

# Finite number and finite size effects in relativistic Bose-Einstein condensation

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Bose-Einstein condensation of a relativistic ideal Bose gas in a rectangular cavity is studied. Finite size corrections to the critical temperature are obtained by the heat kernel method. Using zeta-function regularization of the one-loop effective potential, lower-dimensional critical temperatures are calculated. In the presence of strong anisotropy, the condensation is shown to occur in multisteps. The criteria of this behavior is that critical temperatures corresponding to lower-dimensional systems are smaller than the three-dimensional critical temperature. [S0556-2821(99)02818-0]

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## I. INTRODUCTION

Bose-Einstein condensation (BEC), first predicted in 1924 [1–4], has become an explosive field of research in recent years [5–8]. Because of the temperature dependence of the chemical potential in a Bose-Einstein distribution function, a macroscopic number of bosons start accumulating onto the ground state at the critical temperature. This phenomenon is typically phrased as the phase density of particles being large enough such that all particles characterized by the de Broglie wavelength overlap to form a condensate. Liquid helium, which becomes a superfluid at the transition, has until recently been known as the only substance which shows this behavior. However, the strong interaction between helium atoms has been an obstacle for a complete understanding of the mechanism of condensation.

Recent technological progress in atom cooling techniques made it possible to achieve Bose-Einstein condensation for neutral atoms. The weakly interacting nature of these atoms enables one to understand the condensation within the perturbative framework. It also provides an ideal testing ground for some fundamental aspects of quantum mechanics in a controlled environment. Studies of similar phenomena, stimulated by rapid progress in this subject, are no longer restricted to condensed-matter physics and atom optics but start to involve other areas in physics, such as nuclear-particle physics and astrophysics [5].

In this paper we study the effect of a finite size container on the condensation. For a finite size system, the absence of thermodynamic limit alters various critical behaviors defined and expected for a bulk system [9–11]. Thermodynamic quantities such as the free energy has a surface term which vanishes in the thermodynamic limit, causing a shift in the critical point. Finite size effects in Bose-Einstein condensa-

tion manifest themselves as the rounding-off of the kink in the specific heat at the critical temperature.<sup>1</sup> Off-diagonal long range order stays in a finite range at the critical point.

For a finite size quantum system, the invariant operator of small fluctuations has a discrete spectrum. The relevant dimensionless parameter  $\eta_i$  which characterizes finite size effects near the transition is given by  $\eta_i = \lambda_{\theta dB} / L_i$  for a non-relativistic system, where  $\lambda_{\theta dB}$  is the thermal de Broglie wavelength and  $L_i (i = 1, 2, 3)$  are the system sizes in the three spatial directions.  $\eta_i = \beta / L_i$  for a relativistic system, and  $\eta_i = \beta \omega_i$  for a harmonic oscillator with natural frequencies  $\omega_i (i = 1, 2, 3)$ .<sup>2</sup>

The presence of anisotropy adds more variety to the critical behavior. Suppose  $\eta_i > 1$  for some  $i$  (for a nonrelativistic case, this implies  $L_i < \lambda_{\theta dB}$ ) then only the lowest mode in the  $i$ th direction contributes significantly to the dynamics of the system. The motion in the  $i$ th direction is frozen out and the system has an infrared behavior effectively equivalent to a system with one less dimension in that direction [12–17]. Thus we can classify the dynamics with an effective infrared dimension (EIRD) into the following four cases dependent on the degree of anisotropy.

- Case 1:  $\eta_1, \eta_2, \eta_3 > 1 \rightarrow \text{EIRD} = 0$ ,
- case 2:  $\eta_1, \eta_2 > 1 > \eta_3 \rightarrow \text{EIRD} = 1$ ,
- case 3:  $\eta_1 > 1 > \eta_2, \eta_3 \rightarrow \text{EIRD} = 2$ ,
- case 4:  $1 > \eta_1, \eta_2, \eta_3 \rightarrow \text{EIRD} = 3$ .

In this paper, we mainly study case 4 where modes in all three directions are excitable. As the temperature is lowered, the crossover behavior between higher- and lower-dimensional excitations can be observed. Each mode is labeled by three quantum numbers associated with the excitation energy in each direction. In the presence of strong

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<sup>1</sup>The bulk specific heat of an ideal Bose gas is not divergent, but shows the discontinuity in its derivative.

<sup>2</sup>In this paper, we use the units where  $k_B = \hbar = 1$ , which renders  $\eta_i$  dimensionless. The results in ordinary units can be obtained by replacing  $\omega$  by  $\hbar \omega$  and  $T$  by  $k_B T$ .

anisotropy, with these quantum numbers, it is meaningful to split the whole particle spectrum into either zero-, one-, two-, or three-dimensional excitations. The ground state, being the state with the lowest quantum numbers, is viewed as a zero-dimensional excitation. Let us denote the number of modes excited in the corresponding directions as  $N_0, N_1, N_2, N_3$ , respectively.

We can define an  $n$ -dimensional critical temperature  $T_{nD}$  ( $n=1,2,3$ ) as the temperature at which all the  $n$ -dimensionally excited modes are saturated;

$$\text{three-dimensional temperature: } N = N_3(T_{3D}), \quad (1)$$

$$\text{two-dimensional temperature: } N = N_3(T_{2D}) + N_2(T_{2D}), \quad (2)$$

$$\begin{aligned} \text{one-dimensional temperature: } N = N_3(T_{1D}) + N_2(T_{1D}) \\ + N_1(T_{1D}), \end{aligned} \quad (3)$$

where the thermodynamic limit in each case is taken differently. As we set the total number of particles  $N \rightarrow \infty$ , we obtain  $T_{3D}$  by tuning  $\eta_1, \eta_2$ , and  $\eta_3$  to zero,  $T_{2D}$  by tuning  $\eta_2$  and  $\eta_3$  to zero while  $\eta_1$  fixed, and  $T_{1D}$  by tuning  $\eta_3$  to zero while  $\eta_1$  and  $\eta_2$  fixed. Finite size corrections necessarily modify the above definitions, since they involve excitations in lower dimensions. In Sec. III, we will discuss this aspect in detail.

By changing the edge lengths of a cavity or oscillator frequencies for a magnetic trap, it is possible to control the critical temperature and realize the lower dimensional condensation. In particular, when  $T_{1D} < T_{2D} < T_{3D}$  holds, condensation is expected to occur in three steps: As the temperature is lowered, condensation into two-dimensionally excited modes begins at  $T_{3D}$  when three-dimensionally excited states saturate. At the two-dimensional critical temperature  $T_{2D}$  condensation into one-dimensionally excited modes begins. The condensation onto the ground state does not occur until one dimensional critical temperature  $T_{1D}$  is reached.

In a finite size system, the reduced chemical potential  $\epsilon \equiv \beta(E_0 - \mu)$  does not vanish. From the expression for the ground state contribution, we can still assume  $\epsilon \sim 0$  up to order  $1/N_0$ . This condition is justified as long as  $N_0$  is close to the total number of particles, or equivalently, the temperature is lower than the critical temperature.

Although work on BEC in relativistic systems has a long history, modern treatment using quantum field theory did not begin until 1980s [18,19,21–23]. At relativistic temperatures,  $T > 2m$ , where  $m$  is the mass of the relativistic field, pair creation-annihilation effects become nonnegligible, and the particle number is no longer conserved. However, the global  $U(1)$  gauge symmetry of the Hamiltonian guarantees the existence of a conserved charge based on Noether's theorem. The net charge  $Q$  in relativistic field theory is given by

$$Q = \sum_I \left[ \frac{1}{e^{\beta(E_I - \mu)} - 1} - \frac{1}{e^{\beta(E_I + \mu)} - 1} \right]. \quad (4)$$

Particles and antiparticles have chemical potentials opposite in sign due to the fact that they carry opposite charges. Taking this fact into account, the positive definiteness of the particle number of the particles and antiparticles with energy  $E_N$  requires that  $|\mu| \leq m$ .

Another important point is the relation between spontaneous symmetry breaking (SSB) and Bose-Einstein condensation [19]. Condensation into the ground state results in a nonzero vacuum expectation value of the field. Hence BEC can be interpreted as a SSB of the local gauge symmetry. This argument presumes that the chemical potential reaches its critical value at the critical temperature. However, for a finite system, this is generally not the case.

Bose-Einstein condensation of a relativistic noninteracting quantum field in a rectangular cavity is studied in this paper. Similar aspects for a nonrelativistic ideal Bose gas in anisotropic magnetic traps is discussed in a companion paper [20]. In Sec. II, we derive the effective action which includes one-loop quantum corrections to the classical action and use the  $\zeta$  function regularization to evaluate the grand canonical thermodynamic potential [24–27]. The generalized  $\zeta$  function is written in terms of  $\theta$  functions via a Mellin transformation. We then use the asymptotic expansion of  $\theta$  functions for a large system size and a small massive field to see the finite size correction to the total charge and the critical temperature. This asymptotic expansion is a special case of the more general class of short time expansion of the heat kernel which is used for spectral analysis on an arbitrary differentiable manifold [28]. The terms in the expansions correspond to the volume (Weyl), area, and edge contributions, etc. [26,29,30]. In Sec. III, we consider the effects of accidental degeneracy in a discrete spectrum and show that the highly oscillating behavior of the density of states is large enough to dominate over the higher order terms in the asymptotic expansion. We introduce an infrared cutoff to include the lowest mode contribution properly and estimate the lower-dimensional critical temperature accurately. In the last part of this paper, we discuss the multistep behavior of condensation process in the presence of strong anisotropy [13,31]. The conditions for one-, two-, and three-dimensional condensations are clarified. The relevant critical temperatures are obtained.

As discussed and observed in Refs. [32,33] for a weakly interacting gas, the corrections to bulk ideal-gas ground state occupation number and critical temperature are well explained by the finite size effects. Interaction effects on those quantities are negligibly small. Hence we expect the results discussed in this paper will still hold for weakly interacting gases. On the other hand, interaction effects are known to affect higher moments such as the specific heat significantly and considered to be essential in explaining the observed specific heat data. Extension of our analysis to the strongly interacting case is a nontrivial problem which deserves further careful study.

Bose-Einstein condensation of a relativistic gas could be relevant to cosmology in the dark matter problem [34] or for inflationary universe [35]. Our problem is directly related to condensation of positronium in a cavity discussed in Ref. [36]. Although we restrict our study to a rectangular cavity,

similar results are expected for systems with a finite boundary. In the presence of an irregular boundary the dynamics of the system can be nonintegrable or chaotic. The implication of our result for nonintegrable systems of this kind is of particular interest [37]. The extension to curved spacetimes can also be obtained by similar methods [38–41].

## II. EFFECTIVE ACTION AND HEAT KERNEL

### A. One loop effective action

The action of a free complex scalar field

$$S_0[\phi] = \int d^4x [\partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi] \quad (5)$$

is invariant under the global  $U(1)$  gauge transformation

$$\phi(x) \rightarrow e^{i\eta} \phi(x). \quad (6)$$

The corresponding Noether current is

$$J_\mu(x) = i\phi^\dagger \partial_\mu \phi - i\partial_\mu \phi^\dagger \phi, \quad (7)$$

with the total charge

$$Q = \int_V d^3x J_0(x) = i \int_V d^3x [\phi^\dagger \dot{\phi} - \dot{\phi}^\dagger \phi]. \quad (8)$$

The integration is over the volume  $V$  of the cavity, here assumed to be rectangular of edge lengths  $L_i (i=1,2,3)$ . Decomposing  $\phi(x)$  into real and imaginary parts such that  $\phi(x) = 1/\sqrt{2}[\phi_1(x) + i\phi_2(x)]$ , Eq. (5) becomes

$$S_0[\phi] = \frac{1}{2} \int d^4x [\partial^\mu \phi_1^\dagger \partial_\mu \phi_1 + \partial^\mu \phi_2^\dagger \partial_\mu \phi_2 - m^2 \phi_1^\dagger \phi_1 - m^2 \phi_2^\dagger \phi_2]. \quad (9)$$

The Hamiltonian for this action is

$$H = \frac{1}{2} \int_V d^3x [\pi_1^2 + \pi_2^2 + (\nabla \phi_1)^2 + (\nabla \phi_2)^2 + m^2 \phi_1^\dagger \phi_1 + m^2 \phi_2^\dagger \phi_2], \quad (10)$$

where  $\pi_1 = \dot{\phi}_1$ ,  $\pi_2 = \dot{\phi}_2$  are the momentum fields canonically conjugate to  $\phi_1$  and  $\phi_2$ . The total charge becomes  $Q = \int d^3x (\phi_2 \pi_1 - \phi_1 \pi_2)$ .

The grand canonical partition function for this system when brought in contact with a heat bath at temperature  $T = 1/\beta$  is given by

$$Z = \text{Tr} e^{-\beta(\hat{H} - \mu \hat{Q})}, \quad (11)$$

where  $\hat{H}$  and  $\hat{Q}$  are the Hamiltonian and the total charge operators respectively and  $\mu$  is the chemical potential. Equation (11) in Hamiltonian form has a path integral representation

$$Z = \int D\pi D\phi \exp \left[ \int_0^\beta d\tau \int_V d^3x [i\pi_1 \dot{\phi}_1 + i\pi_2 \dot{\phi}_2 - H + \mu(\phi_2 \pi_1 - \phi_1 \pi_2)] \right], \quad (12)$$

where  $\dot{\phi}_i = \partial_\tau \phi_i$ . In the spirit of (imaginary time) finite temperature field theory, a periodic boundary condition is imposed on  $\phi_i$ , with  $\phi_i(0, \vec{x}) = \phi_i(\beta, \vec{x})$ . We perform an integral over the momentum field and obtain

$$Z = \int D\phi e^{-S[\phi]}, \quad (13)$$

with the action

$$S[\phi] = \int_0^\beta d\tau \int_V d^3x \left[ \frac{1}{2} (\dot{\phi}_1 - i\mu \phi_2)^2 + \frac{1}{2} (\dot{\phi}_2 + i\mu \phi_1)^2 + \frac{1}{2} (\nabla \phi_1)^2 + \frac{1}{2} (\nabla \phi_2)^2 + \frac{1}{2} m^2 (\phi_1^2 + \phi_2^2) \right]. \quad (14)$$

Using the background field decomposition  $\phi = \phi_c + \varphi$  with fluctuation  $\varphi$ , and expanding the action in Eq. (14) around the classical solution  $\phi_c$  which minimizes the action

$$S[\phi] = S[\phi_c] + \frac{1}{2} \sum_{i,j=1}^2 \frac{\delta^2 S}{\delta \phi_i \delta \phi_j} \varphi_i \varphi_j + O(\varphi^3). \quad (15)$$

The partition function can be written as

$$Z = e^{-\Gamma[\phi_c]} = e^{-S[\phi_c]} \int D\varphi e^{-(1/2)\Lambda_{ij}[\phi_c] \varphi_i \varphi_j}, \quad (16)$$

where  $\Gamma[\phi_c]$  is the effective action and  $\Lambda_{ij}[\phi_c] \equiv \delta^2 S[\phi_c] / \delta \phi_i \delta \phi_j$ .

The functional measure  $D\varphi$  is defined as

$$D\varphi = \prod_n \frac{dc_n}{\sqrt{2\pi l}}, \quad (17)$$

where  $c_n$  are the coefficients of an eigenfunction expansion of  $\varphi$  and  $l$  is a constant with unit of length. Then the functional integral in Eq. (16) can be evaluated as

$$\prod_n \frac{1}{\sqrt{2\pi l}} \int_{-\infty}^{\infty} dc_n e^{-(1/2)\Lambda_n c_n^2} = \text{Det}(l^2 \Lambda_{ij}[\phi_c])^{-1/2}. \quad (18)$$

The effective action to one loop order is given by [42–44]

$$\Gamma[\phi_c] = S[\phi_c] + \frac{1}{2} \log \text{Det}(l^2 \Lambda_{ij}[\phi_c]). \quad (19)$$

The second term in Eq. (19) can be split into two parts as

$$\log \text{Det}(l^2 \Lambda_{ij}[\phi_c]) = \log \text{Det}(l^2 \Lambda_+[\phi_c]) + \log \text{Det}(l^2 \Lambda_-[\phi_c]), \quad (20)$$

where  $\Lambda_{\pm} = -(\partial_{\tau} \pm \mu)^2 - \nabla^2 + m^2$ . The eigenvalues of  $\Lambda_{\pm}$  are given by

$$\lambda_{n,N}^{\pm} = \left( \frac{2\pi n}{\beta} \pm i\mu \right)^2 + \omega_N + m^2, \quad (21)$$

where  $n$  is an integer and  $\omega_N$  is the eigenvalue of  $-\nabla^2$ .

### B. $\zeta$ -function regularization

The generalized  $\zeta$  function for an elliptic differential operator  $\mathcal{O}$  is defined by

$$\zeta_{\mathcal{O}}(s) = \text{Tr } \mathcal{O}^{-s} = \sum_N \lambda_N^{-s}, \quad (22)$$

where  $\lambda_N$  are eigenvalues of  $\mathcal{O}$ . From Eq. (22),

$$\zeta'_{\mathcal{O}}(s) = - \sum_N \lambda_N^{-s} \log \lambda_N^{-s}. \quad (23)$$

Thus

$$\log \text{Det}(l^2 \mathcal{O}) = \zeta_{\mathcal{O}}(0) \log l^2 - \zeta'_{\mathcal{O}}(0). \quad (24)$$

Using a Mellin transformation defined by

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} d\tau \tau^{s-1} e^{-\lambda\tau} \quad (25)$$

we can write the generalized  $\zeta$  function for  $\Lambda_{ij}$  as

$$\begin{aligned} \zeta_{\Lambda_{\pm}}(s) &= \text{Tr } \Lambda_{\pm}^{-s} = \sum_N \lambda_{\pm N}^{-s} \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} d\tau \tau^{s-1} \sum_N e^{-\lambda_{\pm N}\tau} \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} d\tau \tau^{s-1} K_{\Lambda_{\pm}}(\tau), \end{aligned} \quad (26)$$

where  $\lambda_{\pm N}$  and  $K_{\Lambda_{\pm}}(\tau)$  are eigenvalues and the heat kernels for  $\Lambda_{\pm}$ .

Here

$$\begin{aligned} K_{\Lambda_{\pm}}(\tau) &= K_0(\tau) = \sum_{n=-\infty}^{\infty} \sum_N \\ &\times \exp \left\{ -\tau \left[ \left( \frac{2\pi n}{\beta} + i\mu \right)^2 + \omega_N + m^2 \right] \right\} \end{aligned} \quad (27)$$

yielding

$$\begin{aligned} \zeta_{\Lambda}(s) &\equiv \zeta_{\Lambda_+}(s) = \zeta_{\Lambda_-}(s) \\ &= \frac{\bar{\beta}^{2s}}{\Gamma(s)} \int_0^{\infty} d\tau \tau^{s-1} K_0(\bar{\beta}^2 \tau), \end{aligned} \quad (28)$$

where  $\bar{\beta} \equiv \beta/2\pi$  [41,45] and

$$K_0(\bar{\beta}^2 \tau) = K(\tau) e^{-(m^2 - \mu^2) \bar{\beta}^2 \tau} \theta_3(\mu \bar{\beta} \tau | i\tau/\pi), \quad (29)$$

where  $\theta_3(z|\tau) = 1 + 2 \sum_{n=1}^{\infty} e^{i\pi n \tau} \cos(2nz)$  is a  $\theta$  function [46]. The heat kernel  $K(\tau)$  for  $-\nabla^2$  is defined by

$$K(\tau) = \sum_N e^{-\bar{\beta}^2 \omega_N \tau}. \quad (30)$$

With this, the effective action can be expressed in terms of  $\zeta_{\Lambda}(s)$  as

$$\Gamma[\phi_c] = S[\phi_c] + \zeta_{\Lambda}(0) \log l^2 - \zeta'_{\Lambda}(0). \quad (31)$$

We first consider Neumann boundary conditions at the boundary of the cavity. The corresponding eigenfunction for  $-\nabla^2$  is

$$\phi_N(x) = \sqrt{\frac{2}{L_1 L_2 L_3}} \cos\left(\frac{\pi n_1 x_1}{L_1}\right) \cos\left(\frac{\pi n_2 x_2}{L_2}\right) \cos\left(\frac{\pi n_3 x_3}{L_3}\right) \quad (32)$$

and the eigenvalue  $\omega_N$  is

$$\omega_N = \left(\frac{\pi n_1}{L_1}\right)^2 + \left(\frac{\pi n_2}{L_2}\right)^2 + \left(\frac{\pi n_3}{L_3}\right)^2, \quad (33)$$

where  $n_i = 0, 1, 2, \dots$  ( $i = 1, 2, 3$ ). The eigenfunction for Dirichlet boundary conditions can be written similarly as

$$\phi_N(x) = \sqrt{\frac{2}{L_1 L_2 L_3}} \sin\left(\frac{\pi n_1 x_1}{L_1}\right) \sin\left(\frac{\pi n_2 x_2}{L_2}\right) \sin\left(\frac{\pi n_3 x_3}{L_3}\right) \quad (34)$$

and the eigenvalue  $\omega_N$  is

$$\omega_N = \left(\frac{\pi n_1}{L_1}\right)^2 + \left(\frac{\pi n_2}{L_2}\right)^2 + \left(\frac{\pi n_3}{L_3}\right)^2, \quad (35)$$

where  $n_i = 1, 2, \dots$  ( $i = 1, 2, 3$ ).

### C. Asymptotic expansion of the heat kernel

The heat kernel for all accessible quantum states is given by

$$\begin{aligned} K(\tau) &= \sum_N e^{-\bar{\beta}^2 \omega_N \tau} \\ &= \sum_{n_1}^{\infty} e^{-\eta_1^2 \pi^2 n_1^2 \tau} \sum_{n_2}^{\infty} e^{-\eta_2^2 \pi^2 n_2^2 \tau} \sum_{n_3}^{\infty} e^{-\eta_3^2 \pi^2 n_3^2 \tau} \\ &= \frac{1}{8} [\theta_3(0 | i\eta_1^2 \pi \tau) \pm 1] [\theta_3(0 | i\eta_2^2 \pi \tau) \pm 1] \\ &\quad \times [\theta_3(0 | i\eta_3^2 \pi \tau) \pm 1], \end{aligned} \quad (36)$$

where  $\eta_i = \bar{\beta}/L_i$  for  $i = 1, 2, 3$ . Positive (negative) signs correspond to Neumann (Dirichlet) boundary conditions. If we assume  $L_i \gg \bar{\beta}$  or  $\eta_i \ll 1$  for  $i = 1, 2, 3$ , we can make use of the asymptotic behavior of the  $\theta$  function as  $\tau \rightarrow 0$

$$\theta_3(0|i\tau) \rightarrow \frac{1}{\sqrt{\tau}} \quad (37)$$

to obtain the following asymptotic property for the heat kernel:

$$\begin{aligned} K(\tau) &\rightarrow \frac{1}{8} \left[ \frac{1}{\sqrt{\eta_1^2 \pi \tau}} \pm 1 \right] \left[ \frac{1}{\sqrt{\eta_2^2 \pi \tau}} \pm 1 \right] \left[ \frac{1}{\sqrt{\eta_3^2 \pi \tau}} \pm 1 \right] \\ &= \frac{1}{8(\pi\tau)^{3/2} \eta_1 \eta_2 \eta_3} \pm \frac{1}{8\pi\tau} \left( \frac{1}{\eta_1 \eta_2} + \frac{1}{\eta_2 \eta_3} + \frac{1}{\eta_3 \eta_1} \right) \\ &\quad + \frac{1}{8\sqrt{\pi\tau}} \left( \frac{1}{\eta_1} + \frac{1}{\eta_2} + \frac{1}{\eta_3} \right) \pm \frac{1}{8} \\ &= \frac{A_3}{\bar{\beta}^3 \tau^{3/2}} + \frac{A_2}{\bar{\beta}^2 \tau} + \frac{A_1}{\bar{\beta} \tau^{1/2}} + A_0, \end{aligned} \quad (38)$$

where

$$\begin{aligned} \pm A_0 &= 1/8, \\ A_1 &= L_1 + L_2 + L_3 / 8\pi^{1/2}, \\ \pm A_2 &= L_1 L_2 + L_2 L_3 + L_3 L_1 / 8\pi, \\ A_3 &= L_1 L_2 L_3 / 8\pi^{3/2}. \end{aligned} \quad (39)$$

Note that the leading finite size correction  $A_2$  has opposite signs for Neumann and Dirichlet boundary conditions. This fact results in the opposite shift in the critical temperature as we will see in Eq. (58), where the coefficient  $b_{3/2}$  is proportional to  $A_2$ . This expansion is equivalent to the short time expansion of the heat kernel used in spectral analysis on an arbitrary Riemannian manifold. The first term in Eq. (38) is the Weyl term, the second term is the boundary contribution, etc. [29,30]. Since the model is integrable, the heat kernel is factorized into contributions from each dimension.

Using the above expression in Eqs. (28) and (29), we obtain the generalized  $\zeta$  function for  $\Lambda_{ij}$

$$\begin{aligned} \zeta_\Lambda(s) &= \frac{\bar{\beta}^{2s}}{\Gamma(s)} \sum_{k=0}^3 \frac{A_k}{\bar{\beta}^k} \\ &\quad \times \int_0^\infty d\tau \tau^{s-k/2-1} e^{-(m^2-\mu^2)\bar{\beta}^2\tau} \theta_3(\mu\bar{\beta}\tau|i\tau/\pi) \\ &= \frac{\bar{\beta}^{2s}}{\Gamma(s)} \sum_{k=0}^3 \frac{A_k}{\bar{\beta}^k} \int_0^\infty d\tau \tau^{s-k/2-1} e^{-(m^2-\mu^2)\bar{\beta}^2\tau} \\ &\quad \times \left[ 1 + 2 \sum_{n=1}^\infty e^{-n^2\tau} \sum_{q=0}^\infty \frac{(-1)^q (2n\mu\bar{\beta}\tau)^{2q}}{(2q)!} \right]. \end{aligned} \quad (40)$$

Let us denote the first term in Eq. (40) on the right hand side of the equality as  $\zeta_1(s)$ , the second term as  $\zeta_2(s)$ . Then

$$\begin{aligned} \zeta_1(s) &= \frac{\bar{\beta}^{2s}}{\Gamma(s)} \sum_{k=0}^3 \frac{A_k}{\bar{\beta}^k} [(m^2-\mu^2)\bar{\beta}^2]^{k/2-s} \int_0^\infty d\tau \tau^{s-k/2-1} \\ &= \sum_{k=0}^3 \frac{\Gamma(s-k/2)}{\Gamma(s)} A_k (m^2-\mu^2)^{k/2-s} \end{aligned} \quad (41)$$

and

$$\begin{aligned} \zeta_2(s) &= \frac{2\bar{\beta}^{2s}}{\Gamma(s)} \sum_{k=0}^3 \frac{A_k}{\bar{\beta}^k} \int_0^\infty d\tau \tau^{s-k/2-1} e^{-(m^2-\mu^2)\bar{\beta}^2\tau} \\ &\quad \times \sum_{n=1}^\infty e^{-n^2\tau} \sum_{q=0}^\infty \frac{(-1)^q (2n\mu\bar{\beta}\tau)^{2q}}{(2q)!} \\ &= \frac{2\bar{\beta}^{2s}}{\Gamma(s)} \sum_{k=0}^3 \frac{A_k}{\bar{\beta}^k} \int_0^\infty d\tau \tau^{s-k/2-1} \\ &\quad \times \sum_{p=0}^\infty \frac{(-1)^p (m^2-\mu^2)^p \bar{\beta}^{2p} \tau^p}{p!} \sum_{n=1}^\infty e^{-n^2\tau} \\ &\quad \times \sum_{q=0}^\infty \frac{(-1)^q (2n\mu\bar{\beta}\tau)^{2q}}{(2q)!} \\ &= \frac{2\bar{\beta}^{2s}}{\Gamma(s)} \sum_{k=0}^3 \frac{A_k}{\bar{\beta}^k} \sum_{p=0}^\infty \sum_{q=0}^\infty \zeta(2s-k+2p+2q) \\ &\quad \times \Gamma(s-k/2+p+2q) \frac{(-1)^{p+q}}{p!(2q)!} (m^2-\mu^2)^p \\ &\quad \times \bar{\beta}^{2p} (2\mu\bar{\beta})^{2q}. \end{aligned} \quad (42)$$

We expand Eq. (42), assuming  $\bar{\beta}/L_i$ ,  $m\bar{\beta}$  ( $\mu\bar{\beta}$ ), and  $(m^2-\mu^2)\bar{\beta}^2$  are small quantities,<sup>3</sup> and obtain

$$\begin{aligned} \zeta_2(s) &= \frac{2\bar{\beta}^{2s}}{\Gamma(s)} \left[ \frac{A_3}{\bar{\beta}^3} \pi^{2s-7/2} \zeta(4-2s) \Gamma(2-s) + \frac{A_2}{\bar{\beta}^2} \pi^{2s-5/2} \right. \\ &\quad \times \zeta(3-2s) \Gamma(3/2-s) + \bar{\beta}^{-1} \pi^{2s-3/2} \zeta(2-2s) \\ &\quad \times \Gamma(1-s) \{A_1 - A_3 [m^2 + (2s-2)\mu^2]\} \\ &\quad \left. + \zeta(2s) \Gamma(s) \{A_0 - A_2 [m^2 + (2s-1)\mu^2]\} \right. \\ &\quad \left. + \bar{\beta} \zeta(2s+1) \Gamma(s+1/2) \{A_3 [(m^2-\mu^2)^2/2 + (s+3/2) \right. \\ &\quad \left. \times (s+1/2) 2\mu^4/3] - A_1 (m^2 + 2s\mu^2)\} \right]. \end{aligned} \quad (43)$$

The one-loop effective action then has the form

<sup>3</sup>For large size ( $L_i \gg \bar{\beta}$ ) and small mass  $m \ll T$ , we do not need to assume higher temperature. See also Ref. [47]

$$\begin{aligned}
\Gamma[\phi_c] &= S[\phi_c] - \zeta'_\Lambda(0) + \zeta_\Lambda(0) \log l^2 \\
&= S[\phi_c] + \frac{c_3}{\bar{\beta}^3} + \frac{c_2}{\bar{\beta}^2} + \frac{c_1}{\bar{\beta}} + c_{1/2} \log \bar{\beta} + c_0 \\
&\quad + c_{-1/2} \bar{\beta} \log \bar{\beta} + \dots,
\end{aligned} \tag{44}$$

where

$$\begin{aligned}
c_3 &= -\frac{\pi^{1/2}}{45} A_3, \\
c_2 &= -\frac{\zeta(3) A_2}{\pi^2}, \\
c_1 &= -\frac{\pi^{1/2}}{3} \{A_1 - A_3 [m^2 - 2\mu^2]\}, \\
c_{1/2} &= 2 \{A_0 - A_2 [m^2 - \mu^2]\}, \\
c_0 &= 2 \log(2\pi) A_0 - \frac{\pi^{1/2}}{2} (m^2 - \mu^2)^{1/2} A_1 \\
&\quad - \{(m^2 - \mu^2) \log(m^2 - \mu^2) + 2 \log(2\pi) m^2 \\
&\quad - [2 \log(2\pi) + 1] \mu^2\} A_2 - \frac{4\sqrt{\pi}}{2} (m^2 - \mu^2)^{3/2} A_3, \\
c_{-1/2} &= -\pi^{-1/2} \{A_3 [(m^2 - \mu^2)^2 + \mu^4] - 2A_1 m^2\}. \tag{45}
\end{aligned}$$

The total charge can likewise be written as

$$\begin{aligned}
Q &= \frac{-1}{\beta} \frac{\partial \Gamma[\phi]}{\partial \mu} \\
&= b_2 T^2 + b_{3/2} T \log T + b_1 T + b_{1/2} \log T + b_0 + \dots,
\end{aligned} \tag{46}$$

where

$$\begin{aligned}
b_2 &= \frac{8\mu\pi^{3/2}}{3} A_3, \\
b_{3/2} &= 4\mu A_2, \\
b_1 &= -\mu \left[ 4\mu\pi^{1/2} (m^2 - \mu^2)^{1/2} A_3 + 2[2 + \log(m^2 - \mu^2)] A_2 \right. \\
&\quad \left. - \frac{2\pi^{1/2} A_1}{(m^2 - \mu^2)^{1/2}} - 2 \frac{A_0}{m^2 - \mu^2} \right], \\
b_{1/2} &= -\frac{\mu C}{2\pi^{1/2}}, \\
b_0 &= \frac{\mu}{2\pi^{1/2}} \left[ 2C[\psi(1/2) + 3\gamma] + \frac{32}{3} A_3 \mu^2 - 8A_1 - \log(l^2) C \right],
\end{aligned} \tag{47}$$

where  $C = A_3(4\mu^2 - m^2)$ ,  $\gamma$  is the Euler constant, and  $\psi$  is the Digamma function [ $\psi(1/2) = -\gamma - 2 \log 2 = -1.96351 \dots$ ]. Note that  $b_2$  gives the bulk term discussed in Ref. [18].

The above derivation is based on an asymptotic expansion of the heat kernel which assumes a continuum spectrum. The same assumption may not be justified in low dimensions where the density of states does not increase as rapidly with energy due to the restricted degrees of freedom. Thus, the continuum spectral approximation has to be modified accordingly. Indeed this type of expansion does not reproduce the bulk term which appeared in [18] in the two-dimensional case [40]. Also it is not straightforward to define one-dimensional critical temperature  $T_{1D}$  with the method described above due to the existence of inverse powers of  $m^2 - \mu^2$  in Eq. (47). It is known that the chemical potential does not reach its critical value for a finite system at the critical temperature. For a relativistic field theory in curved space-time, this aspect has been studied in Ref. [41].

### III. FINITE SIZE EFFECTS AND MULTISTEP CONDENSATION

#### A. Finite size effects and discrete spectrum

In this section, we use an alternative method to treat the discrete spectrum in a more appropriate way. Rewriting the heat kernel for all accessible states in Eq. (36) gives

$$\begin{aligned}
K(\tau) &= \sum_{n_1}^{\infty} e^{-\eta_1^2 \pi^2 n_1^2 \tau} \sum_{n_2}^{\infty} e^{-\eta_2^2 \pi^2 n_2^2 \tau} \sum_{n_3}^{\infty} e^{-\eta_3^2 \pi^2 n_3^2 \tau} \\
&= \sum_{n_1}^{\infty} \sum_{n_2}^{\infty} \sum_{n_3}^{\infty} q_1^{n_1^2} q_2^{n_2^2} q_3^{n_3^2},
\end{aligned} \tag{48}$$

where  $q_i = e^{-\eta_i^2 \pi^2 \tau}$ . Throughout this section, we assume that  $L_3$  is an integral multiple of  $L_1$  and  $L_2$ , such that  $a_1 L_1 = a_2 L_2 = L_3$  for some integers  $a_1, a_2$ .<sup>4</sup>  $K(\tau)$  becomes

$$\begin{aligned}
K(\tau) &= \sum_{n_1}^{\infty} \sum_{n_2}^{\infty} \sum_{n_3}^{\infty} q_3^{a_1^2 n_1^2 + a_2^2 n_2^2 + n_3^2} \\
&= \sum_{n=0}^{\infty} r_3(n) q_3^n,
\end{aligned} \tag{49}$$

where  $r_3(n)$  is the number of solutions of the Diophantine equation  $n = a_1^2 n_1^2 + a_2^2 n_2^2 + n_3^2$  in natural numbers [48].

Let us define the function  $\mathcal{N}_3(\varepsilon)$  which counts the number of points with integer coordinates inside the ellipsoid whose  $x, y, z$  intercept are  $\sqrt{\varepsilon}/a_1, \sqrt{\varepsilon}/a_2, \sqrt{\varepsilon}$ , respectively.<sup>5</sup> In the Appendix, we give a derivation of the exact formula for  $\mathcal{N}_d(\varepsilon)$  for arbitrary dimension  $d$  which gives the

<sup>4</sup>The integer assumption here is not essential but for calculational convenience.

<sup>5</sup>Here we rewrite  $n$  as  $\varepsilon$  (dimensionless energy).

$d$ -dimensional cumulative density of states  $\bar{\mathcal{N}}_d(\varepsilon)$  in the way described below. For  $d=1,2,3$ , they are related to each other by

$$\begin{aligned}\mathcal{N}_3(\varepsilon) &= 8\bar{\mathcal{N}}_3(\varepsilon) \pm 12\bar{\mathcal{N}}_2(\varepsilon) + 6\bar{\mathcal{N}}_1(\varepsilon) \pm 1, \\ \mathcal{N}_2(\varepsilon) &= 4\bar{\mathcal{N}}_2(\varepsilon) \pm 4\bar{\mathcal{N}}_1(\varepsilon) + 1, \\ \mathcal{N}_1(\varepsilon) &= 2\bar{\mathcal{N}}_1(\varepsilon) \pm 1,\end{aligned}\quad (50)$$

where the upper (lower) signs correspond to Dirichlet (Neumann) boundary conditions.

Inverting Eq. (50), we obtain the expression for  $\bar{\mathcal{N}}_d(\varepsilon)$  ( $d=1,2,3$ ) in terms of  $\mathcal{N}_d(\varepsilon)$  ( $d=1,2,3$ ) as

$$\begin{aligned}\bar{\mathcal{N}}_3(\varepsilon) &= \frac{1}{8}[\mathcal{N}_3(\varepsilon) \mp 3\mathcal{N}_2(\varepsilon) + 3\mathcal{N}_1(\varepsilon) \mp 1], \\ \bar{\mathcal{N}}_2(\varepsilon) &= \frac{1}{4}[\mathcal{N}_2(\varepsilon) \mp 2\mathcal{N}_1(\varepsilon) + 1], \\ \bar{\mathcal{N}}_1(\varepsilon) &= \frac{1}{2}[\mathcal{N}_1(\varepsilon) \mp 1].\end{aligned}\quad (51)$$

From Eqs. (51) and (A4), we readily obtain

$$\bar{\mathcal{N}}_3(\varepsilon) = \frac{\pi}{6} \frac{\varepsilon^{3/2}}{a_1 a_2} \mp \frac{\pi \varepsilon}{8} \left[ \frac{1}{a_1 a_2} + \frac{1}{a_1} + \frac{1}{a_2} \right] + \Delta(\varepsilon). \quad (52)$$

The first term in Eq. (52) simply comes from the volume of the ellipsoid, the second term originates from compensating the oversubtracted points on the three coordinate planes. The residual term  $\Delta(\varepsilon)$  includes, in addition to the terms corresponding to higher order contributions in the asymptotic spectral expansion, the error of approximating cubes located on the surface of the sphere by a smooth surface. This error is ascribed to what is known as accidental degeneracies [37,49]. A numerical plot of  $\Delta(\varepsilon)$  given in Fig. 1 shows that this term oscillates rapidly. The fitting of  $\text{Sup}_{\varepsilon' < \varepsilon} \Delta(\varepsilon)$  gives  $\text{Sup}_{\varepsilon' < \varepsilon} \Delta(\varepsilon) \sim \varepsilon^\gamma$ , where  $\gamma=0.6$ . Since  $\gamma < 1$ , the first two terms in Eq. (52) are still dominant as long as  $\varepsilon \gg 1$ . However, the contribution  $\mathcal{N}_1(\varepsilon)$  arising from overcounting the points on coordinate axis is proportional to  $\varepsilon^{1/2}$  and smaller than the second term [50]. Hence the fluctuating part of the cumulative density of states  $\Delta(\varepsilon)$  in Eq. (52) dominates over the contributions from  $A_1$  and  $A_0$  terms in Sec. II C. Similar arguments should hold in any finite size systems regardless of whether the system is integrable or not.

For these reasons, here we properly take into account the lowest energy gap which carries essential information about finite size effects, and use the continuous spectrum approximation above the lowest excited mode. The density of states has the following form:

$$\rho_3(\varepsilon) = \frac{\pi}{4} \frac{\varepsilon^{1/2}}{a_1 a_2} \mp \frac{\pi}{8} \left[ \frac{1}{a_1 a_2} + \frac{1}{a_1} + \frac{1}{a_2} \right] + \dots \quad (53)$$

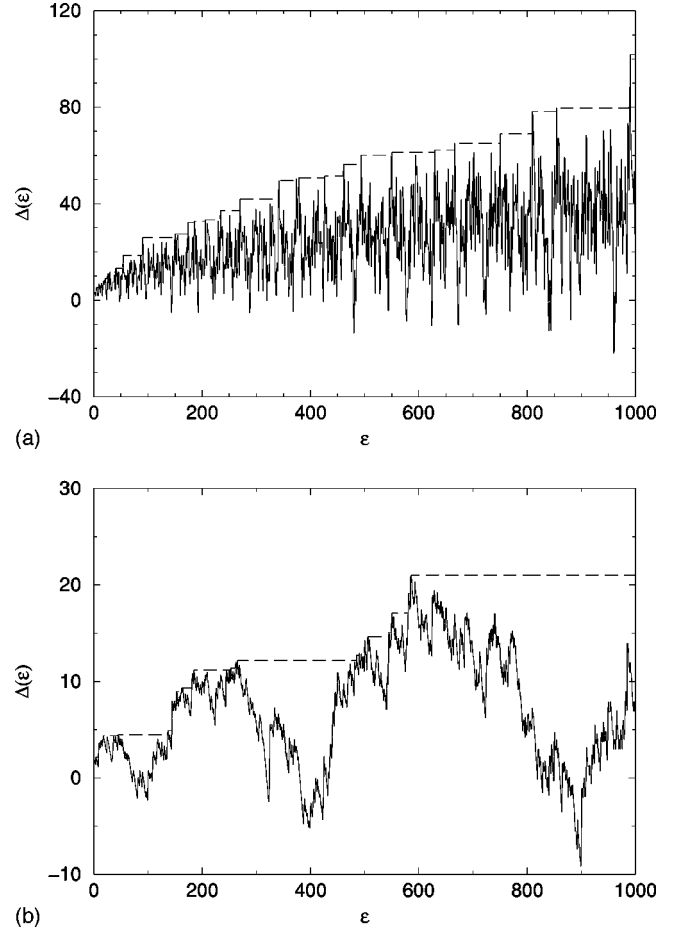


FIG. 1. The residual term  $\Delta(\varepsilon)$  in the cumulative density of states [Eq. (52)] is plotted for the Neumann boundary condition. This term shows the highly oscillating behavior due to accidental degeneracies. (a) is an isotropic case ( $a_1=a_2=1$ ) and (b) is an anisotropic case ( $a_1=10$  and  $a_2=3$ ). Supremum (dashed lines) of both curves show that the rate of increase is proportional to  $\varepsilon^\gamma$  where  $\gamma=0.6$ .

The first term is the Weyl term, the second term is the area contribution from the boundary. We can easily see these terms give the same terms in the heat kernel Eq. (38) related by a Laplace transformation.

Next we write the heat kernel in terms of the density of states as

$$K(\tau) = K_0 + \int_{\varepsilon_1}^{\infty} \rho_3(\varepsilon) q_3^{\varepsilon} d\varepsilon, \quad (54)$$

where  $\varepsilon_1=1$  corresponds to the energy level of the lowest excited mode and  $K_0$  is the contribution in Eq. (49) from the ground state. Due to the presence of a cutoff, one can show that the total charge of all excited modes is

$$Q = b_2 T^2 + \frac{b_{3/2} T}{2} \log \frac{T^2}{\tilde{m}^2} + O(\Delta), \quad (55)$$

where  $\tilde{m}^2 \equiv \pi^2/L_3^2 + m^2 - \mu^2$  and  $O(\Delta)$  is the contribution from the residual term  $\Delta(\varepsilon)$  and will be ignored hereafter.

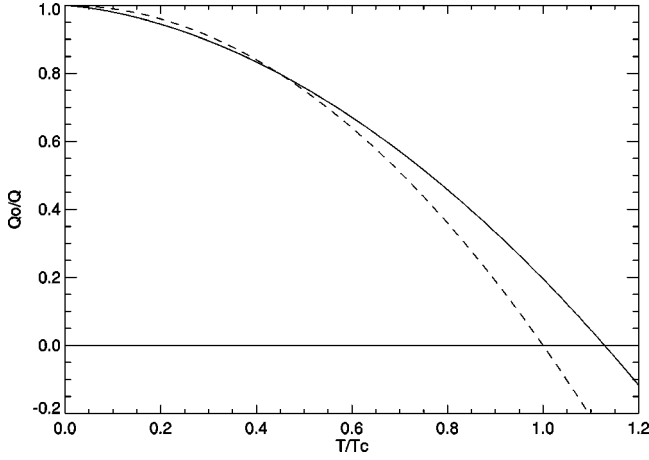


FIG. 2. The condensation fraction  $Q_0/Q$  for the Neumann boundary condition is plotted as a function of the temperature.  $L_1 = 1$ ,  $L_2 = 10$ ,  $L_3 = 100$ ,  $Q = 10\,000$ , and  $m = 0.1$ . The dotted curve shows the bulk contribution. Solid curve includes the finite size correction based on Eq. (59).  $T_c$  denotes the finite size corrected three-dimensional critical temperature defined in Eq. (58).

Then the second term in Eq. (46) is replaced by  $b_{3/2}T \log TL_3$  for large  $L_3$  close to the critical temperature.

Now we evaluate the finite size correction to the critical temperature. The bulk critical temperature  $T_c^{(0)}$  is defined by

$$Q = b_2 T_c^{(0)2}. \quad (56)$$

From Eq. (46) and the above argument, the leading correction to the bulk critical temperature for a finite system manifests as

$$Q = b_2 T_c^2 + b_{3/2} T_c \log T_c L_3. \quad (57)$$

From Eqs. (56) and (57), we obtain the finite number correction to the critical temperature for small  $b_{3/2}$

$$\frac{T_c}{T_c^{(0)}} = 1 - \frac{b_{3/2} \log(QL_3^2/b_2)}{4(b_2 Q)^{1/2}}. \quad (58)$$

The correction to the condensation fraction can be easily obtained as

$$\begin{aligned} \frac{Q_0}{Q} &= 1 - \frac{Q_1}{Q} \\ &= 1 - \left( \frac{T}{T_c^{(0)}} \right)^2 + \frac{b_{3/2} \log(QL_3^2/b_2)}{2(b_2 Q)^{1/2}} \left[ \left( \frac{T}{T_c^{(0)}} \right)^2 - \frac{T}{T_c^{(0)}} \right] \\ &\quad - \frac{b_{3/2}}{(b_2 Q)^{1/2}} \frac{T}{T_c^{(0)}} \log \left( \frac{T}{T_c^{(0)}} \right). \end{aligned} \quad (59)$$

In Fig. 2, we plot the condensation fraction of the ground state as a function of the temperature. As mentioned in Sec. II C, the finite size correction in Eq. (58) gives the opposite shift in the critical temperature whether Neumann or Dirichlet boundary condition is used. In the case of Dirichlet

boundary condition, as we saw in Eq. (53), the surface term decreases the density of states. The smaller density of states requires the excitations with higher energy for the condensation criteria (1) to be met and therefore the condensation has to start at a higher temperature. These results agree with those found in Refs. [10,50]. It is of interest to compare the results obtained here with those for atoms trapped in a harmonic oscillator potential. One can easily see that the boundary effect in such a potential is due to the Neumann boundary condition [20]: the surface term increases the density of states which results in the decrease of the critical temperature from the bulk value as observed in Ref. [33].

## B. Multistep condensation

### 1. One-dimensional condensation

As mentioned in the Introduction, in the presence of strong anisotropy, condensation can occur in multisteps. To see one-dimensional condensation, we require  $L_1 = L_2 \ll L_3$ , equivalently,  $a_1 = a_2 \gg 1$  where  $a_1 L_1 = a_2 L_2 = L_3$  as defined in Sec. III A. In such a case, it is meaningful to split all the excited quantum states into one, two, and three-dimensionally excited modes in the following way. Hereafter we focus on Neumann boundary conditions through the rest of the paper.

The corresponding heat kernels for these states can be defined as

$$\begin{aligned} K_1(\tau) &= \sum_{n_3=1}^{\infty} e^{-\eta_3^2 \pi^2 n_3^2 \tau}, \\ K_2(\tau) &= 2 \sum_{n_2=1}^{\infty} e^{-\eta_2^2 \pi^2 n_2^2 \tau} \sum_{n_3=0}^{\infty} e^{-\eta_3^2 \pi^2 n_3^2 \tau}, \\ K_3(\tau) &= \sum_{n_1=1}^{\infty} e^{-\eta_1^2 \pi^2 n_1^2 \tau} \sum_{n_2=1}^{\infty} e^{-\eta_2^2 \pi^2 n_2^2 \tau} \sum_{n_3=0}^{\infty} e^{-\eta_3^2 \pi^2 n_3^2 \tau}, \end{aligned} \quad (60)$$

respectively. The factor 2 in  $K_2(\tau)$  is due to the symmetry between  $L_1$ - $L_3$  plane and  $L_2$ - $L_3$  plane.

Following the similar steps from Eqs. (50)–(53), we obtain the expression for the three-dimensional density of states as

$$\rho_3(\varepsilon) = \frac{\pi}{4} \frac{\varepsilon^{1/2}}{a_1^2} - \frac{\pi}{24} \left[ \frac{1}{a_1^2} + \frac{2}{a_1} \right] + \dots \quad (61)$$

And the three-dimensional heat kernel in terms of the density of states is given as

$$K_3(\tau) = \int_{\varepsilon_1}^{\infty} \rho_3(\varepsilon) q_3^{\varepsilon} d\varepsilon, \quad (62)$$

where  $\varepsilon_1 = a_1^2$ . This gives the total charge of three-dimensionally excited modes



$$Q_3 = b_2 T^2 - \frac{b_{3/2} T}{6} \log \frac{T^2}{\tilde{m}^2}, \quad (63)$$

where  $\tilde{m}^2 \equiv \pi^2 \varepsilon_1 / L_3^2 + m^2 - \mu^2$ .

For the two-dimensional heat kernel,

$$\begin{aligned} K_2(\tau) &= 2 \sum_{n_2=1}^{\infty} e^{-\eta_2^2 \pi^2 n_2^2 \tau} \sum_{n_3=0}^{\infty} e^{-\eta_3^2 \pi^2 n_3^2 \tau} \\ &= 2 \sum_{n_2=1}^{\infty} \sum_{n_3=0}^{\infty} q_3^{a_1^2 n_2^2 + n_3^2} = \sum_{n=0}^{\infty} 2r_2(n) q_3^n, \end{aligned} \quad (64)$$

where  $r_2(n)$  is the number of solutions of  $n = a_1^2 n_2^2 + n_3^2$  in natural numbers. We write Eq. (64) in an integral form using the density of states  $\rho_2(\varepsilon) = \pi/2a_1$  such that

$$\begin{aligned} K_2(\tau) &= \int_{\varepsilon_1}^{\infty} d\varepsilon \rho_2(\varepsilon) q_3^{\varepsilon} \\ &= \frac{q_3^{\varepsilon_1}}{2\pi a_1 \eta_3^2 \tau}, \end{aligned} \quad (65)$$

where  $\varepsilon_1 = a_1^2$  for the present choice of units. Then from the expression of the zeta function in terms of the heat kernel in Eqs. (28) and (29) we obtain

$$\zeta_{\Lambda}(s) = \frac{L_2 L_3}{4\pi} \frac{\bar{\beta}^{2s-2}}{\Gamma(s)} \int_0^{\infty} d\tau \tau^{s-2} e^{-\tilde{m}^2 \bar{\beta}^2 \tau} \theta_3(\mu \bar{\beta} \tau | i\tau/\pi), \quad (66)$$

where  $\tilde{m}^2$  is the same as in Eq. (63). The total charge of two-dimensionally excited states is

$$Q_2 = \frac{2\mu L_2 L_3 T}{\pi} \log \frac{T^2}{\tilde{m}^2}. \quad (67)$$

For the one-dimensional case, we have

$$\begin{aligned} \zeta_{\Lambda}(s) &= \frac{\bar{\beta}^{2s}}{\Gamma(s)} \int_0^{\infty} d\tau \tau^{s-1} \\ &\times \sum_{n_3=1}^{\infty} q_3^{n_3^2} e^{-(m^2 - \mu^2) \bar{\beta}^2 \tau} \theta_3(\mu \bar{\beta} \tau | i\tau/\pi). \end{aligned} \quad (68)$$

The total charge carried by one-dimensionally excited states has the form

$$Q_1 = \frac{2\mu L_3^2 T}{\pi} \log(2\pi). \quad (69)$$

The three-dimensional critical temperature is reached when all the three-dimensionally excited modes are saturated, namely, sum of all the modes with energy larger than  $\varepsilon = a_1^2$  is equal to the total charge  $Q$ . We write this condition corresponding to Eq. (1) as

$$Q = Q_3(T_{3D}). \quad (70)$$

Thus we obtain

$$Q = b_2 T_{3D}^2 - \frac{b_{3/2} T_{3D}}{3} \log \frac{T_{3D} L_1}{\pi} + \frac{4m L_2 L_3 T_{3D}}{\pi} \log \frac{T_{3D} L_1}{\pi}, \quad (71)$$

where we set  $\mu = m$ . The third term is the contribution from two-dimensionally excited modes with energy larger than  $\varepsilon = a_1^2$ . We will ignore the contribution from  $Q_1$  based on the argument in Sec. III A that this term is dominated by the residual term  $\Delta(\varepsilon)$ .

For one-dimensional condensation to be observable, we must have

$$T_{1D} < T_{3D}. \quad (72)$$

Furthermore for sufficiently large  $L_3 \gg L_1, L_2$ , comparing the expression for  $Q_1$  in Eq. (69) with those for  $Q_2$  and  $Q_3$  in Eqs. (67) and (63), we obtain

$$Q_2(T_{1D}), Q_3(T_{1D}) \ll Q_1(T_{1D}). \quad (73)$$

Thus we obtain  $T_{1D}$  as the temperature at which one-dimensionally excited states saturate, i.e.,

$$Q = Q_1(T_{1D}). \quad (74)$$

This gives  $T_{1D} = \pi Q [2m L_3^2 \log(2\pi)]^{-1}$ .

## 2. Two-dimensional condensation

For two-dimensional condensation, we assume  $L_1 \ll L_2 = L_3$ , whence we split the excited quantum states into

$$K_1(\tau) = 2 \sum_{n_3=1}^{\infty} e^{-\eta_3^2 \pi^2 n_3^2 \tau},$$

$$K_2(\tau) = \sum_{n_2=1}^{\infty} e^{-\eta_2^2 \pi^2 n_2^2 \tau} \sum_{n_3=1}^{\infty} e^{-\eta_3^2 \pi^2 n_3^2 \tau},$$

$$K_3(\tau) = \sum_{n_1=1}^{\infty} e^{-\eta_1^2 \pi^2 n_1^2 \tau} \sum_{n_2=0}^{\infty} e^{-\eta_2^2 \pi^2 n_2^2 \tau} \sum_{n_3=0}^{\infty} e^{-\eta_3^2 \pi^2 n_3^2 \tau}. \quad (75)$$

The integer  $a_1 \gg 1$  defined by  $a_1 L_1 = L_2 = L_3$  will be used. The factor 2 in  $K_1(\tau)$  accounts for the symmetry between  $L_2$  direction and  $L_3$  direction.

The three-dimensional density of states becomes

$$\rho_3(\varepsilon) = \frac{\pi}{4} \frac{\varepsilon^{1/2}}{a_1^2} + \frac{\pi}{24} \left[ \frac{2}{a_1} + 1 \right] + \dots \quad (76)$$

This gives the total charge carried by three-dimensionally excited modes

$$Q_3 = b_2 T^2 + \frac{b_{3/2} T}{6} \log \frac{T^2}{\tilde{m}^2}, \quad (77)$$

where  $\tilde{m}^2 \equiv \pi^2 \varepsilon_1 / L_3^2 + m^2 - \mu^2$  and  $\varepsilon_1 = a_1^2$ .

The two-dimensional heat kernel is given by the density of states  $\rho_2(\varepsilon) = \pi/4$  as

$$K_2(\tau) = \int_{\varepsilon_1}^{\infty} d\varepsilon \rho_2(\varepsilon) q_3^\varepsilon = \frac{q_3^{\varepsilon_1}}{4\pi\eta_3^2\tau}, \quad (78)$$

where  $\varepsilon_1 = 1$ . The total charge of two-dimensionally excited states is

$$Q_2 = \frac{\mu L_2 L_3 T}{2\pi} \log \frac{T^2}{\tilde{m}^2}. \quad (79)$$

The total charge carried by one-dimensionally excited states is given by  $Q_1 = 4\mu L_3^2 T \log(2\pi)/\pi$ .

The three-dimensional critical temperature is reached under the same condition as in one-dimensional condensation,  $Q = Q_3(T_{3D})$ . Thus we obtain

$$Q = b_2 T_{3D}^2 + \frac{b_{3/2} T_{3D}}{3} \log \frac{T_{3D} L_1}{\pi} + \frac{m L_3^2 T_{3D}}{\pi} \log \frac{T_{3D} L_1}{\pi}, \quad (80)$$

where the third term is the contribution from the two-dimensional modes as in Eq. (71).

For two-dimensional condensation to be observable, we must have

$$T_{2D} < T_{3D}. \quad (81)$$

Furthermore, for sufficiently large  $L_2, L_3 \gg L_1$ , by comparing the expression in Eqs. (77) and (79), we have

$$Q_3(T_{2D}) \ll Q_2(T_{2D}). \quad (82)$$

Thus we obtain  $T_{2D}$  as the temperature at which two-dimensionally excited states saturate,

$$Q = Q_2(T_{2D}). \quad (83)$$

Then  $T_{2D} = Q[\tilde{b}_{3/2} \log(QL_3/\tilde{b}_{3/2})]^{-1}$  for large  $Q$ , where  $\tilde{b}_{3/2} = mL_2 L_3 / \pi = mL_3^2 / \pi$ .

### 3. Three-step condensation

To show three-step condensation, we assume  $L_1 \ll L_2 \ll L_3$ , or equivalently,  $a_1 \gg a_2 \gg 1$ . The corresponding heat kernels for these states can be defined as

$$K_1(\tau) = \sum_{n_3=1}^{\infty} e^{-\eta_3^2 \pi^2 n_3^2 \tau} = \frac{1}{2} [\theta_3(0|i\eta_3^2 \pi \tau) - 1],$$

$$K_2(\tau) = \sum_{n_2=1}^{\infty} e^{-\eta_2^2 \pi^2 n_2^2 \tau} \sum_{n_3=0}^{\infty} e^{-\eta_3^2 \pi^2 n_3^2 \tau} \\ = \frac{1}{4} [\theta_3(0|i\eta_2^2 \pi \tau) - 1] [\theta_3(0|i\eta_3^2 \pi \tau) + 1],$$

$$K_3(\tau) = \sum_{n_1=1}^{\infty} e^{-\eta_1^2 \pi^2 n_1^2 \tau} \sum_{n_2=0}^{\infty} e^{-\eta_2^2 \pi^2 n_2^2 \tau} \sum_{n_3=0}^{\infty} e^{-\eta_3^2 \pi^2 n_3^2 \tau} \\ = \frac{1}{8} [\theta_3(0|i\eta_1^2 \pi \tau) - 1] [\theta_3(0|i\eta_2^2 \pi \tau) + 1] \\ \times [\theta_3(0|i\eta_3^2 \pi \tau) + 1]. \quad (84)$$

The asymptotic behavior when  $1 \gg \eta_1 \gg \eta_2 \gg \eta_3$  can be derived in a similar way as in Eq. (38).

The three-dimensional density of states becomes

$$\rho_3(\varepsilon) = \frac{\pi}{4} \frac{\varepsilon^{1/2}}{a_1 a_2} + \frac{\pi}{24} \left[ \frac{1}{a_1 a_2} + \frac{1}{a_1} + \frac{1}{a_2} \right] + \dots \quad (85)$$

and the total charge of three-dimensionally excited modes is given by the same form as in Eq. (77).

From the two-dimensional heat kernel, we obtain the two-dimensional density of states  $\rho_2(\varepsilon) = \pi/4a_2$  such that

$$K_2(\tau) = \int_{\varepsilon_1}^{\infty} d\varepsilon \rho_2(\varepsilon) q_3^\varepsilon \\ = \frac{q_3^{\varepsilon_1}}{4\pi a_2 \eta_3^2 \tau}, \quad (86)$$

where  $\varepsilon_1 = a_2^2$  for the present case. This will give us the total charge of two-dimensionally excited states as

$$Q_2 = \frac{\mu L_2 L_3 T}{2\pi} \log \frac{T^2}{\tilde{m}^2}. \quad (87)$$

The total charge carried by one-dimensionally excited states has the same form as in Eq. (69).

The three-dimensional critical temperature is obtained as

$$Q = b_2 T_{3D}^2 + \frac{b_{3/2} T_{3D}}{3} \log \frac{T_{3D} L_1}{\pi} + \frac{m L_2 L_3 T_{3D}}{\pi} \log \frac{T_{3D} L_1}{\pi}. \quad (88)$$

To observe three-step condensation, we must have

$$T_{1D} < T_{2D} < T_{3D}. \quad (89)$$

In addition, for large anisotropy  $L_1 \ll L_2 \ll L_3$ , comparison of explicit formulas for  $Q_1$ ,  $Q_2$ , and  $Q_3$  in Eqs. (69), (87), and (77) gives

$$Q_3(T_{2D}) \ll Q_2(T_{2D}) \quad (90)$$

and

$$Q_3(T_{1D}) \ll Q_2(T_{1D}) \ll Q_1(T_{1D}), \quad (91)$$

where  $T_{2D}$  ( $T_{1D}$ ) is obtained by the saturation of the two- (one)-dimensionally excited states as  $Q = Q_2(T_{2D})$  and  $Q = Q_1(T_{1D})$ .  $T_{2D}$  is given in the leading order by

$$Q = \tilde{b}_{3/2} T_{2D} \log(L_2 T_{2D}), \quad (92)$$

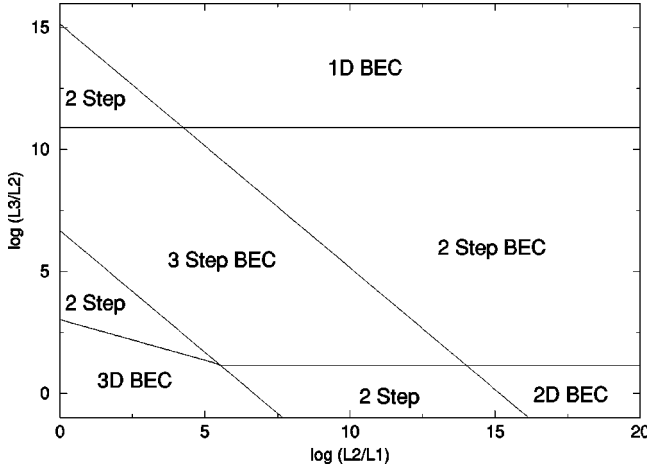


FIG. 3. Different multistep behaviors corresponding to different anisotropy parameters  $L_2/L_1$  and  $L_3/L_2$  are indicated. The logarithmic scale is used for both axis.  $\tilde{Q} \equiv \pi Q/mL_2 = 10\,000$  is fixed. Multistep BEC can only happen in an intermediate yet strongly anisotropic regime.

where  $\tilde{b}_{3/2} = mL_2L_3/\pi$ . We thus obtain  $T_{2D} = Q[\tilde{b}_{3/2} \log(QL_2/\tilde{b}_{3/2})]^{-1}$  for large  $Q$ .  $T_{1D}$  has the same form as in the one-dimensional condensation case.

From Eqs. (89) to (91), we obtain the following inequalities for anisotropy parameters and the total charge corresponding to (A)  $T_{1D} < T_{2D}$ , (B)  $T_{2D} < T_{3D}$ , and (C)  $T_{1D} < T_{3D}$  as

$$\frac{L_3}{L_2} \gg \frac{\log(\tilde{Q})}{2 \log(2\pi)}, \quad (93)$$

$$\frac{L_3}{L_1} \gg \frac{\pi\tilde{Q}}{3[\log\tilde{Q}]^2}, \quad (94)$$

$$\frac{L_3}{L_1} \left[ \frac{L_3}{L_2} \right]^2 \gg \frac{\pi\tilde{Q}}{12[\log(2\pi)]^2}, \quad (95)$$

where  $\tilde{Q} \equiv \pi Q/mL_2$ . In Fig. 3, different multistep behaviors corresponding to various ranges of anisotropy parameters are shown. Three-step BEC can be seen in a wide region of parameter space. For an extremely strong anisotropy, dynamics along the short edge length will freeze out ( $EIRD < 3$ ) before BEC into  $Q_2$  or  $Q_1$  sets in. In such a case, we only observe two or one-step BEC. For a quasilinear cavity along the vertical axis, three-step BEC can still be observed, while for a quasiplanar cavity along the horizontal axes, only up to two-step BEC can happen.

In Fig. 4, the condensation fractions

$$Q_0/Q, Q_1/Q, Q_2/Q, Q_3/Q$$

as a function of the temperature are plotted. In the isotropic case [Fig. 4(a)], condensation is only into the ground state. Due to the finite size effects, condensation occurs before the critical temperature is reached. In strongly anisotropic cases,

condensation occurs in steps. In Fig. 4(b), one-dimensional condensation is seen.  $T_{3D}$  determines the onset of condensation into one-dimensionally excited states. Note that at  $T_{3D}$ , the ground state fraction is negligibly small. Condensation into the ground state occurs at a much lower temperature. In Fig. 4(c), two-dimensional condensation is plotted. At  $T_{3D}$ , two-dimensional condensation manifests itself. The critical condition  $\mu = m$  is satisfied well in Figs. 4(a)–4(c). In Fig. 4(d), three-step condensation is shown. Three-dimensionally excited modes dominant in higher temperature are condensed into two, one, and the ground state as the temperature is lowered. The deviation of  $T_{3D}$  and the onset of two-dimensional components reflects the fact that the condition  $\mu = m$  is not satisfied for the parameters chosen in Fig. 4(d) at  $T_{3D}$ . This is another manifestation of finite size effects and the result should improve near the thermodynamic limit ( $\eta_i \rightarrow 0$ ,  $Q \rightarrow \infty$ ). The similarity between each condensation process becomes evident in the logarithmic  $T$  scale as can be seen in Fig. 4(e). In conclusion, finite size effects on the Bose-Einstein condensation of a charged scalar field can lead to the multistep condensation in the presence of strong anisotropy.

In this paper, we started from calculating the effective action to one-loop order using zeta function regularization. Large volume and small mass conditions are assumed to facilitate an asymptotic expansion of the heat kernel and finite size corrections corresponding to the surface term, corner term, etc., are obtained. We proceeded beyond the continuum spectrum approximation and showed that the higher order terms in the standard asymptotic expansion are dominated by the contribution from the fluctuating part of the density of states due to accidental degeneracy. The lowest-energy gap is shown to play the crucial role in determining the critical temperatures for one- and two-dimensional systems. The corresponding low-dimensional critical temperatures are calculated. The energy spectrum and the associated heat kernel can be partitioned into parcels of eigenmodes excitable in dimensions 3, 2, 1, or 0. As the temperature is lowered, modes in different parcels behave quite similarly in the presence of strong anisotropy. When  $T_{1D}, T_{2D} < T_{3D}$  are satisfied, condensation occurs first into the lower-dimensionally excited states at  $T_{3D}$  following the ground state condensation at lower-dimensional critical temperature. Experimental observation of these phenomena can in principle be realized in an anisotropic harmonic potential traps.

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## APPENDIX A: CUMULATIVE DENSITY OF STATES

In this appendix, the exact formula for  $\mathcal{N}_d(\varepsilon)$  for arbitrary dimension  $d$  is derived.  $d$ -dimensional cumulative den-

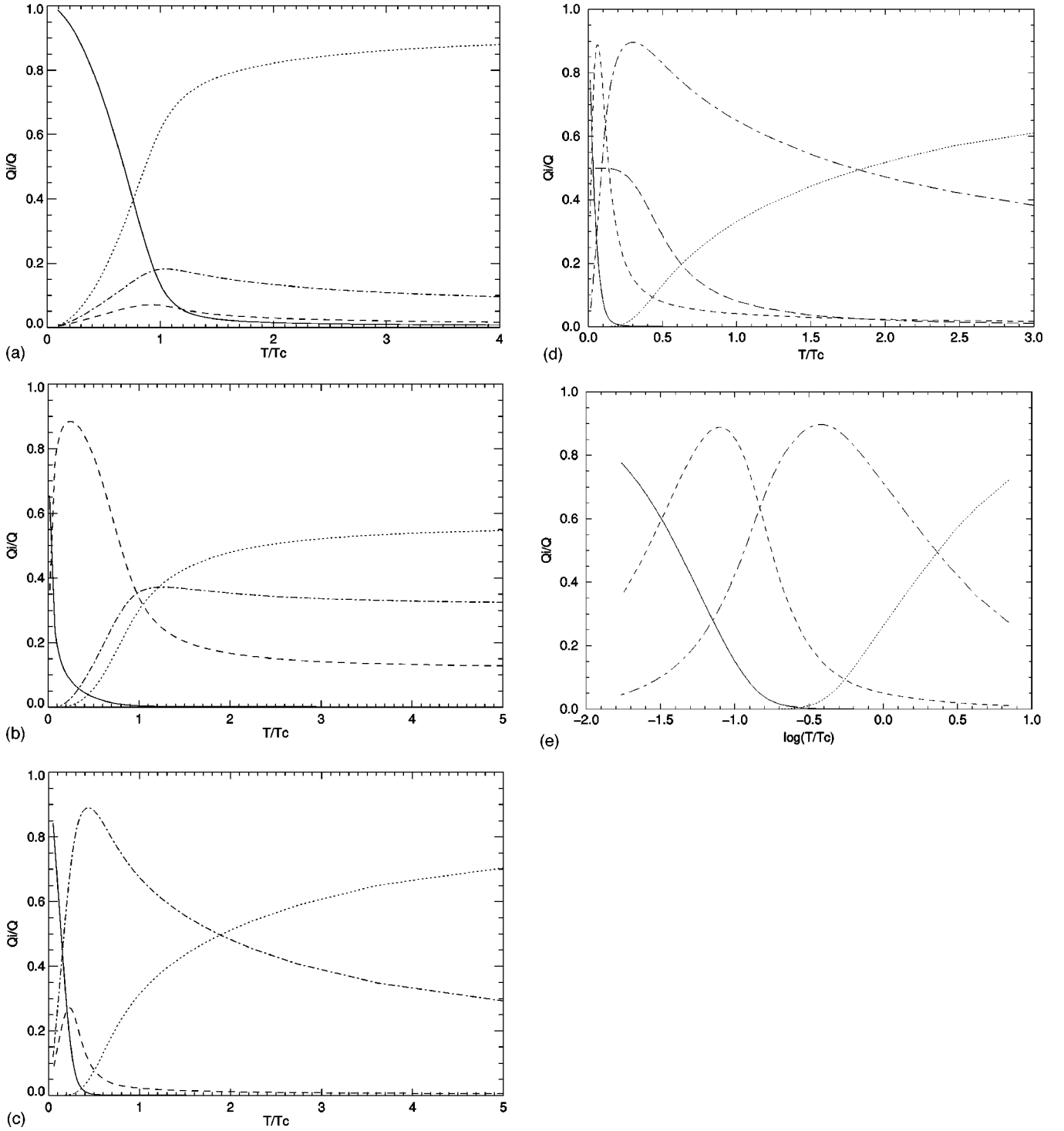


FIG. 4. The condensation fractions  $Q_0/Q$  (solid curve),  $Q_1/Q$  (dashed curve),  $Q_2/Q$  (dot-dashed curve),  $Q_3/Q$  (dotted curve) as a function of the temperature are plotted for the Neumann boundary condition. Isotropic case ( $L_1=L_2=L_3=3$ ,  $Q=100$ , and  $m=2$ ) are plotted in (a). Condensation is only into the ground state.  $T_c=1.97$  is the critical temperature in Eq. (57). In (b)–(e), anisotropic cases are shown. In (b),  $L_1=2$ ,  $L_2=2$ ,  $L_3=300$ ,  $Q=2000$ , and  $m=1$  are chosen. One-dimensional condensation occurs in this case.  $T_c=2.03$  is the three-dimensional critical temperature in Eq. (71). In (c),  $L_1=2$ ,  $L_2=200$ ,  $L_3=200$ ,  $Q=8000$ , and  $m=0.5$  are chosen.  $T_c=0.98$  is the three-dimensional critical temperature in Eq. (80). Two-dimensional condensation can be seen. In (d), condensation occurs in three steps.  $L_1=2$ ,  $L_2=100$ ,  $L_3=600$ ,  $Q=4000$ , and  $m=0.5$  are used. The long dashed line is the chemical potential  $\mu$ .  $T_c=0.79$  is the three-dimensional critical temperature in Eq. (88). The logarithmic  $T$  scale is used in (e) for the parameters in (d).

sity of states  $\bar{\mathcal{N}}_d(\varepsilon)$  can be obtained straightforwardly from  $\mathcal{N}_d(\varepsilon)$  as we showed in Sec. III. Suppose  $\mathcal{N}_d(\varepsilon)$  counts the number of integer solutions  $\vec{n}$  of the equation  $\varepsilon = a_1^2 n_1^2 + \dots + a_d^2 n_d^2$  where  $\vec{a} \equiv (a_1, \dots, a_d)$  is a constant  $d$ -dimensional vector with integer coordinates. Then  $\mathcal{N}_d(\varepsilon)$  can be written as

$$\begin{aligned} \mathcal{N}_d(\varepsilon) &= \sum_{\vec{n}} \theta(\varepsilon - |\vec{a}\vec{n}|^2) \\ &= \sum_{\vec{n}} \int d\vec{u} \theta(\varepsilon - |\vec{u}|^2) \delta^d(\vec{u} - \vec{a}\vec{n}) \\ &= A^{-1} \sum_{\vec{l}} \int d\vec{u} \theta(\varepsilon - |\vec{u}|^2) e^{2\pi i \vec{u} \cdot (\vec{l}/\vec{a})} \\ &= \sum_{\vec{l}} C(\vec{l}), \end{aligned} \quad (\text{A1})$$

where  $A \equiv a_1 \dots a_d$ ,  $\vec{a}\vec{n} \equiv (a_1 n_1, \dots, a_d n_d)$ , and  $\vec{l}/\vec{a} \equiv (l_1/a_1, \dots, l_d/a_d)$ . Summation is over all  $d$ -dimensional vectors with interger coordinates. Poisson's summation formula is used to obtain the third line and

$$\begin{aligned} C(\vec{l}) &\equiv A^{-1} \int d\vec{u} \theta(\varepsilon - |\vec{u}|^2) e^{2\pi i \vec{u} \cdot (\vec{l}/\vec{a})} \\ &= A^{-1} \int d\vec{u} \theta(\varepsilon - |\vec{u}|^2) e^{2\pi i |\vec{u}| |\vec{l}/\vec{a}| \cos \theta}, \end{aligned} \quad (\text{A2})$$

where  $\theta$  is the angle between  $\vec{u}$  and  $\vec{l}/\vec{a}$ . We make an orthogonal coordinate transformation from  $\vec{u}$  to  $\vec{v}$  such that  $v_1 = |\vec{u}| \cos \theta$  and write Eq. (A2) as

$$\begin{aligned} C(\vec{l}) &= A^{-1} \int d\vec{v} \theta(\varepsilon - |\vec{v}|^2) e^{2\pi i v_1 |\vec{l}/\vec{a}|} \\ &= A^{-1} \frac{\pi^{(d-1)/2} \varepsilon^{(d-1)/2}}{\Gamma[(d+1)/2]} \\ &\quad \times \int_{-1}^1 dv_1 (1 - v_1^2)^{(d-1)/2} e^{2\pi i v_1 |\vec{l}/\vec{a}|}. \end{aligned} \quad (\text{A3})$$

This yields

$$C(\vec{l}) = \begin{cases} A^{-1} \varepsilon^{d/4} J_{d/2}(2\pi |\vec{l}/\vec{a}| \sqrt{\varepsilon}) / |\vec{l}/\vec{a}|^{d/2} & \text{for } \vec{l} \neq 0, \\ A^{-1} \varepsilon^{d/4} V_{d-1} & \text{for } \vec{l} = 0, \end{cases}$$

where  $J_d(x)$  is the Bessel function and  $V_d = \pi^{d/2} / \Gamma(d/2 + 1)$  is the volume of a  $d$ -dimensional sphere with unit radius. Hence we obtain

$$\bar{\mathcal{N}}_d(\varepsilon) = A^{-1} \varepsilon^{d/4} V_{d-1} + A^{-1} \varepsilon^{d/4} \sum_{\vec{l}} \frac{J_{d/2}(2\pi |\vec{l}/\vec{a}| \sqrt{\varepsilon})}{|\vec{l}/\vec{a}|^{d/2}}. \quad (\text{A4})$$

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