very effectively solved in the framework of proper time superfield technique [15,32]. For functional integration over quantum chiral fields, which arise after splitting of fields on quantum and background ones, it is convenient to introduce unconstrained superfields $\phi=\bar{D}^2\psi$ and $\bar{\phi}=D^2\bar{\psi}$. In principle this introduces a new gauge invariance into the action, but in the absence of background gauge fields, the ghost associated with this gauge fixing are decoupled. Another procedure transforming the path integral over the chiral superfields into a path integral over general superfields has been developing in Ref. [15]. The functional integration over $\psi,\bar{\psi}$ leads to a determination of the effective action in the form $-\frac{1}{2}\mathrm{Tr}\ln[\hat{H}(x,\theta,D)]$ with the kinetic operator for the given model

$$\hat{H}=\begin{pmatrix}\lambda & \bar{D}^2\\ D^2 & \bar{\lambda}\end{pmatrix}\begin{pmatrix}\bar{D}^2 & 0\\ 0 & D^2\end{pmatrix},\quad \text{where}\quad \lambda=m+g\phi_{(BG)}. \tag{48}$$

Except a functional trace, the operation Tr means a matrix trace as usual. There are many techniques of calculation the Kählerian potential which is an analogue of the Coleman-Weinberg potential [32,19].

We can implement the Fourier transformation in superspace, though it is not necessary, since the δ function of Grassmanian coordinates is explicitly known, as well as the action on the $D,\bar{D}$ derivatives

$$\delta(z-z')=\int\frac{d^4p}{(2\pi)^4}d^2\psi d^2\bar{\psi}e^{i(x-x')\cdot p+\psi^{\alpha}(\theta-\theta')_{\alpha}+\bar{\psi}^{\dot{\alpha}}(\bar{\theta}-\bar{\theta}')_{\dot{\alpha}}}. \tag{49}$$

We use the superspace agreements from Ref. [16] and we will omit the obvious indexes. Commutating exponents on the left through the differential operators we find in the coincidence limits the standard replacements

$$\begin{aligned}D_{\theta}&=\partial_{\theta}+i/2\bar{\theta}\partial\rightarrow\psi-\frac{1}{2}p\bar{\theta}+D_{\theta},\\ \bar{D}_{\bar{\theta}}&=\partial_{\bar{\theta}}+i/2\partial\theta\rightarrow\bar{\psi}-\frac{1}{2}\theta p+\bar{D}_{\bar{\theta}}.\end{aligned} \tag{50}$$

To obtain the covariant symbols of the operators $D,\bar{D}$ in momentum representation we use identities

$$U\left(D_{\theta}+\psi-\frac{1}{2}\bar{\theta}p\right)U^{-1}=\psi-\frac{1}{2}\partial_{\bar{\psi}}p=D_p, \tag{51}$$

$$U\left(\bar{D}_{\bar{\theta}}+\bar{\psi}-\frac{1}{2}\theta p\right)U^{-1}=\bar{\psi}-\frac{1}{2}\partial_{\psi}p=\bar{D}_p,$$

where parallel translation operator was chosen in the form

$$U=e^{i\partial_p\cdot\partial_x}e^{1/2\theta p\partial_{\bar{\psi}}-1/2\partial_{\psi}p\bar{\theta}}e^{\partial_{\psi}D_{\theta}+\partial_{\bar{\psi}}\bar{D}_{\bar{\theta}}}. \tag{52}$$

The anticommutator $\{D_p,\bar{D}_p\}=-p$ and, naturally, all useful algebraic relations for D_p have the same form as in D_{θ} algebra. In addition, we have a transformation for a general superfield

$$\phi^p=U\phi U^{-1}=\phi(x+i\partial_p,\theta+\partial_{\psi},\bar{\theta}+\partial_{\bar{\psi}}) \tag{53}$$

which is the finite degree polynomial in $\partial_{\psi},\partial_{\bar{\psi}}$ with factors $D_{\theta}\cdots D_{\bar{\theta}}\phi$.

Let us note that other arrangement of exponents in Eq. (52) related to the corresponding replacement of the normal coordinates. For example, the same transformations with

$$U=e^{i\partial_p\cdot\partial_x}e^{\partial_\psi D_\theta}e^{1/2\theta p\partial_{\bar\psi}-1/2\partial_\psi p\bar\theta}e^{\partial_{\bar\psi}\bar D_{\bar\theta}}$$

give us

$$\psi-\partial_{\bar\psi}p=D_p,\quad \bar\psi=\bar D_p.$$

The steps described above from the operators to the pseudodifferential operators on the phase superspace are conventional (see Ref. [24]). It should be mentioned, that the final result for the trace of the operator does not depend on selection (49) which reflects the chosen ordering scheme. The replacements (51), (53) actually correspond to the transition from the operators to their symbols and can be justified by the arguments similar to those described in the second section.

Limiting ourselves to a problem of calculation of the first correction to the potential in decomposition over Grassmanian derivatives, we split the pseudodifferential operator H, acting on phase superspace, in two parts

$$H=H_0+\begin{pmatrix}\Lambda\bar D_p^2 & 0\\ 0 & \bar\Lambda D_p^2\end{pmatrix},\tag{54}$$

where $\Lambda=\partial_\psi^\alpha D_{\theta^\alpha}\lambda$, $\bar\Lambda=\partial_{\bar\psi}^{\dot\alpha}\bar D_{\theta^{\dot\alpha}}\bar\lambda$ and H_0 copies the form (48). In the following steps we will write $D,\bar D$ instead of $D_p,\bar D_p$. This must not confuse, because $D_\theta,\bar D_{\bar\theta}$ are contained in $\Lambda,\bar\Lambda$ only. Then the Kählerian potential and the correction can be split:

$$\operatorname{Tr}\ln H_0+\operatorname{Tr}\ln\left[1+\begin{pmatrix}\Lambda\bar D^2 & 0\\ 0 & \bar\Lambda D^2\end{pmatrix}H_0^{-1}\right].$$

For calculation of Tr ln(H_0) we take out and omit the ''free'' part of the operator

$$\begin{pmatrix}0 & \bar D^2D^2\\ D^2\bar D^2 & 0\end{pmatrix}.$$

It is clear, that in the expression

$$\operatorname{Tr}\ln\left[1+\begin{pmatrix}0 & \dfrac{1}{D^2\bar D^2}\bar\lambda D^2\\ \dfrac{1}{\bar D^2D^2}\lambda\bar D^2 & 0\end{pmatrix}\right]\tag{55}$$

the nonzero contribution will give only even degrees of the logarithm decomposition. Unfolding the matrix part of the trace we get

$$-\frac12\ln\left(1-\frac{1}{D^2\bar D^2}\bar\lambda D^2\frac{1}{\bar D^2D^2}\lambda\bar D^2-\frac{1}{\bar D^2D^2}\lambda\bar D^2\frac{1}{D^2\bar D^2}\bar\lambda D^2\right).$$

Further, using the decomposition of unit in front of the logarithm in the form

$$1=\frac{\{D^2,\bar D^2\}-D^\alpha\bar D^2D_\alpha}{\Box}$$

one can convert all spinor derivatives in ''boxes'' $\Box=-p^2$. After that we obtain

$$K^{(1)}=\int\frac{d^4p}{(2\pi)^4}\frac{1}{2p^2}\ln\left(1+\frac{\lambda\bar\lambda}{p^2}\right),\tag{56}$$

which gives the known result [32,19] after integration and renormalization of a wave function by the condition $\partial^2K/\partial\phi\partial\bar\phi|_{\phi=\phi_0;\bar\phi=\bar\phi_0}=1$.

For calculation of the next nonvanishing contribution in the EA expansion, we rewrite H_0^{-1} in the form

$$\begin{pmatrix}\dfrac{1}{D^2\bar D^2}D^2 & 0\\ 0 & \dfrac{1}{\bar D^2D^2}\bar D^2\end{pmatrix}\begin{pmatrix}-\dfrac{1}{\Box_+}\bar\lambda & \dfrac{1}{\Box_+}\bar D^2\\ \dfrac{1}{\Box_-}D^2 & -\dfrac{1}{\Box_-}\lambda\end{pmatrix},$$

where $\Box_+=\bar D^2D^2-\lambda\bar\lambda$, $\Box_-=D^2\bar D^2-\lambda\bar\lambda$. Then

$$\begin{pmatrix}\Lambda\bar D^2 & 0\\ 0 & \bar\Lambda D^2\end{pmatrix}(H_0^{-1})=\begin{pmatrix}-\Lambda P_2\dfrac{1}{\Box_+}\bar\lambda & \Lambda\bar D^2\dfrac{1}{\Box_-}\\ \bar\Lambda D^2\dfrac{1}{\Box_+} & -\bar\Lambda P_1\dfrac{1}{\Box_-}\lambda\end{pmatrix},\tag{57}$$

where $P_1=D^2\bar D^2/\Box$, $P_2=\bar D^2D^2/\Box$ are the projectors in momentum representation. The first nonvanishing contribution in the decomposition of the logarithm gives trace of the fourth degree of the matrix (we keep in mind the properties of integration over $d^2\psi d^2\bar\psi$).

Moreover, among 16 terms the zero contribution automatically comes from terms containing powers more than two of $\Lambda,\bar\Lambda$ and also from terms containing $\bar\Lambda D^2$ and $\Lambda\bar D^2$ from the right, because the derivatives $\partial_{\bar\psi}$ and ∂_ψ contained in $\Lambda,\bar\Lambda$ act on nothing. We are left with

$$\frac{\lambda\bar\lambda}{4\Box\Box_\lambda^4}(\Lambda\bar D^2D^2\overline{\Lambda\Lambda}\Lambda\bar D^2D^2+\bar\Lambda D^2\bar D^2\Lambda\Lambda\bar\Lambda D^2\bar D^2),$$

$$\Box_\lambda=\Box-\lambda\bar\lambda.$$

We shall transfer $\partial_\psi, \partial_{\bar\psi}$ to the right, using Heisenberg relation $\{\partial_\psi, D_p\}=1$. The trivial integration over Grassmanian and usual momenta gives us immediately the known result for the non-Kählerian terms [32,19], leading to quantum deformations of classical vacuum of the theory

$$F^{(1)}=\frac{1}{3\times 2^7}\,\frac{D^\alpha\lambda D_\alpha\lambda\bar{D}^{\dot\alpha}\bar\lambda\bar{D}_{\dot\alpha}\bar\lambda}{\lambda^2\bar\lambda^2}, \tag{58}$$

where the factor 2^{-7} is caused by the superagreements. This kind of one-loop quantum correction is called the effective potential of auxiliary fields. Certainly, such quantum corrections are important in $N=1,2$ supersymmetrical models, since they lead potentially to the removal of degeneration in classical vacua of the theory. This method should be quite general and has important applications for other interesting cases, for example, for models with explicitly broken supersymmetry.

VII. HEISENBERG-EULER LAGRANGIAN IN SQED

In this section we develop manifestly supersymmetrical gauge invariant strategy of calculations of one-loop effective action for the most general renormalizable $N=1$ models including Yang-Mills fields and chiral supermuliplets

$$\mathcal{S}=\mathrm{tr}\int d^6 z W^2+\int d^8 z\bar\Phi e^V\Phi+\left[\int d^6 z P(\Phi)+\mathrm{H.c.}\right].$$

In more detail we consider the one-loop diagrams only with external Abelian superfields and the expansion in terms of spinor covariant derivatives of superfields $W,\bar W$ which cannot be reduced to usual space-time derivatives. This approximation corresponds to generalization of the Heisenberg-Euler Lagrangian of usual QED. The background field method in superspace [16,15] allows us to treat both vector supermultiplets and matter superfield on the equal footing and in an explicitly gauge-invariant way. However, in contrast to ordinary gauge theories the gauge connections are not independent objects and are expressed in terms of the prepotentials.

The basics of the method in its ''quantum-chiral background-vector'' representation are given in Ref. [16]. This approach implies that higher loop contributions can be arranged in such a way that background fields appears in ∇_A, W_A, Φ only. After expansion of full action, including gauge-fixing and ghost terms, in powers of quantum fields, the quadratic part determines a matrix of the kinetic operator acting in the space of all quantum fields. The physical quantities depend on particular gauge invariant combinations of the gauge superfields only, such as the field strength and derivatives thereof.

As in the previous section, the replacement of the operators by their symbols gives $\nabla\to\psi-\frac{1}{2}p\bar\theta+\nabla$ with manifest dependence on grassmanian coordinates. To obtain gauge-invariant and manifestly supercovariant symbols of operators, we use identities (51) with replacement of the flat D's by covariant ones.

Using the known notations and conventions from Ref. [16], we find the expansion of the symbols $\nabla^p=U(\psi-\frac{1}{2}p\bar\theta+\nabla)U^{-1}$ in superspace ''normal'' coordinates

$$\nabla^p_\alpha=\psi_\alpha-\frac{1}{2}\bar\partial^{\dot\alpha}p_{\alpha\dot\alpha}+\frac{i}{4}\bar\partial^{\dot\alpha}(\partial^{\dot\beta}_\alpha f_{\dot\beta\dot\alpha}+\partial^\beta_{\dot\alpha}f_{\beta\alpha})-\frac{1}{3}\partial_\alpha\bar\partial^{\dot\alpha}i\bar W_{\dot\alpha}, \tag{59}$$

$$+\frac{1}{3}\bar\partial^2 iW_\alpha+\frac{1}{4}\partial_\alpha\bar\partial^2 D'+\frac{3}{4!}\bar\partial^2\partial^\beta i f_{\beta\alpha}+\cdots,$$

$$\nabla^p_{\dot\alpha}=\psi_{\dot\alpha}-\frac{1}{2}\partial^\alpha p_{\alpha\dot\alpha}+\frac{i}{4}\partial^\alpha(\partial^{\dot\beta}_\alpha f_{\dot\beta\dot\alpha}+\partial^\beta_{\dot\alpha}f_{\beta\alpha})+\frac{1}{3}\partial^2 i\bar W_{\dot\alpha}$$

$$-\frac{1}{3}\bar\partial_{\dot\alpha}\partial^\alpha iW_\alpha-\frac{1}{4}\bar\partial_{\dot\alpha}\partial^2 D'+\frac{3}{4!}\partial^2\bar\partial^{\dot\beta}if_{\dot\beta\dot\alpha}+\cdots.$$

We do not specify here obvious indexes $\psi,\bar\psi,p$ in the ∂ representation of normal supercoordinates. The quantities f, D' are the standard notation for superfields $f_{\alpha\beta}=\frac{1}{2}\nabla_{(\alpha}W_{\beta)}$, $D'=-(i/2)\nabla^\alpha W_\alpha$, $\nabla^\alpha W_\alpha+\bar\nabla^{\dot\alpha}\bar W_{\dot\alpha}=0$. Here the dots mean the expansion in $\nabla_{\alpha\dot\alpha}$ derivatives, which we shall omit keeping in mind problems on the constant background which is independent on space-time coordinates, but with arbitrary dependence on Grassmanian coordinates. By the construction, the normal coordinate expansion used gives the connection decomposition in the Wess-Zumino gauge.

Similarly, for a vector derivative we have

$$\nabla^p_{\alpha\dot\alpha}=ip_{\alpha\dot\alpha}+\frac{1}{2}(\partial^\beta_{\dot\alpha}f_{\alpha\beta}+\partial^{\dot\beta}_\alpha f_{\dot\alpha\dot\beta})+\partial_\alpha\bar W_{\dot\alpha}+\bar\partial_{\dot\alpha}W_\alpha$$

$$+\frac{1}{2}(\partial_\alpha\bar\partial^{\dot\beta}f_{\dot\beta\dot\alpha}+\bar\partial_{\dot\alpha}\partial^\beta f_{\beta\alpha})+i\partial_\alpha\bar\partial_{\dot\alpha}D'. \tag{60}$$

It is not difficult to check up the validity of the identical correspondence of the algebra of covariant symbols to the algebra of covariant derivatives

$$\{\nabla^p_\alpha,\nabla^p_{\dot\alpha}\}=i\nabla^p_{\alpha\dot\alpha},\quad [\nabla^p_{\alpha\dot\alpha},\nabla^p_{\beta\dot\beta}]=i(C_{\dot\beta\dot\alpha}f_{\beta\alpha}+C_{\beta\alpha}f_{\dot\beta\dot\alpha}), \tag{61}$$

$$[\nabla^p_{\dot\beta},\nabla^p_{\alpha\dot\alpha}]=C_{\dot\beta\dot\alpha}W^p_\alpha,\quad [\nabla^p_{\dot\alpha},W^p_\alpha]=0,$$

where $W^p_\alpha=UW_\alpha U^{-1}=W_\alpha+\partial^\beta f_{\beta\alpha}-i\partial_\alpha D'$. This is the verification that the gauge connection given by Eqs. (59), (60) indeed gives rise to the field strength.

It is convenient to present $\nabla^p_{\alpha(\dot\alpha)}$ in a remarkably simple form

$$\nabla^p_\alpha=\tilde\psi_\alpha+\frac{i}{2}\bar\partial^{\dot\alpha}\nabla^p_{\alpha\dot\alpha},\qquad \bar\nabla^p_{\dot\alpha}=\tilde{\bar\psi}_{\dot\alpha}+\frac{i}{2}\partial^\alpha\nabla^p_{\alpha\dot\alpha},$$

where

$$\tilde{\psi}_\alpha=\psi_\alpha+\frac{1}{3!}\partial_\alpha\overline{\partial}^{\dot\alpha}i\overline{W}_{\dot\alpha}-\frac{1}{3!}\overline{\partial}^2 iW_\alpha-\frac{1}{8}\overline{\partial}^2\partial^\beta if_{\beta\alpha}-\frac{1}{4}\partial_\alpha\overline{\partial}^2 D',$$

$$\tilde{\overline{\psi}}=\overline{\psi}+\frac{1}{3!}\overline{\partial}_{\dot\alpha}\partial^\alpha iW_\alpha-\frac{1}{3!}\partial^2 i\overline{W}_{\dot\alpha}-\frac{1}{8}\partial^2\overline{\partial}^{\dot\beta}if_{\dot\beta\dot\alpha}+\frac{1}{4!}\overline{\partial}_{\dot\alpha}\partial^2 D'.$$

So, we have obtained the connection decomposition in normal supercoordinates, which naturally can be called a supergeneralization of the Fock-Schwinger gauge. For some disscussion about this subject see Ref. [51].

Let us consider a particular example of calculation of the one-loop contributions of chiral superfields in the diagrams with external vector legs. It is known [16] that such contributions in the full EA are determined by the expression $\mathrm{Tr}\{\ln(\nabla^2\overline{\nabla}^2-m^2)+\mathrm{H.c.}\}$ Using $\int d^4\theta=\int d^2\theta\overline{\nabla}^2$, we obtain the known basic chiral expression

$$\int d^2\theta\ln(\overline{\nabla}^2\nabla^2-m^2)\overline{\nabla}^2\delta^{(8)}+\mathrm{H.c.}=\int d^2\theta\ln(\Box_+-m^2)\overline{\nabla}^2\delta^{(8)}, \tag{62}$$

where $\Box_+=\Box-iW^\alpha\nabla_\alpha-i/2(\nabla W)$ with covariant $\Box$. The transition to the momentum representation consists in replacement of the assumed operators and fields by corresponding pseudodifferential operators and the additional integration $\int[d^4p/(2\pi)^4]d^2\psi d^2\overline{\psi}$. Obviously, all ∂_ψ, $\overline{\partial}_{\overline{\psi}}$ symbols from right-hand side of $\overline{\nabla}_p^2$ can be omitted, since they act on nothing. Having in mind the property of the Grassmanian integration, it is also possible to omit all $\overline{\partial}_{\overline{\psi}}$ acting on $\overline{\psi}^2$ inside the logarithm and to perform integration over $d^2\overline{\psi}$.

Further, it is convenient to proceed to the proper-time representation for the logarithm of operator and to use the appropriate ζ regularization $\Gamma_{(1)}\sim-\zeta'(0)$. The next step in our strategy, which helps us to get the final result practically without computations, consists in separation of exponents of the operators ∇_p and the covariant ''box''

$$K(T)=e^{-TD'}\int\frac{d^4p}{(2\pi)^4}d^2\psi e^{-T\nabla_p^\alpha iW_\alpha^p}e^{T\Box_p+\cdots}, \tag{63}$$

where the omitted terms are $W\overline{W}$ and $W^2\overline{W}^2$, since the factor in front of the integral obviously will be W^2. Moreover, in the considered $U(1)$ gauge effective theory, they do not give the contribution. With the purpose to reduce the problem of performing the trivial integration over $d^2\psi$, we extract from the T exponent

$$\exp T(\psi_\alpha iW^\alpha+\psi_\alpha\partial^\beta N^\alpha_\beta),\quad N^\alpha_\beta=if^\alpha_\beta+\delta^\alpha_\beta D',$$

the operator of affine transformations, i.e., $\exp(\psi_\alpha\partial^\beta)$. Using

$$[\psi_\alpha\partial^\beta,\psi_\gamma]=\delta^\beta_\gamma\psi_\alpha,\quad \exp(\psi_\alpha\partial^\beta N^\alpha_\beta)\cdot 1=1,$$

and the identity (16), we get

$$K(T)=\int\frac{d^4p}{(2\pi)^4}d^2\psi e^{-TD'}\times\exp\left\{-iW^\alpha\left(\frac{e^{TN}-1}{N}\right)_\alpha^{\ \beta}\psi_\beta\right\}e^{T\Box_p}=W^2e^{-TD'}\,\mathrm{tr}\left(\frac{e^{TN}-1}{N}\right)\int\frac{d^4p}{(2\pi)^4}e^{T\Box_p}. \tag{64}$$

The last factor is the Schwinger result (39) for a scalar loop. For calculations of a factor which modify the Heisenberg-Euler Lagrangian, we diagonalize the matrix N and find directly

$$\Gamma_{(1)}=\int d^8zW^2\int_0^\infty\frac{dT}{T}\times e^{-Tm^2}\frac{\cosh(TD')-\cosh(T\mathcal{H}_-)}{D'^2-\mathcal{H}_-^2}K(T)_{\mathrm{Schw}}, \tag{65}$$

where $\mathcal{H}_-$ was defined in Eq. (40). Note the coincidence of this result with the result of Refs. [47,33], obtained by essentially different methods. Certainly, there is an ultraviolet divergence, which can be excluded with the help of a wave function renormalization. It is important to note, that the corrections to W^2 contain nonholomorphic, in the sense of Seiberg, terms $f_{\dot\alpha\dot\beta}$. The superfield action (65) reproduces correctly the results of the calculations on the component level [13].

In the last example we will consider contributions from only the quantum gauge field V. After splitting the field into a background and quantum part, the SYM action in Fermi-Feynman gauge is

$$\mathcal{S}=-\frac{1}{2g^2}\mathrm{Tr}[(e^{-V}\nabla^\alpha e^V)\overline{\nabla}^2(e^{-V}\nabla_\alpha e^V)+V(\overline{\nabla}^2\nabla^2+\nabla^2\overline{\nabla}^2)V].$$

The quadratic action has the form

$$\mathcal{S}_0=-\frac{1}{2g^2}\mathrm{Tr}(V[\Box-iW^\alpha\nabla_\alpha-i\overline{W}^{\dot\alpha}\overline{\nabla}_{\dot\alpha}]V).$$

All the dependence on the background fields is through the connection coefficients and through the background field strength. Further, we use the heat kernel representation of the EA and change all quantities by pseudodifferential operators as before. In this case $\Box_V=\Box_p+i\nabla^p_\alpha W^\alpha_p+i\overline{\nabla}^p_{\dot\alpha}\overline{W}^{\dot\alpha}_p$. Here all one-loop background graphs are finite in super QCD theories, but they potentially have an infrared singularity, that is an attribute of an unstable mode. We consider $U(1)$ gauge theory case. Following our strategy, we set all three operators in separate exponents

$$e^{T\nabla^p_\alpha iW^\alpha_p}e^{T\overline{\nabla}^p_{\dot\alpha}i\overline{W}^{\dot\alpha}_p}L(W,\overline{W})e^{T\Box_p}.$$

where $L(W,\overline{W})$ is the function of the superfields $W,\overline{W}$ and the operator $\nabla^p_{\alpha\dot\alpha}$. For SQED, where the power W is limited by 2, the function does not give the contribution to the EA. Now, as well as in the previous example of this section, we have nonzero contributions to $d^2\psi d^2\overline{\psi}$ integrals

$$\int d^2\psi d^2\overline{\psi} e^{T\psi_\alpha i W^\alpha_p} e^{T\overline{\psi}_{\dot\alpha} i \overline{W}^{\dot\alpha}_p} \int \frac{d^4p}{(2\pi)^4} e^{T\Box_p}$$

and we can, using results of the previous calculations, show at once the final result

$$K(T)=W^2\overline{W}^2 \det\left(\frac{e^{TN}-1}{N}\right)\det\left(\frac{e^{T\overline{N}}-1}{\overline{N}}\right) \times\frac{1}{(4\pi T)^2}\left[\det\frac{TF}{\sinh(TF)}\right]^{1/2}, \tag{66}$$

where $N_\alpha^\beta = iD_\alpha W^\beta$, $\overline{N}^{\dot\beta}_{\dot\alpha} = i\overline{D}_{\dot\alpha}\overline{W}^{\dot\beta}$. To check this result, we could use the technique of correlator calculation [47], which we have already demonstrated in Sec. V.

As well as for covariant constant YM background, the condition $[\nabla_{\alpha\dot\alpha}, W_A]=0$ leads to the anticommutator $\{W^a_\alpha, \overline{W}^a_{\dot\alpha}\}=0$, i.e., in this approximation the superfields $W,\overline{W}$ are effective Abelian, and we can use the results for the EA super QED with certain changes. Full DEEA on a SYM background and chiral superfields both in adjoint and in fundamental representation demands a more detailed consideration. The complication originates from $\mathcal{S}^2_{\rm mix}$ and mass terms in the operator $\Box_V$, which depends on chiral fields.

VIII. SUMMARY

In the present paper we develop elegant and effective technique based on noncommutative geometry of deformation quantization for calculation of the expansion in the derivatives of background fields for the one-loop effective action. It is important that the supersymmetrical and gauge invariant form is conserved through all stages of calculations. We use the simple idea of exploiting a canonical transformation that leads to the normal coordinate expansion of symbols. It is the well known realization of equivalence principle which requires the existence of such a reference frame at every point that the effects of gauge fields can be locally neglected.

To test the approach suggested we focused on comprehensively investigated models, though all constructions could be applied straightforwardly to the QFT models involving difficulties in the quantization. In all examples considered, the results of the proposed computing scheme coincide completely with the known ones. The suggested approach allows ''manual'' manipulations to be effectively replaced by computer methods to get all next HMDS coefficients in the expansion of the one-loop effective action.

It can be also said that the approach shows the problem from another side and extends our knowledges about the structure of the path integrals. Other applications of the presented method and its modifications for nonflat and harmonic superspace will be given in the subsequent papers.

ACKNOWLEDGMENTS

We thank V. G. Serbo and A. I. Vainstein for useful discussions. The authors are also grateful to V. P. Gusynin and I. Avramidi for informing us about several papers. This work was partially supported by RFBR Grant No. 96-02-19079.

[1] B. S. De Witt, *Dynamical Theory of Groups and Fields* (Gordon and Breach, New York, 1965).
[2] S. Coleman and E. Weinberg, Phys. Rev. D **7**, 1888 (1973).
[3] A. O. Barvinsky and G. A. Vilkovisky, Phys. Rep. C **119**, 1 (1985); G. A. Vilkovisky, in *Quantum Theory of Gravity*, edited by S. Christensen (Hilger, Bristol, 1983), p. 169.
[4] R. D. Ball, Phys. Rep. C **182**, 1 (1989).
[5] J. Schwinger, Phys. Rev. **82**, 664 (1951).
[6] I. G. Avramidi, Nucl. Phys. **B355**, 712 (1991).
[7] A. O. Barvinsky and G. A. Vilkovisky, Nucl. Phys. **B282**, 163 (1987); **B333**, 471 (1990).
[8] E. S. Fradkin, Nucl. Phys. **76**, 588 (1966); E. S. Fradkin, D. M. Gitman, and S. M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum* (Nauka, Moscow, 1990); H. B. Nielsen and Rorlich, Nucl. Phys. **B299**, 471 (1988); A. P. Balachandran, G. Marmo, B. S. Skagerstam, and A. Stern, *Gauge Symmetry and Fibre Bundles, Vol. 188 of Lecture notes in Physics* (Springer-Verlag, Berlin, 1983).
[9] Z. Bern and D. A. Kosower, Phys. Rev. Lett. **66**, 1669 (1991); Nucl. Phys. **B379**, 451 (1992); M. J. Strassler, *ibid.* **B385**, 145 (1992); M. Reuter, M. G. Schmidt, and C. Schubert, Ann. Phys. (N.Y.) **259**, 313 (1997).
[10] S. de Boer, B. Peeters, K. Skenderis, and P. van Nieuwenhuizen, hep-th/9511141.
[11] F. A. Berezin and M. A. Shubin, *Schrödinger Equation* (Moskow State University Press, Moskov, 1983); F. A. Berezin and M. A. Shubin, in Hilbert Space Operators and Operator Algebras, proceedings of the International Conference, Tihany, 1970.
[12] P. West, Phys. Lett. B **285**, 375 (1991); I. Jack, P. R. T. Jones, and P. West, *ibid.* **285**, 382 (1991); I. L. Buchbinder, S. M. Kuzenko, and A. Yu. Petrov, *ibid.* **321**, 372 (1994); and general discussion in Ref. [15].
[13] M. Hug, Phys. Rev. D **16**, 1733 (1977); R. Miller, Nucl. Phys. **B241**, 535 (1984); P. Di Vecchia, R. Musto, F. Nicodemi, and R. Pettorino, *ibid.* **B252**, 635 (1985).
[14] M. T. Grisaru, F. Riva, and D. Zanon, Nucl. Phys. **B214**, 465 (1983); A. D'Adda, A. C. Davis, P. Di Vecchia, and P. Salomonson, *ibid.* **B222**, 45 (1983).
[15] I. Buchbinder and S. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity* (IOP, Bristol, 1995).
[16] S. J. Gates, M. T. Grisaru, M. Roček, and W. Siegel, *Superspace* (Benjamin/Cummings, Reading, MA, 1985).
[17] N. Seiberg and E. Witten, Nucl. Phys. **B426**, 19 (1994); **B430**,

485(E) (1994); **B431**, 484 (1994); N. Seiberg, hep-th/9506077; hep-th/9408013.

[18] B. de Wit, M. T. Grisaru, and M. Roček, Phys. Lett. B **374**, 297 (1996); M. T. Grisaru, M. Roček, and R. von Unge, *ibid.* **383**, 415 (1996).

[19] A. Pickering and P. West, Phys. Lett. B **374**, 297 (1996).

[20] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky, and E. Sokatchev, Class. Quantum Grav. **1**, 469 (1984).

[21] I. L. Buchbinder, E. I. Buchbinder, E. A. Ivanov, S. M. Kuzenko, and B. A. Ovrut, Phys. Lett. B **412**, 309 (1997); I. L. Buchbinder, S. M. Kuzenko, and B. A. Ovrut, *ibid.* **433**, 335 (1998).

[22] L. Alvarez-Gaume, J. Distler, C. Kounnas, and M. Marino, Int. J. Mod. Phys. A **11**, 4745 (1996).

[23] L-H. Chan, Phys. Rev. Lett. **54**, 1222 (1985); L-H. Chan, ''Derivative expansion for the one-loop effective actions with internal symmetry,'' CTP Report No. 41345, 1986; O. Cheyette, Phys. Rev. Lett. **55**, 2394 (1985); C. M. Fraser, Z. Phys. C **28**, 101 (1985); J. Caro and L. L. Salcedo, Phys. Lett. B **309**, 359 (1993).

[24] M. Gaillard, Nucl. Phys. **B268**, 669 (1986); V. Jain and A. Kotchev, Report No. LAPP-TH-285/90.

[25] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerovicz, and D. Sternheimer, Ann. Phys. (N.Y.) **111**, 61 (1978); **111**, 111 (1978).

[26] H. Weyl, Z. Phys. **46**, 1 (1927); E. P. Wigner, Phys. Rev. **40**, 749 (1932); J. E. Moyal, Proc. Cambridge Philos. Soc. **45**, 99 (1949).

[27] I. A. Batalin and I. V. Tyutin, Nucl. Phys. **B345**, 645 (1990).

[28] D. Sternheimer, Report No. math.QA/19809056.

[29] V. A. Novikov, M. A. Schifman, A. I. Vainstein, V. I. Zakharov, Fortschr. Phys. **32**, 585 (1985); JETP Lett. **35**, 323 (1982); Sov. J. Nucl. Phys. **39**, 77 (1984).

[30] D. Fliegner, P. Haberl, M. G. Schmidt, and C. Schubert, hep-th/9411177; Ann. Phys. (N.Y.) **264**, 51 (1998); D. Fliegner, M. G. Schmidt, and C. Schubert, Z. Phys. C **64**, 111 (1994).

[31] V. P. Gusynin and I. A. Shovkovy, Can. J. Phys. **74**, 282 (1996); hep-th/9804143.

[32] I. Buchbinder, S. Kuzenko, and J. Yarevskaya, Phys. At. Nucl. **56**, 202 (1993); I. Buchbinder, S. Kuzenko, and A. Petrov, Phys. Lett. B **321**, 372 (1974); I. L. Buchbinder, S. Kuzenko, and Zh. Yarevskaya, Nucl. Phys. **B411**, 665 (1994).

[33] K. Shizuya and Y. Yasui, Phys. Rev. D **29**, 1160 (1984); Th. Ohrndorf, Nucl. Phys. B **273**, 165 (1986); Phys. Lett. B **176**, 421 (1986).

[34] F. Antonsen, quant-ph/9802031.

[35] N. P. Landsman, J. Geom. Phys. **12**, 93 (1993); M. A. Robson, hep-th/9406041.

[36] R. L. Stratonovich, Dokl. Akad. Nauk SSSR **109**, 72 (1956).

[37] M. Bordemann, N. Neumaier, and S. Waldmann, Report No. q-alg/9707030; Report No. q-alg/9711016.

[38] E. S. Fradkin and D. M. Gitman, Phys. Rev. D **44**, 3230 (1991); D. G. C. McKeon and A. Rebhan, *ibid.* **48**, 2891 (1993).

[39] E. D'Hoker and D. G. Gagné, Nucl. Phys. **B467**, 272 (1996); **B467**, 297 (1996); M. Mondragón, L. Nellen, M. G. Schmidt, and C. Shubert, Phys. Lett. B **351**, 200 (1995); *ibid.* **366**, 212 (1996).

[40] D. G. C. McKeon and C. S. Shubert, Phys. Lett. B **440**, 101 (1998).

[41] L. L. Salcedo and E. Ruiz Arriola, Ann. Phys. (N.Y.) **250**, 1 (1996).

[42] J. Hauknes, Ann. Phys. (N.Y.) **156**, 303 (1984); H. W. Lee, P. Y. Pac, and H. K. Shin, Phys. Rev. D **40**, 4202 (1989); D. Cangemi, E. D'Hoker, and G. Dunne, *ibid.* **51**, 2513 (1995); D. G. C. McKeon, *ibid.* **55**, 7989 (1997).

[43] T. G. Gargett and I. N. McArthur, J. Math. Phys. **39**, 4430 (1998).

[44] Photon '97, Proceedings of the International Workshop on Gamma-Gamma Collision; R. N Lee, A. I. Milstein, and V. A. Strakhovenko, Novosibirsk, IYF Report No. BUDKERINP-97-35, hep-ph/9704230; I. F. Ginzburg, G. L. Kotkin, and S. I. Polityko, Sov. J. Nucl. Phys. **37**, 222 (1983); G. L. Kotkin and V. G. Serbo, Phys. Lett. B **413**, 122 (1997).

[45] V. I. Ritus, The Lagrangian Function of an Intense Electromagnetic Field and Quantum Electrodynamics at short Distances, in Issues in Intense Field Quantum Electrodynamics Proceedings of the Lebedev Physics Institute, 1987, Vol. 168.

[46] I. G. Avramidi, Phys. Lett. B **305**, 27 (1993).

[47] I. N. Mc Arthur and T. D. Gargett, Nucl. Phys. **B494**, 525 (1997).

[48] L. D. Landau and I. M. Lifschitz (Pergamon Oxford, England), Vol. 2.

[49] I. A. Batalin and A. E. Shabad, Sov. Phys. JETP **33**, 483 (1971).

[50] J. Bagger and J. Wess, *Supersymmetry and Supergravity* (Princeton University Press, Princeton, 1992).

[51] T. Ohrndorf, Report No. CERN-TH.4198/85.